Tight Security Analysis of 3-Round Key-Alternating Cipher with A Single Permutation

Yusai Wu[1], Liqing Yu[1], Zhenfu Cao[1,2,3(✉)], and Xiaolei Dong[1]

[1] Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai, China
yusaiwu@126.com, lqyups@126.com, zfcao@sei.ecnu.edu.cn, dong-xl@sei.ecnu.edu.cn
[2] Cyberspace Security Research Center, Peng Cheng Laboratory, Shenzhen, China
[3] Shanghai Institute of Intelligent Science and Technology, Tongji University, China

Abstract. The tight security bound of the KAC (Key-Alternating Cipher) construction whose round permutations are independent from each other has been well studied. Then a natural question is how the security bound will change when we use fewer permutations in a KAC construction. In CRYPTO 2014, Chen et al. proved that 2-round KAC with a single permutation (2KACSP) has the same security level as the classic one (i.e., 2-round KAC). But we still know little about the security bound of incompletely-independent KAC constructions with more than 2 rounds. In this paper,we will show that a similar result also holds for 3-round case. More concretely, we prove that 3-round KAC with a single permutation (3KACSP) is secure up to $\Theta(2^{\frac{3n}{4}})$ queries, which also caps the security of 3-round KAC. To avoid the cumbersome graphical illustration used in Chen et al.'s work, a new representation is introduced to characterize the underlying combinatorial problem. Benefited from it, we can handle the knotty dependence in a modular way, and also show a plausible way to study the security of rKACSP. Technically, we abstract a type of problems capturing the intrinsic randomness of rKACSP construction, and then propose a high-level framework to handle such problems. Furthermore, our proof techniques show some evidence that for any r, rKACSP has the same security level as the classic r-round KAC in random permutation model.

1 Introduction

In provable-security setting, the construction of a practical cipher is often abstracted into a reasonable model with certain assumptions (e.g., the underlying primitives are random functions/permutations and independent from each other). Under those assumptions, we try to prove that the abstract construction is immune to all (known or unknown) attacks executed by an adversary with specific abilities. Then the provable-security results provide some heuristic support for the underlying design-criteria of the cipher, since the practical underlying primitives do not satisfy the assumptions in general.

As aforementioned, the provable-security results are closely related to the abstract assumptions. If the assumptions are closer to the actual implementations, then the corresponding results will be more persuasive. For example, most of the existing work reduces the security of SPN block ciphers to the classic KAC construction (see Eq.(2)), in which the underlying round permutations as well as the round keys are random and independent from each other. Unfortunately, most KAC-based practical ciphers use the same round function and generate the round keys from a shorter master-key (i.e., the underlying round permutations and round keys are not independent from each other at all). Thus, there is still a big gap between the existing provable-security results and the practical ciphers.

Opposite to the KAC construction with independent round permutations and round keys (i.e., the classic KAC construction), we refer to the one whose round permutations or round keys are not independent from each other as *incompletely-independent* KAC or KAC *with dependence*. It is well known that r-round KAC is $\Theta(2^{\frac{r}{r+1}n})$-secure in the random permutation model [CS14,HT16]. To characterize the actual SPN block ciphers, we should abstract a natural KAC construction (with dependence) satisfying two requirements: all the round permutations are the same and the round keys are generated from a shorter master-key by a certain deterministic algorithm. Hence, the ultimate question is whether there exists such a r-round incompletely-independent KAC construction which can still achieve $\Theta(2^{\frac{r}{r+1}n})$-security. In other words, we want to know whether the required randomness of KAC construction can be minimized without a significant loss of security.

Up to now, people know little about the incompletely-independent KAC constructions (even with very small number of rounds), since it becomes much more complicated when either the underlying round permutations or round keys are no longer independent. To our knowledge, the best work about the KAC with dependence was given by Chen et al.[CLL$^+$18]. They proved that several types of 2-round KAC with dependence have almost the same security level as 2-round KAC construction. However, it is still open about the security of incompletely-independent KAC with more than 2 rounds in provable-security setting.

In this paper, we initiate the study on the incompletely-independent KAC with more than 2 rounds. Here, we mainly focus on a special class of KAC, in which all the round permutations are the same and the round keys are still independent from each other, and refer to it as *KACSP construction*. Given a permutation $P : \{0,1\}^n \to \{0,1\}^n$, as well as $r+1$ round keys $k_0, \ldots, k_r$, the r-round KACSP construction rKACSP$[P; k_0, \ldots, k_r]$ maps a message $x \in \{0,1\}^n$ to

$$k_r \oplus P\Big(k_{r-1} \oplus P\big(\cdots P(x \oplus k_0) \cdots\big)\Big). \tag{1}$$

Before turning into the results, we review the related existing work on classic KAC and KAC with dependence, respectively.

<u>RESULTS ON CLASSIC KAC.</u> KAC construction is the generalization of the Even-Mansour construction [EM97] over multiple rounds. As one of the most popular

ways to construct a practical cipher, the KAC construction captures the high-level structure of many SPN block ciphers, such as AES [DR02], PRESENT [BKL$^+$07], LED [GPPR11] and so on. Given r permutations $P_1, \ldots, P_r \colon \{0,1\}^n \to \{0,1\}^n$, as well as $r+1$ round keys $k_0, \ldots, k_r$, the r-round KAC construction $r\mathrm{KAC}[P_1, \ldots, P_r;\, k_0, \ldots, k_r]$ maps a message $x \in \{0,1\}^n$ to

$$k_r \oplus P_r\Big(k_{r-1} \oplus P_{r-1}\big(\cdots P_1(x \oplus k_0)\cdots\big)\Big). \tag{2}$$

In the random permutation model, it was proved by Even and Mansour [EM97] that an adversary needs roughly $2^{\frac{n}{2}}$ queries to distinguish the 1-round KAC construction from a true random permutation. Their bound was matched by a distinguishing attack [Dae91] which needs about $2^{\frac{n}{2}}$ queries in total. Many years later, Bogdanov et al. [BKL$^+$12] proved that r-round KAC is secure up to $2^{\frac{2n}{3}}$ queries and the result is tight for $r = 2$. Besides, they also conjectured that the security for r-round KAC should be $2^{\frac{rn}{r+1}}$ because of a simple generic attack. After that, Steinberger [Ste12] improved the bound to $2^{\frac{3n}{4}}$ queries for $r \geq 3$ by modifying the way to upper bound the statistical distance between two product distributions. In the same year, Lampe et al. [LPS12] used coupling techniques to show that $2^{\frac{rn}{r+1}}$ queries and $2^{\frac{rn}{r+2}}$ queries are needed for any nonadaptive and any adaptive adversary, respectively. The first asymptotically tight bound was proved by Chen et al. [CS14] through an elegant path-counting lemma. Recently, Hoang and Tessaro [HT16] refined the H-coefficient technique (named as the expectation method) and gave the first exact bound of KAC construction. At this point, the security bound of the classic KAC construction is solved perfectly.

Results on KAC with dependence. The development in the field of incompletely-independent KAC is much slower, since it usually becomes very involved when the underlying components are no longer independent from each other. Dunkelman et al. [DKS12] initiated the study of minimizing 1-round KAC construction, and showed that several strictly simpler variants provide the same level of security. After that, the best work was given by Chen et al. [CLL$^+$14] in CRYPTO 2014. They proved that several types of incompletely-independent 2-round KAC have almost the same security level as the classic one. The result even holds when only a single permutation and a n-bit master-key are used, where n is the length of a plaintext/ciphertext. In their work, a generalized *sum-capture theorem* [1] is used to upper bound the probability of bad transcripts. And the probability calculation related to good transcripts is reduced to a combinatorial

[1] Informally, the type of sum-capture theorems state that when choosing a random subset A of $\mathbb{Z}_2^n$ of size q, the value

$$\mu(A) = \max_{\substack{U,V \subseteq \mathbb{Z}_2^n \\ |U|=|V|=q}} |\{(a,u,v) \in A \times U \times V : a = u \oplus v\}|$$

is close to the expected value q^3/N. In the extended version of [CLL$^+$18], the set A can be produced by a set of query-answer pairs, and an automorphism transformation is also allowed.

problem. Using the similar techniques, Cogliati and Seurin [CS18] obtained the security bound of the single-permutation encrypted Davies-Meyer construction. Nevertheless, their work is still limited in the scope of 2-round constructions.

Recently, Dai et al. [DSST17] proved that the 5-round KAC with a non-idealized key-schedule is indifferentiable from an ideal cipher. The model employed in their work is however orthogonal to ours and hence the result is not directly comparable.

Our Contributions. In this paper, we initiate the study on the incompletely-independent KAC with more than 2 rounds and give a tight security bound of 3KACSP construction. Our contributions are conceptually novel and mainly two-fold:

1. We prove the tight security bound $\Theta(2^{\frac{3n}{4}})$ queries of 3KACSP, which is an open problem (proposed in [CLL$^+$18]) for incompletely-independent KAC with more than 2 rounds. That is, we can use only one instead of three distinct permutations to construct 3-round KAC without a significant loss of security. Notably, our proof framework is general and theoretically workable for any rKACSP. Following the ideas of analyzing 3KACSP, we strongly believe that rKACSP is also $\Theta(2^{\frac{r}{r+1}n})$-secure in random permutation model, provided that the input/output size n is sufficiently large.
2. We develop a lot of general techniques to handle the dependence. Firstly, a new representation (see Section 3.3) is introduced to circumvent the cumbersome graphical illustration used in [CLL$^+$18]. Benefited from it, we can handle the underlying combinatorial problem in a natural and intuitive way. Secondly, we abstract a type of combinatorial problems (i.e., Problem 1) capturing the intrinsic randomness of rKACSP, and also propose a high-level framework (see Section 5.1) to solve such problems. To instantiate the framework, we introduce some useful notions such as Core, *target-path*, *shared-edge*, and so on (see Section 3.3). Combining with the methods for constructing multiple shared-edges, we solve successfully the key problem in 3KACSP (see Section 5.2). At last, we also develop some new tricks (see Section 6 in the full version of this paper [WYCD20]) which are crucial in analyzing rKACSP ($r \geq 3$). Such tricks are not needed in 2KACSP, since it is relatively simple and does not have much dependence to handle.

It is rather surprising that the randomness of a single random permutation can provide such high level of security. From our proof, we can know an important reason is that, the information obtained by adversary is actually not so much. For instance, assume that n is big enough and an adversary can make $\Theta(2^{\frac{3n}{4}})$ queries to the random permutation, then the ratio of known points (i.e., roughly $2^{-\frac{n}{4}}$) is still very small. Furthermore, our work means a lot more than simply from 2 to 3, and we now show something new compared to Chen et al.'s work.

1. It is the first time to convert the analysis of rKACSP into a type of combinatorial problems, thus we can study the higher-round constructions in a

modular way. To solve such problems, we propose a general counting framework, and also successfully instantiate it for a 3-round case which is much more involved than the 2-round cases.

2. An important discovery is that we can adapt the tricks used in 2KACSP to solve the corresponding subproblems in 3KACSP, by designing proper assigning-strategy and *RoC*s(Range of Candidates, see Notation 5). We believe that the similar properties also hold in the analysis of general rKACSP.
3. A very big challenge in rKACSP($r \geq 3$) is to combine all the subproblems together into a desired bound. We do not need to consider that problem in the case of 2KACSP, since there is only one 2-round case in it. As a result, we develop some useful techniques to handle the dependence between the subproblems. Particularly, the key-points as shown at the beginning of Section 6 in the full version [WYCD20] are also essential in rKACSP($r \geq 4$).

Combining all above findings together, we point out that a plausible way to analyze rKACSP is by induction, and what's left is only to solve a single r-round case of Problem 1. That is, we actually reduce an extremely complex (maybe intractable) problem into a single combinatorial problem, which can be solved by our framework theoretically. From the view of induction, Chen et al. [CLL^{+}18] proved the basis step, while we have done largely the non-trivial work of the inductive step. Besides the conceptually important results, the new notions and ideas used in our proof are rather general and not limited in the rKACSP setting. We hope that they can be applied to analyze more different cryptographic constructions with dependence.

Outline of This Paper. We start in Section 2 by setting the basic notations, giving the necessary background on the H-coefficient technique, and showing some helpful lemmas. In Section 3, we state the main result of this paper and introduce the new representation used throughout the paper. After that, the main result is proved in Section 4 where we also illustrate the underlying combinatorial problem and give two technical lemmas. The core part is Section 5, where we propose the general framework and also show the high-level technical route to handle the key subproblem in 3KACSP. At last, we conclude and give some extra discussion in Section 6.

2 Preliminaries

2.1 Basic Notations

In this paper, we use capital letters such as $A, B, \ldots$ to denote sets. If A is a finite set, then $|A|$ denotes the cardinality of A, and $\overline{A}$ denotes the complement of A in the universal set (which will be clear from the context). For a finite set S, we let $x \leftarrow_{\$} S$ denote the uniform sampling from S and assigning the value to x. Let A and B be two sets such that $|A| = |B|$, then we denote $Bjt(A \rightarrow B)$ as the set of all bijections from A to B. If g and h are two well-defined bijections, then let $g \circ h(x) = h\big(g(x)\big)$. Fix an integer $n \geq 1$, let $N = 2^n$, $I_n = \{0,1\}^n$, and $\mathcal{P}_n$ be the set of all permutations on $\{0,1\}^n$, respectively. If two integers s, t satisfy $1 \leq s \leq t$, then we will write $(t)_s = t(t-1)\cdots(t-s+1)$ and $(t)_0 = 1$ by convention.

Given $\mathcal{Q} = \{(x_1, y_1), \ldots, (x_q, y_q)\}$, where the x_i's (resp. y_i's) are pairwise distinct n-bit strings, as well as a permutation $P \in \mathcal{P}_n$, we say that the permutation P extends the set $\mathcal{Q}$, denoting $P \vdash \mathcal{Q}$, if $P(x_i) = y_i$ for $i = 1, \ldots, q$. Let $X = \{x \in I_n : (x, y) \in \mathcal{Q}\}$ and $Y = \{y \in I_n : (x, y) \in \mathcal{Q}\}$. We call X and Y respectively the domain and range of the set $\mathcal{Q}$.

Definition 1 ($\mathcal{Q}'$ is strongly-disjoint with $\mathcal{Q}$). *Let $\mathcal{Q} = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ and $\mathcal{Q}' = \{(x'_1, y'_1), \ldots, (x'_n, y'_n)\}$. We denote X,Y,X',Y' as the domains and ranges of $\mathcal{Q}$ and $\mathcal{Q}'$, respectively. Then we say that **$\mathcal{Q}'$ is strongly-disjoint with $\mathcal{Q}$** if $X \cap X' = \emptyset$ and $Y \cap Y' = \emptyset$, and denote it as **$\mathcal{Q}' \perp \mathcal{Q}$**.*

2.2 Indistinguishability Framework

We will focus on the provable-security analysis of block ciphers in random permutation model, which allows the adversary to get access to the underlying primitives of the block ciphers. Consider the rKACSP construction (see Eq.(1)), a distinguisher $\mathcal{D}$ can interact with a set of 2 *permutation oracles* on n bits that we denote as (P_O, P_I). There are two worlds in terms of the instantiations of the 2 permutation oracles. If P is a random permutation and the round keys $\boldsymbol{K} = (k_0, \ldots, k_r)$ are randomly chosen from $I_{(r+1)n}$, we refer to $(r\text{KACSP}[P; \boldsymbol{K}], P)$ as *the "real" world.* If E is a random permutation independent from P, we refer to (E, P) as *the "ideal" world.* We usually refer to the first permutation P_O (instantiated by $r\text{KACSP}[P; \boldsymbol{K}]$ or E) as *the outer permutation*, and to permutation P_I (instantiated by P) as *the inner permutation.* Given a certain number of the queries to the 2 permutation oracles, the distinguisher $\mathcal{D}$ should distinguish whether the "real" world or the "ideal" world it is interacting with. The distinguisher $\mathcal{D}$ is adaptive such that it can query both sides of each permutation oracle, and also can choose the next query based on the query results it received. There is no computational limit on the distinguisher, thus we can assume *wlog* that the distinguisher is deterministic (with a priori query which maximizes its advantage) and never makes redundant queries (which means that it never repeats a query, nor makes a query $P_i(x)$ for $i \in \{I, O\}$, if it receives x as an answer of a previous query $P_i^{-1}(y)$, or vice-versa).

The *distinguishing advantage* of the adversary $\mathcal{D}$ is defined as

$$Adv(\mathcal{D}) = \left| \Pr[\mathcal{D}^{r\text{KACSP}[P;\boldsymbol{K}],P} = 1] - \Pr[\mathcal{D}^{E,P} = 1] \right|, \tag{3}$$

where the first probability is taken over the random choice of P and $\boldsymbol{K}$, and the second probability is taken over the random choice of P and E. $\mathcal{D}^{(\cdot)}$ denotes that $\mathcal{D}$ can make both forward and backward queries to each permutation oracle according to the random permutation model described before.

For non-negative integers q_e and q_p, we define the insecurity of rKACSP against any adaptive distinguisher (even with unbounded computational source) who can make at most q_e queries to the outer permutation oracle (i.e., P_O) and q_p queries to the inner permutation oracle (i.e., P_I) as

$$Adv^{cca}_{r\mathrm{KACSP}}(q_e, q_p) = \max_{\mathcal{D}} \ Adv(\mathcal{D}), \tag{4}$$

where the maximum is taken over all distinguishers $\mathcal{D}$ making exactly q_e queries to the outer permutation oracle and q_p queries to the inner permutation oracle.

2.3 The H-Coefficient Method

H-coefficient method [Pat08],[CS14] is a powerful framework to upper bound the advantage of $\mathcal{D}$ and has been used to prove a number of results. We record all interactions between the adaptive distinguisher $\mathcal{D}$ and the oracles as an ordered list of queries which is also called a *transcript*. Each query in a transcript has the form of (i, b, z, z'), where $i \in \{I, O\}$ represents which permutation oracle being queried, b is a bit indicating whether this is a forward or backward query, z is the value queried and z' is the corresponding answer. For a fixed distinguisher $\mathcal{D}$, a transcript is called *attainable* if exists a tuple of permutations $(P_O, P_I) \in \mathcal{P}_n^2$ such that the interactions among $\mathcal{D}$ and (P_O, P_I) yield the transcript. Recall that the distinguisher $\mathcal{D}$ is deterministic and makes no redundant queries, thus we can convert a transcript into 2 following *lists of directionless queries* without loss of information

$$\mathcal{Q}_E = \{(x_1, y_1), \ldots, (x_{q_e}, y_{q_e})\},$$
$$\mathcal{Q}_P = \{(u_1, v_1), \ldots, (u_{q_p}, v_{q_p})\}.$$

We can reconstruct the transcript exactly through the 2 lists, since $\mathcal{D}$ is deterministic and each of its next action is determined by the previous oracle answers (which can be known from those lists) it has received. As a side note, the 2 lists contain the description of the deterministic distinguisher/algorithm $\mathcal{D}$ implicitly. Therefore, the above two representations of an attainable transcript are equivalent with regard to a fixed deterministic distinguisher $\mathcal{D}$. Based on Eq.(3), our goal is to know the values of the two probabilities. It can be verified that the first probability (i.e., the one related to the "real" world) is only determined by the number of coins which can produce the above 2 directionless lists, and the probability is irrelevant to the order of each query in the original transcript. Thus, it seems that the adaptivity of $\mathcal{D}$ is "dropped" (More details can be found in [CS14]). Through this conceptual transition, upper bounding the advantage of $\mathcal{D}$ is often reduced to certain probability problems. That is why the H-coefficient method works well in lots of provable-security problems, especially for an information-theoretic and adaptive adversary.

As what [CS14],[CLL$^+$18] did, we will also be generous with the distinguisher $\mathcal{D}$ by giving it the actual key $\boldsymbol{K} = (k_0, \ldots, k_r)$ when it is interacting with the "real" world or a dummy key $\boldsymbol{K} \leftarrow_{\$} I_{(r+1)n}$ when it is interacting with the "ideal" world at the end of its interaction. This treatment is reasonable since it will only increase the advantage of $\mathcal{D}$. Hence, a transcript τ we consider actually is a tuple $(\mathcal{Q}_E, \mathcal{Q}_P, \boldsymbol{K})$. We refer to $\widehat{\tau} = (\mathcal{Q}_E, \mathcal{Q}_P)$ as the *permutation transcript* of τ and say that a transcript τ is *attainable* if its corresponding permutation transcript $\widehat{\tau}$ is attainable. Let $\mathcal{T}$ denote the set of attainable transcripts. We denote T_{re}, resp. T_{id}, as the probability distribution of the transcript τ induced by the "real" world, resp. the "ideal" world. It should be pointed out that the two probability distributions depend on the distinguisher $\mathcal{D}$, since its description is embedded in the conversion between the aforementioned two representations. And we also use the same notation to denote the random variable distributed according to each distribution.

The H-coefficient method has lots of variants. In this paper, we will employ the standard "good versus bad" paradigm. More concretely, the set of attainable transcripts $\mathcal{T}$ is partitioned into a set of *"good" transcripts* $\mathcal{T}_1$ such that the probability to obtain some $\tau \in \mathcal{T}_1$ are close in the "real" world and in the "ideal" world, and a set of *"bad" transcripts* $\mathcal{T}_2$ such that the probability to obtain any $\tau \in \mathcal{T}_2$ is small in the "ideal" world. Finally, a well-known H-coefficient-type lemma is given as follows.

Lemma 1 (Lemma 1 of [CLL$^+$18]). *Fix a distinguisher $\mathcal{D}$. Let $\mathcal{T} = \mathcal{T}_1 \sqcup \mathcal{T}_2$ be a partition of the set of attainable transcripts. Assume that there exists ε_1 such that for any $\tau \in \mathcal{T}_1$, one has*

$$\frac{\Pr[T_{re} = \tau]}{\Pr[T_{id} = \tau]} \geq 1 - \varepsilon_1,$$

and that there exists ε_2 such that $\Pr[T_{id} \in \mathcal{T}_2] \leq \varepsilon_2$. Then $Adv(\mathcal{D}) \leq \varepsilon_1 + \varepsilon_2$.

2.4 An Useful Lemma

Lemma 2 (3KACSP version, Lemma 2 of [CLL$^+$18]). *Let $\tau = \big(\mathcal{Q}_E, \mathcal{Q}_P, \boldsymbol{K} = (k_0, k_1, k_2, k_3)\big) \in \mathcal{T}$ be an attainable transcript. Let $p(\tau) = \Pr\big[P \leftarrow_\$ \mathcal{P}_n : \mathrm{3KACSP}[P; \boldsymbol{K}] \vdash \mathcal{Q}_E \mid P \vdash \mathcal{Q}_P\big]$. Then*

$$\frac{\Pr[T_{re} = \tau]}{\Pr[T_{id} = \tau]} = (N)_{q_e} \cdot p(\tau).$$

Following Lemma 2, it is reduced to lower-bounding $p(\tau)$ if we want to determine the value of ε_1 in Lemma 1. In brief, $p(\tau)$ is the probability that 3KACSP$[P; \boldsymbol{K}]$ extends $\mathcal{Q}_E$ when P is a random permutation extending $\mathcal{Q}_P$.

3 The Main Result and New Representation

3.1 3-Round KAC with A Single Permutation

Let n be a positive integer, and let $P : I_n \to I_n$ be a permutation on I_n. On input $x \in I_n$ and round keys $\boldsymbol{K} = (k_0, k_1, k_2, k_3) \in I_{4n}$, the block cipher 3KACSP returns $y = P\Big(P\big(P(x \oplus k_0) \oplus k_1\big) \oplus k_2\Big) \oplus k_3$. See Fig.1 for an illustration of the construction of 3KACSP.

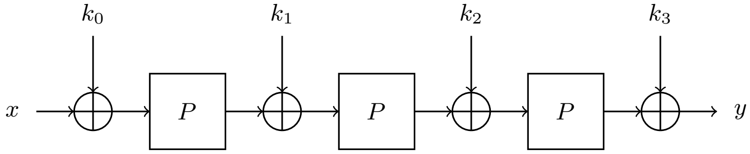


Fig. 1. Illustration of 3KACSP

3.2 Statement of the Result and Discussion

Since 3KACSP is a special case of 3-round KAC construction, its security is also capped by a distinguishing attack with $O(2^{\frac{3n}{4}})$ queries. We will show that the bound is tight by establishing the following theorem, which gives an asymptotical security bound of 3KACSP. Following the main theorem, we also give some comments. The proof of Theorem 1 can be found in Section 4, where we also illustrate the underlying combinatorial problem and give two technical lemmas.

Theorem 1 (Security Bound of 3KACSP). *Consider the 3KACSP construction, in which the underlying round permutation P is uniformly random sampled from $\mathcal{P}_n$ and the round keys $\boldsymbol{K} = (k_0, k_1, k_2, k_3)$ are uniformly random sampled from I_{4n}. Assume that $n \geq 32$ is sufficiently large, $\frac{28(q_e)^2}{N} \leq q_p \leq \frac{q_e}{5}$ and $2q_p + 5q_e \leq \frac{N}{2}$, then for any $6 \leq t \leq \frac{N^{1/2}}{8}$, the following upper bound holds:*

$$Adv^{cca}_{\text{3KACSP}}(q_e, q_p) \leq 98t \cdot \left(\frac{q_e}{N^{3/4}}\right) + 10t^2 \cdot \left(\frac{q_e}{N}\right) + \zeta(q_e),$$

where $\zeta(q_e) = \begin{cases} \frac{32}{t^2}, & \text{if } q_e \leq \frac{t}{6}N^{1/2} \\ \frac{9N}{q_e^2}, & \text{if } q_e \geq \frac{7t}{6}N^{1/2} \end{cases}.$

<u>Obtaining a concrete upper bound.</u> Due to the special form of error term $\zeta(q_e)$, a single constant t cannot optimize the bound for all q_e's simultaneously. The above result gives an upper bound for a range of q_e's once t is chosen, thus different constants t will give different upper bounds for a fixed q_e. That is, for each q_e, we can make the error term $\zeta(q_e)$ be arbitrarily small by choosing a proper t, as long as the n is big enough. In general, we prefer to choose a small t to obtain the bound, since the first two terms in it are proportional to t. As an explanatory example, we next will show how to choose the constant t, assume that the threshold value of $\zeta(q_e)$ is set to 0.01.

Firstly, we should determine the range of q_e's which are suitable for the minimum $t = 6$. It is easy to verify that, for the range of big $q_e \geq 30N^{1/2}$, it must has $\zeta(q_e) \leq 0.01$, since $\zeta(q_e) = \frac{9N}{q_e^2}$ for $q_e \geq 7N^{1/2}$ (when setting $t = 6$). But for a small q_e it needs a larger t, since we will use the function $\zeta(q_e) = \frac{32}{t^2}$ to obtain a desired $\zeta(q_e)$. For simplicity, we can set $t = 60$ for each $q_e \leq 10N^{1/2}$, because it has $\zeta(q_e) = \frac{32}{60^2} < 0.01$. Now what's left is to choose a proper t for covering the remain range of $10N^{1/2} < q_e < 30N^{1/2}$. Using again the function $\zeta(q_e) = \frac{32}{t^2}$, we can crudely set $t = 180$, which implies that $\zeta(q_e) = \frac{32}{180^2} < 0.001$

for all $q_e \leq 30N^{1/2}$. As a side note, a slightly better choice is to choose $t = 6c$ for $q_e = cN^{1/2}$, where $10 < c < 30$.

From the above process, we obtain a concrete upper bound as follows.

$$Adv^{cca}_{\text{3KACSP}}(q_e,q_p) \leq \begin{cases} 588\left(\frac{q_e}{N^{3/4}}\right)+360\left(\frac{q_e}{N}\right)+0.01, & \text{for } q_e \geq 30N^{1/2} \quad \text{(Set } t=6\text{)} \\ 17640\left(\frac{q_e}{N^{3/4}}\right)+324000\left(\frac{q_e}{N}\right)+0.001, & \text{for } 10N^{1/2} < q_e < 30N^{1/2} \quad \text{(Set } t=180\text{)} \\ 5880\left(\frac{q_e}{N^{3/4}}\right)+36000\left(\frac{q_e}{N}\right)+0.01, & \text{for } q_e \leq 10N^{1/2} \quad \text{(Set } t=60\text{)} \end{cases}$$

It is easy to see that $t = 6$ is available for almost all of the q_e's (i.e., except the fraction of $\frac{30}{N^{1/2}}$). That is, the bound $Adv \leq 588\left(\frac{q_e}{N^{3/4}}\right) + 360\left(\frac{q_e}{N}\right) + \frac{9N}{q_e^2}$ is suitable for almost q_e's. We also stress here that Theorem 1 is an asymptotical result (for sufficiently large n) and we are not focusing on optimizing parameters. The point is that it actually shows that $\Omega(N^{3/4})$ queries are needed to obtain a significant advantage against 3KACSP. Combining with the well-known matching attack, we conclude that the 3KACSP construction is $\Theta(2^{\frac{3n}{4}})$-secure.

<u>Discussion about the result.</u> It should be pointed out that the deviation term $\zeta(q_e)$ and the assumption on q_p in Theorem 1 are artifacts of our proof, and have no effect on the final result.

1. The $\zeta(q_e)$ is simply caused by the inaccuracy of Chebyshev's Inequality (i.e., Lemma 5), rather than our proof methods nor the intrinsic flaws of 3KACSP. It is well-known that Chebyshev's Inequality is rather coarse and there must exist a more accurate tail-inequality (e.g., Chenoff Bound). The $\zeta(q_e)$ and t will disappear, as long as a bit more accurate tail-inequality is applied during the computation of Eq.(95) in full version [WYCD20]. That is, just by replacing with a better tail-inequality, our proof techniques actually can obtain a concrete bound such like $Adv \leq 98\left(\frac{q_e}{N^{3/4}}\right) + 10\left(\frac{q_e}{N}\right)$, i.e., $t = 1$ and $\zeta(q_e) = 0$ in Theorem 1. But to our knowledge, there is no explicit expression of the moment generating function for a hypergeometric distribution, hence we now have no idea how to obtain a Chernoff-Type bound.
2. The assumption on q_e and q_p is determined by the assigning-strategy and all the *RoC*s (there are dozens in total) designed in the formal proof. It means that a better choice corresponds to a weaker assumption. Theoretically, there exist choices which can eliminate the assumption without changing our proof framework. However, optimizing such a choice is rather unrealistic, since it is extremely hard to find even one feasible solution (as provided in our formal proof).

In a word, our results and proof techniques are strong enough to show that 3KACSP is $\Theta(N^{3/4})$-secure in random permutation model.

3.3 New Representation

In this subsection, we will propose a new representation which will be used throughout the paper. The representation improves our understanding of the underlying combinatorial problem, and is very helpful to handle the dependence caused by the single permutation. From our proof, it can be found that this new representation is natural to capture the intrinsic combinatorial problem, and the complicated graphical illustration used in [CLL$^+$18] can also be avoided. More specifically, the new representation consists of several definitions.

Definition 2 (Directed-Edge). *Let A denote a set and $\mathsf{a}, \mathsf{b} \in A$. If a permutation Ψ on A maps a to b, then we denote it as $\mathsf{a} \xrightarrow{\Psi} \mathsf{b}$ and say that there is a **$\boldsymbol{\Psi}$-directed-edge** (or simply directed-edge if Ψ is clear from the context) from a to b. We also use $\mathsf{a} \xrightarrow{\Psi} \mathsf{b}$ to denote the ordered query-answer pair (a, b) of the permutation oracle Ψ. That is, if we make queries $\Psi(\mathsf{a})$ (resp. $\Psi^{-1}(\mathsf{b})$), then b (resp. a) will be the answer.*

*For a directed-edge $\mathsf{a} \xrightarrow{\Psi} \mathsf{b}$, we refer to a as the **previous-point** of b under Ψ, and to b as the **next-point** of a under Ψ, respectively. Naturally, the notation $\mathsf{a} \xrightarrow{\Psi}$ means that the next-point of a under Ψ is undefined, and the notation $\xrightarrow{\Psi} \mathsf{b}$ means that the previous-point of b under Ψ is undefined.*

Definition 2 aims to view the binary relation under a permutation as a set of directed-edges. Consider a permutation $P \in \mathcal{P}_n$, the list of directionless queries $\mathcal{Q}_P = \{(u_1, v_1), \ldots, (u_q, v_q)\}$ can be written as the set of P-directed-edges $\{u_1 \xrightarrow{P} v_1, \ldots, u_q \xrightarrow{P} v_q\}$. From now on, we will not distinguish the two representations.

Definition 3 (Directed-Path and Core). *Let $\varphi[\cdot] : \mathcal{P}_n \to \mathcal{P}_n$ be a blockcipher construction invoking one permutation $P \in \mathcal{P}_n$. Fix an attainable transcript $\tau = (\mathcal{Q}_E, \mathcal{Q}_P, \boldsymbol{K})$, where $\mathcal{Q}_E$ and $\mathcal{Q}_P$ are the lists of directionless queries of the outer and inner permutation oracle, respectively.*

*For a specific $P \in \mathcal{P}_n$ and a string $\mathsf{a} \in I_n$, the steps related to P in the calculation of $\varphi[P](\mathsf{a})$ can be denoted as a chain of P-directed-edges and has the form of $\langle f(\mathsf{a}) \xrightarrow{P} \mathsf{a}_1, \ldots, \mathsf{a}_m \xrightarrow{P} g^{-1}(\varphi[P](\mathsf{a}))\rangle$, where $f(\cdot)$ and $g(\cdot)$ are invertible operations before the first invocation of P and after the last invocation of P in the construction $\varphi[\cdot]$, respectively.[2] We refer to such a chain as a $\boldsymbol{\big((\mathsf{a}, \varphi[P](\mathsf{a})), \varphi[P]\big)}$**-directed-path**, where a and $\varphi[P](\mathsf{a})$ are called as the source and destination of the directed-path, respectively. We may simply say a directed-path for convenience, if all things are clear from the context.*

*Let $\mathcal{Q}_E = \{(x_1, y_1), \ldots, (x_q, y_q)\}$, where x_i's (resp. y_i's) are pairwise distinct n-bit strings. We say a permutation $P \in \mathcal{P}_n$ is $\varphi[\cdot]$**-correct with respect to** $\mathcal{Q}_E$, if $\varphi[P] \vdash \mathcal{Q}_E$. That is, the $\varphi[P]$-directed-path starting from x_i must end at y_i (i.e., $y_i = \varphi[P](x_i)$) for a correct permutation P, where $i = 1, \cdots, q$. We refer to the set of P-directed-edges used in above q directed-paths as a $\varphi[P]$-Core with respect to $\mathcal{Q}_E$, and denote it as $\boldsymbol{\mathsf{Core}(\varphi[P] \vdash \mathcal{Q}_E)}$. In addition, we use the notation $\mathsf{Core}(\varphi[\cdot] \vdash \mathcal{Q}_E)$ to denote a certain $\varphi[P]$-Core in general. And we may simply say a Core for convenience, if $\varphi[\cdot]$ and $\mathcal{Q}_E$ are clear from the context.*

[2] In this paper, $f(\cdot)$ and $g(\cdot)$ are often the identity functions.

Definition 3 aims to highlight the steps related to P when calculating the value of $\varphi[P](a)$. In fact, the form of a directed-path is only determined by the construction $\varphi[\cdot]$.[3] That is, each $\big((*,*),\varphi[P]\big)$-directed-path consists of m P-directed-edges, where m is the invoking number of P in the construction $\varphi[\cdot]$. Thus, we often use the notation $\varphi[\cdot]$-*directed-path* to denote a directed-path of the form in general. In addition, the calculation steps independent of P (e.g., the operations $f(\cdot)$ and $g(\cdot)$ in Def.3) are always omitted, since we only care about the assignments of P. Of course, those omitted steps can still be inferred from the directed-path since they are deterministic. For instance, the calculation of $P\big(P(x)\boxplus 1\big) = y$ can be denoted as the directed-path $\langle x \xrightarrow{P} P(x), P(x)\boxplus 1 \xrightarrow{P} y\rangle$, in which the step from $P(x)$ to $P(x)\boxplus 1$ is omitted but can still be known from it. Next, we will give an explanatory example for the above definitions.

Example 1. Let P denote a permutation on $\mathcal{Z}_5 = \{0,1,2,3,4\}$, as well as $\mathcal{Q}_E = \{(0,4),(1,0)\}$, $\mathcal{Q}_P = \emptyset$ and $\varphi[P](x) = P\big(P(x)\boxplus 1\big)$, where $\boxplus$ represents the modulo-5 addition.

Case 1: If $P = \{0 \xrightarrow{P} 1, 1 \xrightarrow{P} 2, 2 \xrightarrow{P} 3, 3 \xrightarrow{P} 4, 4 \xrightarrow{P} 0\}$, then all directed-paths constructed by $\varphi[P]$ are $\langle 0 \xrightarrow{P} 1, 2 \xrightarrow{P} 3\rangle$, $\langle 1 \xrightarrow{P} 2, 3 \xrightarrow{P} 4\rangle$, $\langle 2 \xrightarrow{P} 3, 4 \xrightarrow{P} 0\rangle$, $\langle 3 \xrightarrow{P} 4, 0 \xrightarrow{P} 1\rangle$, and $\langle 4 \xrightarrow{P} 0, 1 \xrightarrow{P} 2\rangle$. That is, the permutation $\varphi[P]$ maps 0 to 3, 1 to 4, 2 to 0, 3 to 1 and 4 to 2, respectively. Obviously, the P is not $\varphi[\cdot]$-correct with respect to $\mathcal{Q}_E$, since the $\varphi[P]$-directed-path $\langle 0 \xrightarrow{P} 1, 2 \xrightarrow{P} 3\rangle$ leads 0 to 3 which is inconsistent with the source-destination pair $(0,4) \in \mathcal{Q}_E$.

Case 2: If $P = \{0 \xrightarrow{P} 2, 1 \xrightarrow{P} 1, 2 \xrightarrow{P} 0, 3 \xrightarrow{P} 4, 4 \xrightarrow{P} 3\}$, then we have $\varphi[P] \vdash \mathcal{Q}_E$ because the directed-paths $\langle 0 \xrightarrow{P} 2, 3 \xrightarrow{P} 4\rangle$ and $\langle 1 \xrightarrow{P} 1, 2 \xrightarrow{P} 0\rangle$ lead 0 to 4 and 1 to 0, respectively. Also, we can know that $\mathsf{Core}(P) = \{0 \xrightarrow{P} 2, 1 \xrightarrow{P} 1, 2 \xrightarrow{P} 0, 3 \xrightarrow{P} 4\}$, and thus $|\mathsf{Core}(P)| = 4$.

Case 3: If $P = \{0 \xrightarrow{P} 3, 1 \xrightarrow{P} 0, 2 \xrightarrow{P} 1, 3 \xrightarrow{P} 2, 4 \xrightarrow{P} 4\}$, then it is easily to verify that $\varphi[P] \vdash \mathcal{Q}_E$, as well as $\mathsf{Core}(P) = \{0 \xrightarrow{P} 3, 1 \xrightarrow{P} 0, 4 \xrightarrow{P} 4\}$ and $|\mathsf{Core}(P)| = 3$.

Case 4: Similarly, if $P = \{0 \xrightarrow{P} 0, 1 \xrightarrow{P} 4, 2 \xrightarrow{P} 1, 3 \xrightarrow{P} 2, 4 \xrightarrow{P} 3\}$, then $\varphi[P] \vdash \mathcal{Q}_E$. Furthermore, it has $\mathsf{Core}(P) = \{0 \xrightarrow{P} 0, 1 \xrightarrow{P} 4\}$ and $|\mathsf{Core}(P)| = 2$.

Statement. For convenience, we will simply use the terms *edge* and *path* instead of *directed-edge* and *directed-path*, respectively. In addition, if x_α^β denotes the source of a path (where α and β are some symbols), then the notation y_α^β always denotes the corresponding destination of the path and vice-versa, and the correspondence can be easily inferred from the context.

We have known that a path can be used to denote a complete calculation given the construction, source and P. In fact, we often confront an incomplete path

[3] Recall that the adversary can obtain the keys after the querying phrase in our proof setting.

whose source and destination are fixed, provided that the permutation P is partially defined.[4] Namely, there are some edges *missing* in such a path. Particularly, we most interest in a special form of incomplete path which is called *target-path.*

Definition 4 (Target-Path). *Assume that P is partially defined, then a* $\boldsymbol{\big((\mathsf{a},\mathsf{b}),}$ $\boldsymbol{\varphi[P]\big)}$***-target-path*** *is a $\varphi[\cdot]$-path in which all the inner-nodes are undefined while the source a and the destination b are fixed. Thus, a target-path always has the form of*[5]

$$\langle \mathsf{a} \xrightarrow{P} \quad , \quad \xrightarrow{P} \quad , \cdots , \quad \xrightarrow{P} \quad , \quad \xrightarrow{P} \mathsf{b}\rangle.$$

In essence, the proof of main result is reduced to the task of completing a group of target-paths (i.e., Problem 1). That is why we refer to such type of paths as target-paths. In general, it is convenient to consider a group of (target-)paths having the same form. Then, the notion of *shared-edge* can also be introduced naturally.

Definition 5 (Group of Paths and Shared-Edge). *Fix a permutation P, which can be partially defined.*

We call the paths $\big((x_1, y_1), \varphi[P]\big)$-path, $\ldots$, $\big((x_q, y_q), \varphi[P]\big)$-path as a group of $\varphi[\cdot]$-paths, and denote it as $\boldsymbol{(\mathcal{Q}_E, \varphi[P])}$***-paths****, where $\mathcal{Q}_E = \{(x_1, y_1), \ldots, (x_q, y_q)\}$ is the set of source-destination pairs. Also, we may simply use the notation $\mathcal{Q}_E$-paths if $\varphi[P]$ is clear from the context.*

Similarly, we call the target-paths $\big((\mathsf{a}_1, \mathsf{b}_1), \varphi[P]\big)$-target-path, $\ldots$, $\big((\mathsf{a}_q, \mathsf{b}_q),$ $\varphi[P]\big)$-target-path as a group of $\varphi[\cdot]$-target-paths, and denote it as $\boldsymbol{(\mathcal{Q}, \varphi[P])}$***-target-paths****, where $\mathcal{Q} = \{(\mathsf{a}_1, \mathsf{b}_1), \ldots, (\mathsf{a}_q, \mathsf{b}_q)\}$ is the set of source-destination pairs.*

If an edge is used in at least 2 different paths, then we refer to it as a ***shared-edge****.*

From now on, we can use Definition 5 to denote a group of (target-)paths conveniently. And it should be pointed out that the *shared-edge* is a key primitive in our proof, though the concept is rather simple and natural. Moreover, the notion of *partial-P* will be useful, since P is often partially defined.

Definition 6 (Partial-*P* and Partially-Sample). *Let P be a permutation on I_n, and let A be a subset of I_n. Then we refer to the set of edges $\{x_i \xrightarrow{P} P(x_i) : x_i \in A\}$ as the* ***partial-P*** *from A to $P(A)$.*

Let S and T be two sets of elements whose next-points and previous-points are undefined under P, respectively. If $|S| = |T|$, then we can sample randomly a bijection $f \leftarrow_{\$} Bjt(S \to T)$ and define $x \xrightarrow{P} f(x)$ for each $x \in S$. We refer to the above process as ***sample partial-P randomly*** *from S to T, or* ***P is partially-sampled randomly*** *from S to T.*

[4] Informally, we say a permutation P is *partially defined*, if the correspondence of some points are undefined.

[5] For simplicity, we assume here that the operations $f(\cdot)$ and $g(\cdot)$ in construction $\varphi[\cdot]$ are both identity functions.

It should be pointed out that a partial-P is a subset of P, and also a set of P-edges. Now let's reconsider the sampling $P \leftarrow_{\$} \mathcal{P}_n$ conditioned on $P \vdash \mathcal{Q}_P$, where $\mathcal{Q}_P = \{u_1 \xrightarrow{P} v_1, \ldots, u_q \xrightarrow{P} v_q\}$. If we denote $S = I_n \setminus \{u_1, \ldots, u_q\}$ and $T = I_n \setminus \{v_1, \ldots, v_q\}$, then the above sampling is equivalent to sample partial-P randomly from S to T. Furthermore, it is natural to view $\mathcal{Q}_P$ as the priori information of P. That is, we can fix the q edges of $\mathcal{Q}_P$ in advance, and then sample partial-P randomly from S to T.

4 Proof of Theorem 1

In this section, we will use the standard H-Coefficient method (i.e., Lemma 1) to prove our main result. That is, all attainable transcripts $\mathcal{T}$ should be partitioned into two disjoint parts: a set of "good" transcripts denoted as $\mathcal{T}_1$ and a set of "bad" transcripts denoted as $\mathcal{T}_2$. Determining the partition is often a subtle task, since it is intrinsically a trade-off between ε_1 and ε_2. If we add more conditions on good transcripts to make they have better property (i.e., with smaller ε_1), then the set of bad transcripts becomes larger accordingly (i.e., ε_2 becomes larger), or vice-versa.

Intuitively, the chance to obtain any $\tau \in \mathcal{T}_1$ in "real" world should be very close to the chance in "ideal" world, and it should be very rare to obtain any $\tau \in \mathcal{T}_2$ in the "ideal" world. For an attainable transcript $\tau = \big(\mathcal{Q}_E, \mathcal{Q}_P, \boldsymbol{K} = (k_0, k_1, k_2, k_3)\big)$, we know that (from Lemma 2) the quotient of $\Pr[T_{re} = \tau]$ and $\Pr[T_{id} = \tau]$ is determined by the value of

$$p(\tau) = \Pr\big[P \leftarrow_{\$} \mathcal{P}_n : \mathrm{3KACSP}[P; \boldsymbol{K}] \vdash \mathcal{Q}_E \mid P \vdash \mathcal{Q}_P\big]. \qquad (5)$$

That is, a transcript τ is whether "good" or not, can also be determined by the value of $p(\tau)$.

Therefore, we firstly illustrate the meaning of $p(\tau)$ through our new representation, and then give the definition of "bad" transcripts. In fact, it is also a good example to show that the knotty dependence can be sorted out if we use a proper representation. At the end of this section, we will prove Theorem 1 directly by combining two technical lemmas together.

4.1 Transcripts and $p(\tau)$

In this subsection, we firstly expound the meaning of $p(\tau)$ for a fixed transcript τ, and then give the concrete definition of "bad" transcripts. To reduce the complexity of notations, we now rewrite the $p(\tau)$ into another equivalent form.

A CONCEPTUAL TRANSFORMATION. For an attainable transcript $\tau = (\mathcal{Q}_E, \mathcal{Q}_P, \boldsymbol{K})$, we modify the inner permutation P and its permutation transcript $\widehat{\tau} = (\mathcal{Q}_E, \mathcal{Q}_P)$ as follows:

$$P' = P \oplus k_1,$$

$$\mathcal{Q}'_E = \{(x \oplus k_0, y \oplus k_1 \oplus k_3) : (x, y) \in \mathcal{Q}_E\},$$
$$\mathcal{Q}'_P = \{(u, v \oplus k_1) : (u, v) \in \mathcal{Q}_P\}.$$

Let

$$X = \{x' \in I_n : (x', y') \in \mathcal{Q}'_E\}, \quad Y = \{y' \in I_n : (x', y') \in \mathcal{Q}'_E\},$$
$$U = \{u' \in I_n : (u', v') \in \mathcal{Q}'_P\}, \quad V = \{v' \in I_n : (u', v') \in \mathcal{Q}'_P\}$$

denote the domains and the ranges of $\mathcal{Q}'_E$ and $\mathcal{Q}'_P$, respectively. Thus, $|\mathcal{Q}_E| = |\mathcal{Q}'_E| = |X| = |Y| = q_e$, and $|\mathcal{Q}_P| = |\mathcal{Q}'_P| = |U| = |V| = q_p$.

Accordingly, we also transform the 3KACSP construction into the 3KACSP′ construction (as shown in Fig.2), i.e., $P' \circ P' \circ (\oplus k_1 \oplus k_2) \circ P'$. The above modification is reasonable, since we show the actual key used in 3KACSP after the distinguisher $\mathcal{D}$ finishing the query phase (i.e., after obtaining $\mathcal{Q}_E$ and $\mathcal{Q}_P$). Thus, it is simply a conceptual transformation and only the notations should be changed. That is, we can consider that the distinguisher $\mathcal{D}$ is querying the outer permutation and inner permutation oracles instantiated by 3KACSP′ and P', respectively. Then the resulting transcript is $\tau' = (\mathcal{Q}'_E, \mathcal{Q}'_P, \boldsymbol{K})$. From now on, we will not distinguish the transcripts τ and τ', since they can transform from each other easily. Thus, we have

$$p(\tau) = p(\tau') = \Pr[P' \leftarrow_{\$} \mathcal{P}_n : \text{3KACSP}'[P'; \boldsymbol{K}] \vdash \mathcal{Q}'_E | P' \vdash \mathcal{Q}'_P]. \tag{6}$$

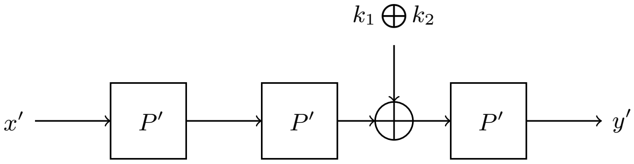


Fig. 2. 3KACSP′: A Conceptual Transformation of 3KACSP

Notation 1 (Abbreviation) *Let A be a set of n-bit strings, and a be an element of A. From now on, we will abbreviate the expression $\mathsf{a} \oplus k_1 \oplus k_2$ as $\mathsf{a}_\oplus$ for convenience. Similarly, we also denote that $A_\oplus = \{\mathsf{a}_\oplus : \mathsf{a} \in A\}$.*

Illustration of $p(\tau')$. Next, we will show the underlying combinatorial problem of $p(\tau')$ intuitively. Fix arbitrarily a transcript $\tau' = (\mathcal{Q}'_E, \mathcal{Q}'_P, \boldsymbol{K})$, the event $\text{3KACSP}'[P'; \boldsymbol{K}] \vdash \mathcal{Q}'_E$ means that for each pair $(x', y') \in \mathcal{Q}'_E$, the 3KACSP′-path starting from x' ends exactly at y'. A complete 3KACSP′-path consists of 3 P'-edges, and has the form of

$$\langle x' \xrightarrow{P'} *_1, *_1 \xrightarrow{P'} *_2, (*_2)_\oplus \xrightarrow{P'} y' \rangle, \tag{7}$$

where $*_1$ and $*_2$ are the 2 *inner-nodes* should be assigned.

Before turning into the value of $p(\tau')$, we consider a simpler case that $\mathcal{Q}'_P = \emptyset$ as first. Since no edge of the $\mathcal{Q}'_E$-paths has been fixed in advance, our task is simply to complete all the $(\mathcal{Q}'_E, \text{3KACSP}')$-target-paths, by sampling P' uniformly random from $\mathcal{P}_n$. In fact, we will see that it is exactly the Problem 1 instantiated by $\varphi[P'] = P' \circ P'_\oplus \circ P'$, $\mathcal{Q}_1 = \mathcal{Q}'_E$, $\mathcal{Q}_2 = \emptyset$, and can be solved directly by a general framework[6].

Unfortunately, it becomes much more complex when $\mathcal{Q}'_P \neq \emptyset$, since some 3KACSP′-target-paths will be "damaged". More specifically, a path will turn into "some other construction"-target-path, when some edges in it are fixed by $\mathcal{Q}'_P$. We now give some intuition about those paths. Assume that q_e and q_p are $O(N^{3/4})$ and $\boldsymbol{K}$ is uniformly random sampled from I_{4n}, then there are at most 4 types of paths. On average, there are $O(1)$ paths containing 3 fixed edges. Similarly, we know that there exist $O(N^{1/4})$ (resp. $O(N^{1/2})$) paths whose 2 edges (resp. 1 edge) are fixed in advance. And there are $O(N^{3/4})$ paths containing no fixed edge (i.e., they are 3KACSP′-target-paths). It can be found that the circumstances are more involved than before, since the constructions of missing-edges are no longer uniform. In other words, there may exist several different constructions of target-paths to be completed. Thus, we should analyze each of the constructions and complete them in turns.

In fact, we judge a transcript τ' is whether "good" or not, according to the $\mathcal{Q}'_E$-paths and the edges fixed by $\mathcal{Q}'_P$. Firstly, a transcript will be classified into the set of "bad" transcripts, if there exists some $\mathcal{Q}'_E$-path containing 3 fixed edges. Otherwise, we should further study the circumstances of paths and fixed edges determined by the transcript. More specifically, for such a transcript, we can classify the q_e paths between $\mathcal{Q}'_E$ into three groups (see Fig.3 as an illustration) according to the number of fixed edges.

► **Group-2.** The paths containing 2 fixed edges belong to Group-2. More specifically, there are 3 subcases of such paths according to the position of fixed edges. Recall that U and V denote the domain and range of $\mathcal{Q}'_P$, respectively.

- *Group-2.1:* The paths whose first two edges are fixed. That is, Group-2.1 consists of the paths starting from the subset $X_{II} \subset X$, where

$$X_{II} \subset U \ \bigwedge\ \mathcal{Q}'_P(X_{II}) \subset U \ \bigwedge\ \Big(\mathcal{Q}'_P\big(\mathcal{Q}'_P(X_{II})\big)\Big)_\oplus \cap U = \emptyset$$

$$\Longleftrightarrow \forall x \in X_{II}, \exists\, w_1, w_2,\ s.t.\ (x, w_1), (w_1, w_2) \in \mathcal{Q}'_P \ \wedge\ (w_2)_\oplus \notin U.$$

- *Group-2.2:* The paths whose last two edges are fixed. That is, Group-2.2 consists of the paths ending at the subset $Y_B \subset Y$, where

$$Y_B \subset V \ \bigwedge\ \big({\mathcal{Q}'}^{-1}_P(Y_B)\big)_\oplus \subset V \ \bigwedge\ {\mathcal{Q}'}^{-1}_P\Big(\big({\mathcal{Q}'}^{-1}_P(Y_B)\big)_\oplus\Big) \cap V = \emptyset$$

$$\Longleftrightarrow \forall y \in Y_B, \exists\, w_1, w_2,\ s.t.\ (w_1, w_2), \big((w_2)_\oplus, y\big) \in \mathcal{Q}'_P \ \wedge\ w_1 \notin V.$$

- *Group-2.3:* The paths whose first and third edges are fixed. That is, Group-2.3 consists of the paths starting from the subset $D_X \subset X$ to the corresponding $D_Y = \mathcal{Q}'_E(D_X) \subset Y$, where

$$D_X \subset U \ \bigwedge\ D_Y \subset V \ \bigwedge\ \big(\mathcal{Q}'_P(D_X)\big) \cap U = \emptyset \ \bigwedge\ \big((\mathcal{Q}'_P)^{-1}(D_Y)\big)_\oplus \cap V = \emptyset$$

[6] The framework and the technical route can be found in Section 5.1 and 5.2, respectively.

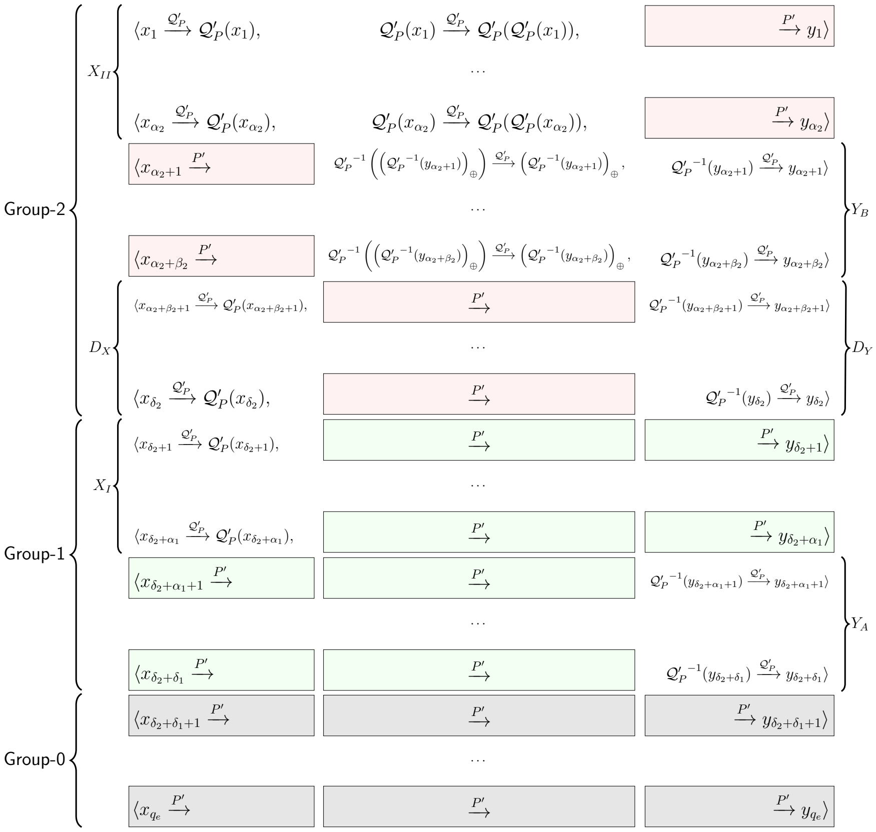


Fig. 3. Illustration of the Missing-Edges in $\mathcal{Q}'_E$-Paths

$$\iff \forall x \in D_X, \exists\, w_1, w_2,\ s.t.\ (x, w_1), \big(w_2, \mathcal{Q}'_E(x)\big) \in \mathcal{Q}'_P \ \wedge\ w_1 \notin U \ \wedge\ (w_2)_\oplus \notin V.$$

Notation 2 (Group-2) *We denote* $|X_{II}| = \alpha_2$, $|Y_B| = \beta_2$, $|D_X| = |D_Y| = \gamma_2$, *and* $\delta_2 = \alpha_2 + \beta_2 + \gamma_2$. *Thus, Group-2 contains* δ_2 *paths in total, where* α_2 *paths belong to Group-2.1,* β_2 *paths belong to Group-2.2 and the other* γ_2 *paths belong to Group-2.3. For convenience, we assume* wlog *that* $X_{II} = \{x_1, \ldots, x_{\alpha_2}\}$, $Y_B = \{y_{\alpha_2+1}, \ldots, y_{\alpha_2+\beta_2}\}$, $D_X = \{x_{\alpha_2+\beta_2+1}, \ldots, x_{\delta_2}\}$ *and* $D_Y = \{y_{\alpha_2+\beta_2+1}, \ldots, y_{\delta_2}\}$.

▶ **Group-1.** The paths containing 1 fixed edge belong to Group-1. More specifically, there are 2 subcases of such paths according to the position of fixed edge. Recall that U and V denote the domain and range of $\mathcal{Q}'_P$, respectively.

- *Group-1.1:* The paths whose first edge is fixed. That is, Group-1.1 consists of the paths starting from the subset $X_I \subset X$, where

$$X_I \subset U \ \bigwedge\ \mathcal{Q}'_P(X_I) \cap U = \emptyset$$

$$\Longleftrightarrow \forall x \in X_I, \exists\ w_1,\ s.t.\ (x, w_1) \in \mathcal{Q}'_P \ \wedge\ w_1 \notin U.$$

- *Group-1.2:* The paths whose third edge is fixed. That is, Group-1.2 consists of the paths ending at the subset $Y_A \subset Y$, where

$$Y_A \subset V \ \bigwedge\ \left(\mathcal{Q}'^{-1}_P(Y_A)\right)_\oplus \cap V = \emptyset$$
$$\Longleftrightarrow \forall y \in Y_A, \exists\ w_1,\ s.t.\ (w_1, y) \in \mathcal{Q}'_P \ \wedge\ (w_1)_\oplus \notin V.$$

Notation 3 (Group-1) *We denote $|X_I| = \alpha_1$, $|Y_A| = \beta_1$ and $\delta_1 = \alpha_1 + \beta_1$. Namely, Group-1 contains δ_1 paths in total, where α_1 paths belong to Group-1.1 and the other β_1 paths belong to Group-1.2. For convenience, we assume* wlog *that $X_I = \{x_{\delta_2+1}, \ldots, x_{\delta_2+\alpha_1}\}$ and $Y_A = \{y_{\delta_2+\alpha_1+1}, \ldots, y_{\delta_2+\delta_1}\}$.*

► **Group-0.** Each path belongs to Group-0 contains no fixed edge.

Notation 4 (Group-0) *We denote $\delta_0 = q_e - \delta_2 - \delta_1$. Thus, Group-0 contains δ_0 paths in total. Let X_0 and Y_0 denote the sets of sources and destinations of Group-0, respectively. For convenience, we assume* wlog *that $X_0 = \{x_i : \delta_2 + \delta_1 + 1 \le i \le q_e\}$ and $Y_0 = \{y_i : \delta_2 + \delta_1 + 1 \le i \le q_e\}$.*

For a fixed transcript τ', its circumstances of $\mathcal{Q}'_E$-paths and fixed edges can be illustrated as Fig.3, where the missing-edges are the ones marked with a colored square. At this point, it is clear that $p(\tau')$ (see Eq.(6)) represents the probability that, all missing-edges are filled by sampling P' uniformly random from the set of permutations extending $\mathcal{Q}'_P$. Furthermore, the above problem becomes more straightforward if we use the notion of target-path (see Def.4).

Definition 7 (Structure of Missing-Edges).

Let E_2 denote the event that the δ_2 paths of Group-2 are completed (i.e., the δ_2 missing-edges in Group-2 are filled).

Let E_{11} denote the event that the $\left(\mathcal{Q}'_{E_{11}}, \varphi_{11}[\cdot]\right)$-target-paths are completed (i.e., the $2\alpha_1$ missing-edges in Group-1.1 are filled), where $\mathcal{Q}'_{E_{11}} = \left\{\left(\mathcal{Q}'_P(x_i), y_i\right) : x_i \in X_I\right\}$ and $\varphi_{11}[P'] = P'_\oplus \circ P'$.

Let E_{12} denote the event that the $\left(\mathcal{Q}'_{E_{12}}, \varphi_{12}[\cdot]\right)$-target-paths are completed (i.e., the $2\beta_1$ missing-edges in Group-1.2 are filled), where $\mathcal{Q}'_{E_{12}} = \left\{\left(x_i, \mathcal{Q}'^{-1}_P(y_i)\right) : y_i \in Y_A\right\}$ and $\varphi_{12}[P'] = P' \circ P'_\oplus$.

Let E_0 denote the event that the $\left(\mathcal{Q}'_{E_0}, \varphi_0[\cdot]\right)$-target-paths are completed (i.e., the $3\delta_0$ missing-edges in Group-0 are filled), where $\mathcal{Q}'_{E_0} = \{(x_i, y_i) : x_i \in X_0\}$ and $\varphi_0[P'] = P' \circ P'_\oplus \circ P'$.

Immediately, we can know that

$$p(\tau') = \Pr[P' \leftarrow_\$ \mathcal{P}_n : E_2 \wedge E_{11} \wedge E_{12} \wedge E_0 | P' \vdash \mathcal{Q}'_P]. \tag{8}$$

Obviously, lower-bounding the value of $p(\tau')$ is reduced to several subproblems which can be applied directly with the counting framework (proposed in Section

5.1). For a "good" transcript, we can successfully obtain an appropriate lower bound for each subproblem.[7]

<u>Definition of Bad Transcripts.</u> Now, we will give the concrete definition of "bad"/"good" transcripts. From the formal proof, it will be seen that each attainable permutation transcript $\widehat{\tau}$ can be extended to a "good" transcript by adding a "good" key $\boldsymbol{K}$. Thus, it is equivalent to study the properties of "bad"/"good" keys for a fixed permutation transcript $\widehat{\tau}$.

Definition 8 (Bad Transcripts and Bad Keys). *Fix arbitrarily a permutation transcript $\widehat{\tau} = (\mathcal{Q}_E, \mathcal{Q}_P)$. If the extended transcript $\tau' = (\mathcal{Q}'_E, \mathcal{Q}'_P, \boldsymbol{K})$ satisfies $\boldsymbol{K} \in \mathrm{BadK} = \bigcup_{1 \le i \le 9} \mathrm{BadK}_i$, then we say the $\tau = (\mathcal{Q}_E, \mathcal{Q}_P, \boldsymbol{K})$ is a "bad" transcript and the $\boldsymbol{K}$ is a "bad" key for $\widehat{\tau}$. Otherwise, we say the τ is a "good" transcript and the $\boldsymbol{K}$ is a "good" key for $\widehat{\tau}$. More specifically, the definitions of* BadK_i $(1 \le i \le 9)$ *are shown as follows.*

$$\begin{aligned}
\boldsymbol{K} \in \mathrm{BadK}_1 &\iff \text{There exists a } \mathcal{Q}'_E\text{-path containing 3 fixed edges.}\\
\boldsymbol{K} \in \mathrm{BadK}_2 &\iff \alpha_2 > \frac{q_e}{N^{1/2}} \bigvee \beta_2 > \frac{q_e}{N^{1/2}} \bigvee \gamma_2 > \frac{q_e}{N^{1/2}}\\
&\qquad\qquad\qquad \bigvee \alpha_1 > \frac{q_e}{N^{1/4}} \bigvee \beta_1 > \frac{q_e}{N^{1/4}}\\
\boldsymbol{K} \in \mathrm{BadK}_3 &\iff U,\ \Big(\mathcal{Q}'_P\big(\mathcal{Q}'_P(X_{II})\big)\Big)_{\oplus},\ \mathcal{Q}'^{-1}_E(Y_B) \text{ and } \mathcal{Q}'_P(D_X) \text{ are not}\\
&\qquad \text{pariwise disjoint} \bigvee\ V,\ \mathcal{Q}'_E(X_{II}),\ \mathcal{Q}'^{-1}_P\Big(\big(\mathcal{Q}'^{-1}_P(Y_B)\big)_{\oplus}\Big) \text{ and}\\
&\qquad \big(\mathcal{Q}'^{-1}_P(D_Y)\big)_{\oplus} \text{ are not pairwise disjoint}\\
\boldsymbol{K} \in \mathrm{BadK}_4 &\iff |\mathcal{Q}'_P(X_I) \setminus (T_{11})_{\oplus}| > \frac{\alpha_1}{N^{1/4}} \bigvee |\mathcal{Q}'_E(X_I) \setminus (S_{11})_{\oplus}| > \frac{\alpha_1}{N^{1/4}}\\
&\qquad \text{when } \alpha_2, \beta_2, \gamma_2 \le \frac{q_e}{N^{1/2}} \text{ and } \alpha_1, \beta_1 \le \frac{q_e}{N^{1/4}}\\
\boldsymbol{K} \in \mathrm{BadK}_5 &\iff |\mathcal{Q}'_P(X_I) \cap \big(\mathcal{Q}'_E(X_I)\big)_{\oplus}| > \frac{\alpha_1}{N^{1/4}} \text{ when } \alpha_2, \beta_2, \gamma_2 \le \frac{q_e}{N^{1/2}}\\
&\qquad \text{and } \alpha_1, \beta_1 \le \frac{q_e}{N^{1/4}}\\
\boldsymbol{K} \in \mathrm{BadK}_6 &\iff |\mathcal{Q}'^{-1}_E(Y_A) \setminus T_{12}| > \frac{\beta_1}{N^{1/4}} \bigvee |\mathcal{Q}'^{-1}_P(Y_A) \setminus (S_{12})_{\oplus}| > \frac{\beta_1}{N^{1/4}}\\
&\qquad \text{when } \alpha_2, \beta_2, \gamma_2 \le \frac{q_e}{N^{1/2}} \text{ and } \alpha_1, \beta_1 \le \frac{q_e}{N^{1/4}}\\
\boldsymbol{K} \in \mathrm{BadK}_7 &\iff |\mathcal{Q}'^{-1}_E(Y_A) \cap \big(\mathcal{Q}'^{-1}_P(Y_A)\big)_{\oplus}| > \frac{\beta_1}{N^{1/4}} \text{ when } \alpha_2, \beta_2, \gamma_2 \le \frac{q_e}{N^{1/2}}\\
&\qquad \text{and } \alpha_1, \beta_1 \le \frac{q_e}{N^{1/4}}\\
\boldsymbol{K} \in \mathrm{BadK}_8 &\iff \quad |X_0 \setminus T_0| > \frac{\delta_0}{N^{1/4}} \bigvee |X_0 \setminus (T_0)_{\oplus}| > \frac{\delta_0}{N^{1/4}}
\end{aligned}$$

[7] In fact, as shown in proof sketch of Lemma 3, we will handle each subproblem with an additional restriction.

$$\bigvee |Y_0 \setminus S_0| > \frac{\delta_0}{N^{1/4}} \bigvee |Y_0 \setminus (S_0)_\oplus| > \frac{\delta_0}{N^{1/4}} \text{ when } \alpha_2, \beta_2, \gamma_2 \leq \frac{q_e}{N^{1/2}} \text{ and } \alpha_1, \beta_1 \leq \frac{q_e}{N^{1/4}}$$

$$\boldsymbol{K} \in \mathrm{BadK}_9 \iff |X_0 \cap Y_0| > \frac{\delta_0}{N^{1/4}} \bigvee |(X_0)_\oplus \cap Y_0| > \frac{\delta_0}{N^{1/4}} \text{ when } \alpha_2, \beta_2, \gamma_2 \leq \frac{q_e}{N^{1/2}} \text{ and } \alpha_1, \beta_1 \leq \frac{q_e}{N^{1/4}},$$

where S_i (resp. T_i) are the sets of n-bit strings whose next-points (resp. previous-points) are undefined when considering the subproblem related to E_i, for $i \in \{11, 12, 0\}$. [8]

It should be pointed out that the conditions for "bad" keys are not designed only for the "real" world. In fact, all discussion in Section 4.1 depends only on the relation between permutation transcript $\widehat{\tau}$ and $\boldsymbol{K}$ (which is dummy in the "ideal" world), and is irrelevant to which world we consider.

4.2 Two Technical Lemmas

In this subsection, we give two technical lemmas to upper-bound the values of ε_1 and ε_2 in Lemma 1, respectively. More specifically, Lemma 3 considers arbitrarily an attainable permutation transcript $\widehat{\tau}$, and lower-bounds the value of $\Pr[T_{re} = (\widehat{\tau}, \boldsymbol{K})])/(\Pr[T_{id} = (\widehat{\tau}, \boldsymbol{K})]$ for any "good" key $\boldsymbol{K}$. This is the major task in our formal proof. And Lemma 4 upper-bounds the value of $\Pr[\boldsymbol{K}$ is bad for $\widehat{\tau}]$ in "ideal" world, where $\widehat{\tau}$ can be any attainable permutation transcript.

Lemma 3. *Consider the 3KACSP construction, and fix arbitrarily an attainable permutation transcript $\widehat{\tau} = (\mathcal{Q}_E, \mathcal{Q}_P)$, where $|\mathcal{Q}_E| = q_e$ and $|\mathcal{Q}_P| = q_p$. Assume that $n \geq 32$, $6 \leq t \leq \frac{N^{1/2}}{8}$, $\frac{28(q_e)^2}{N} \leq q_p \leq \frac{q_e}{5}$ and $2q_p + 5q_e \leq \frac{N}{2}$. Following the Definition 8, if $\boldsymbol{K}$ is a good key for $\widehat{\tau}$, then we have the bound*

$$\frac{\Pr[T_{re} = (\widehat{\tau}, \boldsymbol{K})]}{\Pr[T_{id} = (\widehat{\tau}, \boldsymbol{K})]} \geq 1 - 97t \cdot \left(\frac{q_e}{N^{3/4}}\right) - 10t^2 \cdot \left(\frac{q_e}{N}\right) - \zeta(q_e).$$

<u>Outline of the Proof.</u> From Lemma 2 and the equation (8), we know that

$$\begin{aligned}\frac{\Pr[T_{re} = (\widehat{\tau}, \boldsymbol{K})]}{\Pr[T_{id} = (\widehat{\tau}, \boldsymbol{K})]} &= (N)_{q_e} \cdot \Pr[P' \leftarrow_\$ \mathcal{P}_n : E_2 \wedge E_{11} \wedge E_{12} \wedge E_0 | P' \vdash \mathcal{Q}'_P] \\ &\geq (N)_{q_e} \cdot \Pr[P' \leftarrow_\$ \mathcal{P}_n : E_2 \wedge \widetilde{E_{11}} \wedge \widetilde{E_{12}} \wedge \widetilde{E_0} | P' \vdash \mathcal{Q}'_P] \\ &= \quad (N)_{q_e} \\ &\quad \times \Pr[P' \leftarrow_\$ \mathcal{P}_n : E_2 | P' \vdash \mathcal{Q}'_P] \qquad (9)\end{aligned}$$

[8] For completeness, we give directly the concrete definition here. A more natural way is showing some intuition on "good" transcripts before such a rigorous definition. The interested readers can refer to the Definitions 9 and 13 in full version [WYCD20] for more interpretations about the properties of "bad"/"good" keys.

$$\times \Pr[P' \leftarrow_{\$} \mathcal{P}_n : \widetilde{E_{11}} | P' \vdash \mathcal{Q}'_P \wedge E_2] \tag{10}$$
$$\times \Pr[P' \leftarrow_{\$} \mathcal{P}_n : \widetilde{E_{12}} | P' \vdash \mathcal{Q}'_P \wedge E_2 \wedge \widetilde{E_{11}}] \tag{11}$$
$$\times \Pr[P' \leftarrow_{\$} \mathcal{P}_n : \widetilde{E_0} | P' \vdash \mathcal{Q}'_P \wedge E_2 \wedge \widetilde{E_{11}} \wedge \widetilde{E_{12}}], \tag{12}$$

where $\widetilde{E_{11}}$ denotes the event $E_{11} \bigwedge |\mathsf{Core}(\varphi_{11}[P'] \vdash \mathcal{Q}'_{E_{11}})| \geq (2 - \frac{1}{N^{1/4}})\alpha_1$, $\widetilde{E_{12}}$ denotes the event $E_{12} \bigwedge |\mathsf{Core}(\varphi_{12}[P'] \vdash \mathcal{Q}'_{E_{12}})| \geq (2 - \frac{1}{N^{1/4}})\beta_1$, and $\widetilde{E_0}$ denotes the event $E_0 \bigwedge |\mathsf{Core}(\varphi_0[P'] \vdash \mathcal{Q}'_{E_0})| \geq (3 - \frac{2t}{N^{1/2}})\delta_0$.

We will see that it is easy to calculate the value of (9) when $\boldsymbol{K}$ is a "good" key for $\widehat{\tau}$. Hence what's left is to lower-bound the values of (10)–(12) for any "good" transcript, respectively. Intrinsically, the 3 probabilities belong to the same type of combinatorial problems (i.e., the Problem 1). That means we can view equations (10)–(12) as a 2-round, 2-round and 3-round instantiation of Problem 1, respectively. Interestingly, we find that the techniques used in [CLL$^+$18] can be tailored to obtain desired values of the 2-round cases. Nonetheless, our 2-round cases are more involved and there are some new non-trivial tasks should be solved. Furthermore, the 3-round case is a whole new challenge, and is much more difficult than the 2-round ones. To handle it, we introduce a general framework in Section 5, where we also give the high-level technical route.

However, knowing how to solve (9)–(12) individually is still far from enough. It is a very big challenge to combine all the lower bounds together to obtain an appropriate result, since those subproblems affect each other by sharing the same resource of permutation P. There are numerous technical specifics should be handled, and we defer the formal proof of Lemma 3 to Section 6 of the full version [WYCD20].

Lemma 4. *Consider the "ideal" world, and fix arbitrarily an attainable permutation transcript $\widehat{\tau} = (\mathcal{Q}_E, \mathcal{Q}_P)$, where $|\mathcal{Q}_E| = q_e$ and $|\mathcal{Q}_P| = q_p$. Following the Definition 8, if $q_p \leq \frac{q_e}{5}$, then it has*

$$\Pr[\boldsymbol{K} \leftarrow_{\$} I_{4n} : \boldsymbol{K} \text{ is bad for } \widehat{\tau}] \leq 6 \cdot \left(\frac{q_e}{N^{3/4}}\right).$$

The formal proof of Lemma 4 is deferred to Section 7 of the full version [WYCD20].

4.3 Concluding the Proof of Theorem 1.

At this point,we are ready to complete the proof of Theorem 1. It can be inferred that $\varepsilon_1 = 97t \cdot \left(\frac{q_e}{N^{3/4}}\right) + 10t^2 \cdot \left(\frac{q_e}{N}\right) + \zeta(q_e)$ and $\varepsilon_2 = 6 \cdot \left(\frac{q_e}{N^{3/4}}\right)$ from Lemma 3 and Lemma 4, respectively. Following the H coefficient method and Lemma 1, we finally obtain

$$\begin{aligned} Adv^{cca}_{\mathrm{3KACSP}}(q_e, q_p) &\leq \varepsilon_1 + \varepsilon_2 \\ &\leq 97t \cdot \left(\frac{q_e}{N^{3/4}}\right) + 10t^2 \cdot \left(\frac{q_e}{N}\right) + \zeta(q_e) + 6 \cdot \left(\frac{q_e}{N^{3/4}}\right) \\ &\leq 98t \cdot \left(\frac{q_e}{N^{3/4}}\right) + 10t^2 \cdot \left(\frac{q_e}{N}\right) + \zeta(q_e), \end{aligned}$$

where we use the fact that $t \geq 6$ for the last inequality.

5 A Type of Combinatorial Problem

It is known that the proof of Lemma 3 can be reduced to several subproblems having a similar form. In fact, the analysis of rKACSP can also be reduced to the same type of problems. That is a key perspective to simplify the task of studying the security of rKACSP.

In this section, we will only study how to solve such type of problems individually, while the tricks of balancing all the subproblems are deferred to the formal proof. More specifically, a general framework which can theoretically solve such problems is proposed. For the reason of space, we here only instantiate it for the 3-round case and the full version [WYCD20] also gives the 2-round instance as a warm-up.

First of all, the general definition of aforementioned problems is given as follows.

Problem 1 (Completing A Group of Target-Paths). Consider a group of $(\mathcal{Q}_1, \varphi[\cdot])$-target-paths, where $\mathcal{Q}_1$ is the set of source-destination pairs. Let $\mathcal{Q}_2$ denote the set of fixed edges, and it has $\mathcal{Q}_1 \perp \mathcal{Q}_2$. Then, how to lower-bound the value of

$$p = \Pr[P \leftarrow_\$ \mathcal{P}_n : \varphi[P] \vdash \mathcal{Q}_1 | P \vdash \mathcal{Q}_2]. \tag{13}$$

It should be pointed out that each target-path in the group has the same construction, and hence the same number of missing-edges. This number of missing-edges is the principal character of Problem 1. In addition, we do not care about the specific values of the source-destination pairs in $\mathcal{Q}_1$, as long as they satisfy some "good" properties and $\mathcal{Q}_1 \perp \mathcal{Q}_2$. More importantly, our work shows some evidence that, the problems with the same number of missing-edges can be solved by similar techniques.

Compared to Chen et al's ad-hoc work, our techniques stand in a higher level and unearth something more intrinsic. In a very high level, our method is reduced to constructing a certain number of shared-edges by assigning inner-nodes.

Statement. For simplicity, we assume that all edges defined in this section are well-defined and compatible from each other.

5.1 Counting Framework

In this subsection, we will study how to handle the Problem 1 with at least 2 missing-edges, since the case of 1 missing-edge is trivial. More specifically, a counting framework will be proposed based on the notions of Core (see Definition 3) and shared-edge (see Definition 5). Before that, we will give some intuition about the framework.

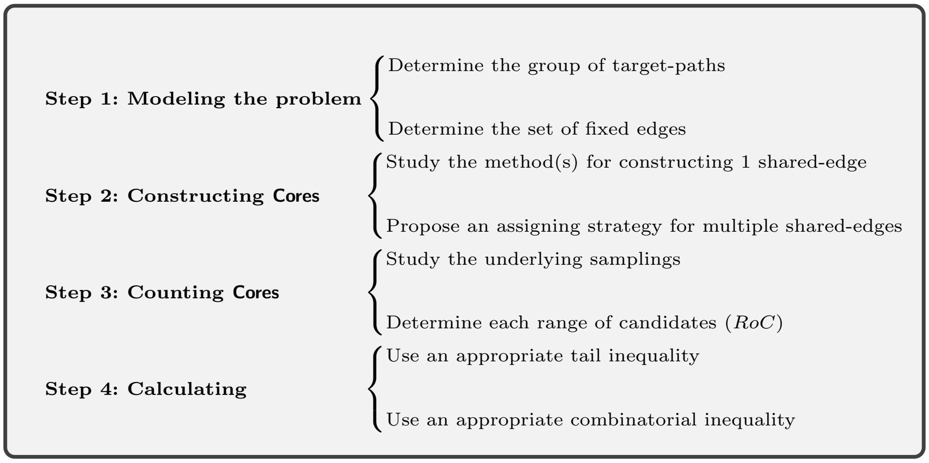
INTUITION. Let U and V denote the domain and range of $\mathcal{Q}_2$, respectively. Then, the sets $S = I_n \setminus U$ and $T = I_n \setminus V$ denote the sets of the strings whose next-points and previous-point are undefined, respectively. In fact, we will only use the edges from S to T to complete all the target-paths. Namely, the Cores we construct must be strongly-disjoint with $\mathcal{Q}_2$. The reason why we can still construct enough Cores is that, the number of known edges (i.e., $|\mathcal{Q}_2|$) is relatively rather small. For example, $|\mathcal{Q}_P| = O(N^{3/4})$ is far more smaller than N (i.e., roughly 1 out of $N^{1/4}$) when n is big enough.

Let $\mathcal{P}_C = \{P \in \mathcal{P}_n : \varphi[P] \vdash \mathcal{Q}_1 \ \wedge \ P \vdash \mathcal{Q}_2\}$ denote the set of all correct permutations extending $\mathcal{Q}_2$, and $\mathcal{C} = \{\widetilde{C} : \widetilde{C} \ \perp \mathcal{Q}_2 \bigwedge \exists \ P \in \mathcal{P}_C \ s.t. \ \widetilde{C} = \mathsf{Core}(\varphi[P] \vdash \mathcal{Q}_1)\}$ denote the set of all possible Cores strongly-disjoint with $\mathcal{Q}_2$. From the definition, we know that each correct permutation $P \in \mathcal{P}_C$ must determine a $\mathsf{Core}(\varphi[P] \vdash \mathcal{Q}_1)$. On the other side, for a specific $\widetilde{C} \in \mathcal{C}$, there exist exactly $(N - |\mathcal{Q}_2| - |\widetilde{C}|)!$ different correct permutations $P \in \mathcal{P}_C$ such that $\mathsf{Core}(\varphi[P] \vdash \mathcal{Q}_1) = \widetilde{C}$. That is because such P must contain the $|\mathcal{Q}_2|$ edges fixed in $\mathcal{Q}_2$ and the $|\widetilde{C}|$ edges fixed in $\widetilde{C}$, while the rest of edges can be defined freely. We can know that the above $(|\mathcal{Q}_2| + |\widetilde{C}|)$ edges are distinct and have no conflict, since it has $\widetilde{C} \ \perp \mathcal{Q}_2$. Additionally, it is easy to know that the size of the sample space is equal to $(N - |\mathcal{Q}_2|)!$, thus we have

$$\begin{aligned}
(13) &= \frac{|\mathcal{P}_C|}{(N - |\mathcal{Q}_2|)!} \\
&\geq \frac{\sum_{\widetilde{C} \in \mathcal{C}} \left|\{P \in \mathcal{P}_C : \mathsf{Core}(\varphi[P] \vdash \mathcal{Q}_1) = \widetilde{C}\}\right|}{(N - |\mathcal{Q}_2|)!} \\
&= \frac{\sum_{\widetilde{C} \in \mathcal{C}} (N - |\mathcal{Q}_2| - |\widetilde{C}|)!}{(N - |\mathcal{Q}_2|)!} \\
&= \frac{\sum_m \sum_{\widetilde{C} \in \mathcal{C} : |\widetilde{C}| = m} (N - |\mathcal{Q}_2| - m)!}{(N - |\mathcal{Q}_2|)!} \\
&= \sum_m \frac{\left|\left\{\widetilde{C} \in \mathcal{C} : |\widetilde{C}| = m\right\}\right|}{(N - |\mathcal{Q}_2|)_m}. \qquad (14)
\end{aligned}$$

Intrinsically, we classify the correct permutations according to the cardinality of the corresponding Core. In fact, we only interest in the Cores strongly-disjoint with $\mathcal{Q}_2$, since they are easier to be counted. From equation (14), it is known that the value of p can be lower-bounded, if we can count the number of Cores with a specific cardinality and also know how to sum all the related terms up.

The Counting Framework. Based on the above intuition, a 4-step counting framework is proposed in Fig.4. Roughly, the first 3 steps aim to lower-bound the number of Cores with a specific cardinality, and the last step will handle the calculation of a summation. As shown in Fig.4, our first task is to instantiate the Problem 1 with specific parameters (i.e., Step 1). Then, we should propose an appropriate assigning strategy for constructing a specific number of shared-edges, and hence obtain the Cores with the specific cardinality (i.e., Step 2). Also, we should count the number of possible assignments which can be constructed from the above strategy (i.e., Step 3). Thus, we actually establish a lower bound for the number of Cores with a specific cardinality. At last, we should calculate a summation to obtain the final result (i.e., Step 4).



Step 1: Modeling the problem
- Determine the group of target-paths
- Determine the set of fixed edges

Step 2: Constructing Cores
- Study the method(s) for constructing 1 shared-edge
- Propose an assigning strategy for multiple shared-edges

Step 3: Counting Cores
- Study the underlying samplings
- Determine each range of candidates (RoC)

Step 4: Calculating
- Use an appropriate tail inequality
- Use an appropriate combinatorial inequality

Fig. 4. Illustration of the Counting Framework

5.2 The Key Subproblem in 3KACSP

In this subsection, we will instantiate the counting framework for the 3-round case to show how it works. For brevity, we here only give the high-level technical route, and all the details will be completed in the formal proof. First of all, we abstract the 3-round subproblem in 3KACSP as follows.

Problem 2 (A Problem with 3 Missing-Edges). Let $\mathcal{Q}_E = \{(x_1, y_1), \ldots, (x_q, y_q)\}$ be the set of source-destination pairs of $\varphi_2[\cdot]$-target-paths, and $\mathcal{Q}_P = \{(u_1, v_1), \ldots, (u_p, v_p)\}$ be the set of known edges, where $\varphi_2[P] = P \circ P_\oplus \circ P$ and $\mathcal{Q}_P \perp \mathcal{Q}_E$. Then, how to lower-bound the value of

$$p_2 = \Pr[P \leftarrow_\$ \mathcal{P}_n : \varphi_2[P] \vdash \mathcal{Q}_E | P \vdash \mathcal{Q}_P]. \tag{15}$$

<u>Modeling The Problem.</u> Following the counting framework in Fig.4, our first task is to make clear the group of target-paths (including the construction and source-destination pairs) and the set of fixed edges. Obviously, Problem 2 is exactly the Problem 1 instantiated by $\varphi[\cdot] = \varphi_2[\cdot]$, $\mathcal{Q}_1 = \mathcal{Q}_E$ and $\mathcal{Q}_2 = \mathcal{Q}_P$. We denote X and Y as the domain and range of $\mathcal{Q}_E$, respectively. And let $S = I_n \setminus \{u_1, \ldots, u_p\}$ and $T = I_n \setminus \{v_1, \ldots, v_p\}$ denote the sets of strings whose next-points and previous-points are undefined, respectively. Then, it has $|S| = |T| = N - p$.

As shown in (16), there are 2 *inner-nodes* (i.e., $*_{i,1}$ and $*_{i,2}$) to be assigned in each $(\mathcal{Q}_E, \varphi_2[\cdot])$-target-path. We refer to the 2 inner-nodes in such a target-path as 1^{th}*-inner-node* and 2^{nd}*-inner-node*, respectively. Since a well-defined assignment of all the inner-nodes (i.e., the tuple of $(*_{1,1}, *_{1,2}, \ldots, *_{q,1}, *_{q,2})$) is equivalent to a $\mathsf{Core}(\varphi_2[\cdot] \vdash \mathcal{Q}_E)$, we will not distinguish them from now on. Moreover, we will count the number of assignments of all the inner-nodes, to lower-bound the value of (15).

$$(\mathcal{Q}_E, \varphi_2[\cdot])\text{-target-paths} \begin{cases} \langle x_1 \xrightarrow{P} *_{1,1},\ *_{1,1} \xrightarrow{P} *_{1,2},\ (*_{1,2})_{\oplus} \xrightarrow{P} y_1 \rangle \\ \qquad\qquad\qquad \ldots \\ \langle x_q \xrightarrow{P} *_{q,1},\ *_{q,1} \xrightarrow{P} *_{q,2},\ (*_{q,2})_{\oplus} \xrightarrow{P} y_q \rangle \end{cases} \tag{16}$$

<ins>CONSTRUCTING Cores.</ins> In essence, constructing Cores with a specific cardinality is equivalent to constructing a specific number of shared-edges. Our goal is to construct Cores with $3q - k - h$ edges, where k and h are variables of positive integers. It means that $k + h$ shared-edges (each of them is used exactly in 2 paths) should be constructed. Naturally, we study 2 assigning methods for constructing such shared-edges at first, and then use them to save k edges and h edges, respectively. In addition, we also need to know how to construct the edges, which are used exclusively in only 1 target-path (i.e., the ones are not shared-edges).

Definition 9 (Exclusive-Element). *We say an inner-node is assigned by an **exclusive-element**, if it not creates any new shared-edge at this moment. In this paper, we always use a notation related to w (e.g., w_i) to denote an exclusive-element.*

Consider the $(\mathcal{Q}_E, \varphi_2[\cdot])$-target-paths in (16), we will only focus on the shared-edge involved exactly in 2 paths (i.e., each such shared-edge will save 1 edge). It is easy to verify that a shared-edge is established once a 1^{th}-inner-node is assigned by an element from X or Y. Similarly, a shared-edge is also established once a 2^{nd}-inner-node is assigned by an element from $X_{\oplus}$ or Y. Therefore, for constructing a shared-edge, we choose a target-path at first, and then assign a proper value to its 1^{th}-inner-node or 2^{nd}-inner-node. In either case, the chosen value determines the other path sharing an edge with the former one. To distinguish them, we refer to the later determined target-path as a negative-path (denoted as $path^{-}$), since it is determined passively by the assigning. Accordingly, we call the former path as a positive-path (denoted as $path^{+}$).

Next, we will further interpret the above process. Since both of the 2 inner-nodes can be used to establish shared-edges, we discuss the 2 cases separately. At first, we will show how to construct shared-edges by assigning 1^{th}-inner-nodes.

► **1^{th}-Inner-Node.** In fact, we will construct exactly 1 shared-edge for each $path^{+}$. That is, the 2^{nd}-inner-node of a $path^{+}$ must be assigned by an exclusive-element. According to the position of the shared-edge(s) in $path^{-}$, there are 3 cases as follows.

- *Case 1:* Fix a target-path from x to y as the $path^{+}$ at first. If we assign $\ddot{x} \in X$ as its 1^{th}-inner-node (i.e., the one with box), then the target-path from $\ddot{x}$ to

$\ddot{y}$ becomes the corresponding *path*$^-$ whose first edge (i.e., the bold one) is the shared-edge. That is, the 1^{th}-inner-node (i.e., the underline one) in *path*$^-$ must be the same exclusive-element (i.e., w) as the 2^{nd}-inner-node in *path*$^+$. Additionally, if $\ddot{y}$ is not assigned to the 1^{th}- nor 2^{nd}- inner-node of any target-path, then we can assign an exclusive-element (i.e., $\widetilde{w}$) to the 2^{nd}-inner-node of *path*$^-$. As a result, we obtain a Type^{1a} sharing-gadget (as shown in (17)) containing 2 paths, 1 shared-edge and 2 exclusive-elements. And we refer to the 2 paths as a Type^{1a}-*path*$^+$ and a Type^{1a}-*path*$^-$, respectively.

$$\begin{array}{c}\text{Type}^{1a}\\ \text{sharing-gadget}\end{array}\begin{cases}\langle x \xrightarrow{P} \boxed{\ddot{x}},\ \boldsymbol{\ddot{x} \xrightarrow{P} w},\ w_{\oplus} \xrightarrow{P} y\rangle & (path^+)\\ \langle \boldsymbol{\ddot{x} \xrightarrow{P} \underline{w}},\ w \xrightarrow{P} \widetilde{w},\ \widetilde{w}_{\oplus} \xrightarrow{P} \ddot{y}\rangle & (path^-)\end{cases} \tag{17}$$

- *Case 2:* Fix a target-path from x to y as the *path*$^+$ at first. If we assign $\ddot{y} \in Y$ as its 1^{th}-inner-node (i.e., the one with box), then the target-path from $\ddot{x}$ to $\ddot{y}$ becomes the corresponding *path*$^-$ whose third edge (i.e., the bold one) is the shared-edge. That is, the 2^{th}-inner-node (i.e., the underline one) in *path*$^-$ must be assigned by $x_{\oplus}$. Additionally, if $\ddot{x}$ is not assigned to the 1^{th}-inner-node of any target-path and $\ddot{x}_{\oplus}$ is not assigned to the 2^{nd}-inner-node of any target-path, then we can assign an exclusive-element (i.e., $\widetilde{w}$) to the 1^{th}-inner-node of *path*$^-$. As a result, we obtain a Type^{1b} sharing-gadget (as shown in (18)) containing 2 paths, 1 shared-edge and 2 exclusive-elements. And we refer to the 2 paths as a Type^{1b}-*path*$^+$ and a Type^{1b}-*path*$^-$, respectively.

$$\begin{array}{c}\text{Type}^{1b}\\ \text{sharing-gadget}\end{array}\begin{cases}\langle \boldsymbol{x \xrightarrow{P} \boxed{\ddot{y}}},\ \ddot{y} \xrightarrow{P} w,\ w_{\oplus} \xrightarrow{P} y\rangle & (path^+)\\ \langle \ddot{x} \xrightarrow{P} \widetilde{w},\ \widetilde{w} \xrightarrow{P} \underline{x_{\oplus}},\ \boldsymbol{x \xrightarrow{P} \ddot{y}}\rangle & (path^-)\end{cases} \tag{18}$$

- *Case 3:* Interestingly, a *path*$^-$ can share edges with 2 different *paths*$^+$ simultaneously. Fix the target-path from x_1 to y_1 as *path*$_1^+$, and fix the target-path from x_2 to y_2 as *path*$_2^+$. If we assign $\ddot{x}$ (resp. $\ddot{y}$) as the 1^{th}-inner-node of *path*$_1^+$ (resp. *path*$_2^+$) (i.e., the ones with box), then the target-path from $\ddot{x}$ to $\ddot{y}$ becomes the *path*$^-$ of *path*$_1^+$ and *path*$_2^+$ simultaneously. That is, the 1^{th}-inner-node in *path*$^-$ must be the same exclusive-element (i.e., w) as the 2^{nd}-inner-node in *path*$_1^+$, and the 2^{nd}-inner-node in *path*$^-$ must be assigned by $(x_2)_{\oplus}$. As a result, we obtain a Type^{1c} sharing-gadget (as shown in (19)) containing 3 paths, 2 shared-edge and 2 exclusive-elements. And we refer to the 3 paths as a Type^{1c}-*path*$_1^+$, a Type^{1c}-*path*$_2^+$ and a Type^{1c}-*path*$^-$, respectively.

$$\begin{array}{c}\text{Type}^{1c}\\ \text{sharing-gadget}\end{array}\begin{cases}\langle x_1 \xrightarrow{P} \boxed{\ddot{x}},\ \boldsymbol{\ddot{x} \xrightarrow{P} w},\quad w_{\oplus} \xrightarrow{P} y_1\rangle & (path_1^+)\\ \langle \boldsymbol{x_2 \xrightarrow{P} \boxed{\ddot{y}}},\ \ddot{y} \xrightarrow{P} \widetilde{w},\quad \widetilde{w}_{\oplus} \xrightarrow{P} y_2\rangle & (path_2^+)\\ \langle \boldsymbol{\ddot{x} \xrightarrow{P} \underline{w}},\ w \xrightarrow{P} \underline{(x_2)_{\oplus}},\ \boldsymbol{x_2 \xrightarrow{P} \ddot{y}}\rangle & (path^-)\end{cases} \tag{19}$$

At this point, we have known how to construct a shared-edge for a *path*$^+$ by assigning its 1^{th}-inner-node. Naturally, we can establish k such shared-edges if k

paths$^+$ are considered. It is easy to verify that the following Method 1 actually establishes k shared-edges for the involved *paths*$^+$ and *paths*$^-$.[9]

Method 1 for constructing k shared-edges:

–Step 1 Choose k proper target-paths as the *paths*$^+$, and then choose k proper elements from $X \cup Y$ as their 1^{th}-inner-nodes, respectively. Namely, we choose a Type^{1a}/Type^{1b}/Type^{1c}-*path*$^-$ for each of the k *paths*$^+$.

–Step 2 For the *paths*$^+$ and *paths*$^-$ determined in Step 1, we assign in turn each undefined inner-node with an exclusive-element.

We can see that Method 1 allows the *paths*$^-$ to be a mixture of Type^{1a}-, Type^{1b}- and Type^{1c}- *paths*$^-$. This is a *key point* to obtain an appropriate lower bound in our proof, since it enlarge the number of candidates for *paths*$^-$ (i.e., roughly double the one involved in 2-round case[10]). It should be pointed out that the number of *paths*$^-$ is determined by the 1^{th}-inner-nodes of the k *paths*$^+$, and is not necessarily equal to k. In a sense, the k *paths*$^+$ with their 1^{th}-inner-nodes determine almost "everything" about the involved *paths*$^+$ and *paths*$^-$.

► **2^{nd}-Inner-Node.** It is similar to construct a shared-edge by assigning the 2^{nd}-inner-node of a *path*$^+$. We will also construct exactly 1 shared-edge for each *path*$^+$. That is, the 1^{th}-inner-node of a *path*$^+$ must be assigned by an exclusive-element. According to the position of the shared-edge(s) in *path*$^-$, there are also 3 cases. For reason of the space, we omit the explanation here, and defer the details to the corresponding part of full version [WYCD20].

- *Case 1:*

$$\begin{array}{c}\text{Type}^{2a}\\ \text{sharing-gadget}\end{array}\begin{cases}\langle x \xrightarrow{P} w,\ w \xrightarrow{P} \boxed{\ddot{x}_\oplus},\ \ \ddot{\boldsymbol{x}} \xrightarrow{\boldsymbol{P}} \boldsymbol{y}\rangle & (\mathit{path}^+)\\ \langle \ddot{\boldsymbol{x}} \xrightarrow{\boldsymbol{P}} \underline{\boldsymbol{y}},\ y \xrightarrow{P} \widetilde{w},\ \ \widetilde{w}_\oplus \xrightarrow{P} \ddot{y}\rangle & (\mathit{path}^-)\end{cases} \tag{20}$$

- *Case 2:*

$$\begin{array}{c}\text{Type}^{2b}\\ \text{sharing-gadget}\end{array}\begin{cases}\langle x \xrightarrow{P} w,\ \boldsymbol{w} \xrightarrow{\boldsymbol{P}} \boxed{\ddot{\boldsymbol{y}}},\ \ddot{y}_\oplus \xrightarrow{P} y\rangle & (\mathit{path}^+)\\ \langle \ddot{x} \xrightarrow{P} \widetilde{w},\ \widetilde{w} \xrightarrow{P} \underline{w_\oplus},\ \ \boldsymbol{w} \xrightarrow{\boldsymbol{P}} \ddot{\boldsymbol{y}}\rangle & (\mathit{path}^-)\end{cases} \tag{21}$$

- *Case 3:*

$$\begin{array}{c}\text{Type}^{2c}\\ \text{sharing-gadget}\end{array}\begin{cases}\langle x_1 \xrightarrow{P} w,\ \ w \xrightarrow{P} \boxed{\ddot{x}_\oplus},\ \ \ \ddot{\boldsymbol{x}} \xrightarrow{\boldsymbol{P}} \boldsymbol{y_1}\rangle & (\mathit{path}_1^+)\\ \langle x_2 \xrightarrow{P} \widetilde{w},\ \ \widetilde{\boldsymbol{w}} \xrightarrow{\boldsymbol{P}} \boxed{\ddot{\boldsymbol{y}}},\ (\ddot{y})_\oplus \xrightarrow{P} y_2\rangle & (\mathit{path}_2^+)\\ \langle \ddot{\boldsymbol{x}} \xrightarrow{\boldsymbol{P}} \underline{\boldsymbol{y_1}},\ y_1 \xrightarrow{P} \underline{\widetilde{w}_\oplus},\ \ \ \widetilde{\boldsymbol{w}} \xrightarrow{\boldsymbol{P}} \ddot{\boldsymbol{y}}\rangle & (\mathit{path}^-)\end{cases} \tag{22}$$

[9] In brief, Step 1 establishes k shared-edges, while Step 2 produces no shared-edge.

[10] More details can be found in the corresponding part of the full version [WYCD20], in which we also give the assigning strategy of 2-round case.

At this point, we have known how to construct a shared-edge for a *path*$^+$ by assigning its 2^{nd}-inner-node. Naturally, we can establish h such shared-edges if h *paths*$^+$ are considered. It is easy to verify that the following Method 2 actually establishes h shared-edges for the involved *paths*$^+$ and *paths*$^-$.

Method 2 for constructing h shared-edges:

−Step 1 Choose h proper target-paths as the *paths*$^+$, and then choose h proper elements from $X_\oplus \cup Y$ as their 2^{nd}-inner-nodes, respectively. Namely, we choose a Type^{2a}/Type^{2b}/Type^{2c}-*path*$^-$ for each of the h *paths*$^+$.

−Step 2 For the *paths*$^+$ and *paths*$^-$ determined in Step 1, we assign in turn each undefined inner-node with an exclusive-element.

Similarly, the number of *paths*$^-$ is determined by the 2^{nd}-inner-nodes of the h *paths*$^+$, and is not necessarily equal to h. In a sense, the h *paths*$^+$ with their 2^{nd}-inner-nodes determine almost "everything" about the involved *paths*$^+$ and *paths*$^-$.

Combing the Method 1 and Method 2, we propose the assigning strategy for constructing a Core with $3q - k - h$ edges. As shown in Fig.5, Step 1 (resp. Step 2) establishes k (resp. h) shared-edges, and Step 3 produces no shared-edge. In brief, we fix k *paths*$^+$ and their 1^{th}-inner-nodes firstly, then other h *paths*$^+$ and their 2^{nd}-inner-nodes. At last, we assign all the undefined inner-nodes with proper exclusive-elements.

Assigning Strategy:

−Step 1 Choose k proper target-paths as the *paths*$^+$, and then choose k proper elements from $X \cup Y$ as their 1^{th}-inner-nodes, respectively.

−Step 2 Apart from the paths involved in Step 1, we choose h proper target-paths as the *paths*$^+$, and then choose h proper elements from $X_\oplus \cup Y$ as their 2^{nd}-inner-nodes, respectively.

−Step 3 Assign in turn each undefined inner-node with an exclusive-element.

Fig. 5. Assigning Strategy for Constructing a Core$(\varphi_2[\cdot] \vdash \mathcal{Q}_E)$ with $3q - k - h$ Edges

<u>Counting Cores.</u> Intrinsically, the assigning strategy consists of several samplings such as the *paths*$^+$, *paths*$^-$, and so on. To lower-bound the number of Cores constructed by Fig.5, we should know how many elements can be chosen for each sampling. For convenience, we introduce the notation RoC to denote the *range of candidates* for a sampling.

Notation 5 (Range of Candidates) *Let A denote a finite set to be sampled, then we write $RoC(A)$ as a set of elements which can be chosen into A.*

That is, we should determine the size of each RoC to count the number of possible assignments. The analysis is rather cumbersome, and we defer it to the formal proof. Here, we can just assume that the lower bound of the number of Cores with $3q-k-h$ edges is given as follows.

$$\#\mathsf{Cores}_{3q-k-h} \ge LB(k,h), \tag{23}$$

where $\#\mathsf{Cores}_{3q-k-h}$ denotes the number of Cores with $3q-k-h$ edges, and $LB(k,h)$ is a function of k and h.

<u>Calculating the lower bound.</u> At this point, we are ready to calculate a lower bound of (15). Since $|S| = |T| = N - p$, and from the equations (14) and (23), we finally obtain that

$$\begin{aligned} p_2 &\ge \sum_{k,h} \frac{LB(k,h)}{(N-p)_{3q-k-h}} \\ &\ge \sum_{\substack{0\le k\le M\\ 0\le h\le M}} \frac{LB(k,h)}{(N-p)_{3q-k-h}} \\ &= \underbrace{\sum_{\substack{0\le k\le M\\ 0\le h\le M}} MHyp_{N,a,b,c}(k,h)}_{\substack{\text{Use Lemma 5 to obtain}\\ \text{a proper lower bound}}} \cdot \underbrace{(\text{Major Terms})}_{\substack{\text{Use Lemma 6 to obtain}\\ \text{a proper lower bound}\\ \text{independent of } k \text{ and } h}} \cdot \underbrace{(\text{Minor Terms})}_{\substack{\text{Obtain directly a}\\ \text{proper lower bound}\\ \text{independent of } k \text{ and } h}}, \end{aligned}$$

where $MHyp_{N,a,b,c}$ is a multivariate hypergeometric distribution random variable. It can be seen that a tail inequality (i.e., Lemma 5) and a combinatorial inequality (i.e., Lemma 6) will be used during the calculation.

Lemma 5 (Chebyshev's Inequality). *Let $X \sim Hyp_{N,a,b}$ be a hypergeometric distribution random variable, that is,* $\Pr[X = k] = \frac{\binom{b}{k}\binom{N-b}{a-k}}{\binom{N}{a}} = \frac{(a)_k (b)_k (N-b)_{a-k}}{k!(N)_a}$. *Then we have*

$$\Pr[X > \lambda] \le \frac{ab(N-a)(N-b)}{(\lambda N - ab)^2\,(N-1)}. \tag{24}$$

Lemma 6. *Let N, a, b, c, d be positive integers such that $c+d = 2b$ and $2a+2b \le N$. Then*

$$\frac{(N)_a\,(N-2b)_a}{(N-c)_a\,(N-d)_a} \times \frac{(N-\frac{b}{2})_a}{(N-b)_a} \ge 1 - \frac{8ab^3}{N^3}. \tag{25}$$

6 Conclusion and Discussion

The practical block-ciphers often iterate the same round function and use a key-schedule algorithm to produce round-keys, while there are a few theoretical results supporting such designing philosophy. Particularly, only a little provable-security work considers the dependence between components, since it always becomes very complicated.

In this paper, we study a family of KAC construction with dependence, and finally prove that 3KACSP construction has the same security level as the classic 3KAC construction. It means that the randomness of one random permutation and a random $4n$-bit string is enough to make the 3KAC construction achieve the ideal security. To our knowledge, it is the first time to obtain a tight bound about an incompletely-independent KAC construction with more than 2 rounds.

Besides the tight security analysis of 3KACSP, our most valuable contributions are the insights into the general rKACSP. Before our work, there is no proof method handling the knotty dependence in a high level. Compared to Chen et al.'s techniques, ours are more general and highly modular so that they can be easily generalized. More concretely, we abstract a type of combinatorial problems capturing the intrinsic randomness of rKACSP construction. To solve such problems, we also propose a general counting framework and successfully apply it to the cases with 2 and 3 missing-edges. Following the proof ideas in this work, we give some intuition on the analysis of rKACSP.

Intuition on rKACSP. Intuitively, when handling a "good" transcript, the paths between $\mathcal{Q}_E$ can be classified into r groups according to the number of fixed edges. Similar to 3KACSP, we denote the Group-i as the group of paths whose i edges are fixed by $\mathcal{Q}_P$, where $0 \leq i \leq r-1$. The subproblem of completing the paths in Group-i can be instantiated by Problem 1 with $r-i$ missing-edges. Inspired by the analysis of 3KACSP, the tricks used in $(r-i)$KACSP can be tailored to solve the corresponding subproblems related to Group-i, where $1 \leq i \leq r-2$. By induction, what's left is only to solve a single r-round instance of Problem 1. Our counting framework, as well as the notions of shared-edges and assigning strategy can still work, but the circumstances of analysis would be very complicated.

To our conjecture, rKACSP construction is also $\Theta(2^{\frac{r}{r+1}n})$-secure in the random permutation model, which is a well-known result for classic rKAC construction.

Conjecture 1. Consider the rKACSP construction (see Eq.(1)), if P is a random permutation, as well as the round keys $\boldsymbol{K} = (k_0, \ldots, k_r)$ are random and independent from each other, then rKACSP is $\Theta\left(2^{\frac{r}{r+1}n}\right)$-secure in the random permutation model.

In fact, the bottleneck of pushing our work to higher-round case is simply the computational power. Following our ideas, the technical roadmap for analyzing rKACSP is rather clear, and one can solve it given sufficient energy. Honestly, we consider that the complexity of proof specifics will increase very fast (maybe exponentially) so that the proof may not be explicitly written out, but we strongly believe that Conjecture 1 is intrinsically correct. If the conjecture is true, then it is exactly a powerful support for the aforementioned broadly-used designing philosophy. Moreover, the proof complexity may just reveal the reason why there often exist gaps between the practical and theoretical results.

Open Problems. Currently, our results only apply when the round keys are random and independent from each other. Thus, it is unknown that whether we can reduce the randomness of round keys without a significant loss of security. Another challenging open problem is of course to generalize our results to larger number of rounds. In addition, the new representation and counting framework are rather generic, therefore we hope that they can be used in more scenarios.

Acknowledgements. This work has been submitted to several conferences successively, we would like to thank all the anonymous reviewers for their valuable comments which helped to improve the presentation of this paper.

This work was supported in part by the National Natural Science Foundation of China (Grant No.61632012 and 61672239), in part by the Peng Cheng Laboratory Project of Guangdong Province (Grant No. PCL2018KP004), and in part by "the Fundamental Research Funds for the Central Universities".

References

BKL$^+$07. Andrey Bogdanov, Lars R. Knudsen, Gregor Leander, Christof Paar, Axel Poschmann, Matthew J. B. Robshaw, Yannick Seurin, and C. Vikkelsoe. PRESENT: an ultra-lightweight block cipher. In *Cryptographic Hardware and Embedded Systems - CHES 2007, 9th International Workshop, Vienna, Austria, September 10-13, 2007, Proceedings*, pages 450–466, 2007. https://doi.org/10.1007/978-3-540-74735-2_31.

BKL$^+$12. Andrey Bogdanov, Lars R. Knudsen, Gregor Leander, François-Xavier Standaert, John P. Steinberger, and Elmar Tischhauser. Key-alternating ciphers in a provable setting: Encryption using a small number of public permutations - (extended abstract). In *Advances in Cryptology - EUROCRYPT 2012 - 31st Annual International Conference on the Theory and Applications of Cryptographic Techniques, Cambridge, UK, April 15-19, 2012. Proceedings*, pages 45–62, 2012. https://doi.org/10.1007/978-3-642-29011-4_5.

CLL$^+$14. Shan Chen, Rodolphe Lampe, Jooyoung Lee, Yannick Seurin, and John P. Steinberger. Minimizing the two-round even-mansour cipher. In *Advances in Cryptology - CRYPTO 2014 - 34th Annual Cryptology Conference, Santa Barbara, CA, USA, August 17-21, 2014, Proceedings, Part I*, pages 39–56, 2014. https://doi.org/10.1007/978-3-662-44371-2_3.

CLL$^+$18. Shan Chen, Rodolphe Lampe, Jooyoung Lee, Yannick Seurin, and John P. Steinberger. Minimizing the two-round even-mansour cipher. *J. Cryptology*, 31(4):1064–1119, 2018. https://doi.org/10.1007/s00145-018-9295-y.

CS14. Shan Chen and John P. Steinberger. Tight security bounds for key-alternating ciphers. In *Advances in Cryptology - EUROCRYPT 2014 - 33rd Annual International Conference on the Theory and Applications of Cryptographic Techniques, Copenhagen, Denmark, May 11-15, 2014. Proceedings*, pages 327–350, 2014. https://doi.org/10.1007/978-3-642-55220-5_19.

CS18. Benoît Cogliati and Yannick Seurin. Analysis of the single-permutation encrypted davies-meyer construction. *Des. Codes Cryptography*, 86(12):2703–2723, 2018. https://doi.org/10.1007/s10623-018-0470-9.

Dae91. Joan Daemen. Limitations of the even-mansour construction. In *Advances in Cryptology - ASIACRYPT '91, International Conference on the Theory* *and Applications of Cryptology, Fujiyoshida, Japan, November 11-14, 1991, Proceedings*, pages 495–498, 1991. https://doi.org/10.1007/3-540-57332-1_46.

DKS12. Orr Dunkelman, Nathan Keller, and Adi Shamir. Minimalism in cryptography: The even-mansour scheme revisited. In *Advances in Cryptology - EUROCRYPT 2012 - 31st Annual International Conference on the Theory and Applications of Cryptographic Techniques, Cambridge, UK, April 15-19, 2012. Proceedings*, pages 336–354, 2012. https://doi.org/10.1007/978-3-642-29011-4_21.

DR02. Joan Daemen and Vincent Rijmen. *The Design of Rijndael: AES - The Advanced Encryption Standard*. Information Security and Cryptography. Springer, 2002. https://doi.org/10.1007/978-3-662-04722-4.

DSST17. Yuanxi Dai, Yannick Seurin, John P. Steinberger, and Aishwarya Thiruvengadam. Indifferentiability of iterated even-mansour ciphers with non-idealized key-schedules: Five rounds are necessary and sufficient. In *Advances in Cryptology - CRYPTO 2017 - 37th Annual International Cryptology Conference, Santa Barbara, CA, USA, August 20-24, 2017, Proceedings, Part III*, pages 524–555, 2017. https://doi.org/10.1007/978-3-319-63697-9_18.

EM97. Shimon Even and Yishay Mansour. A construction of a cipher from a single pseudorandom permutation. *J. Cryptology*, 10(3):151–162, 1997. https://doi.org/10.1007/s001459900025.

GPPR11. Jian Guo, Thomas Peyrin, Axel Poschmann, and Matthew J. B. Robshaw. The LED block cipher. In *Cryptographic Hardware and Embedded Systems - CHES 2011 - 13th International Workshop, Nara, Japan, September 28 - October 1, 2011. Proceedings*, pages 326–341, 2011. https://doi.org/10.1007/978-3-642-23951-9_22.

HT16. Viet Tung Hoang and Stefano Tessaro. Key-alternating ciphers and key-length extension: Exact bounds and multi-user security. In *Advances in Cryptology - CRYPTO 2016 - 36th Annual International Cryptology Conference, Santa Barbara, CA, USA, August 14-18, 2016, Proceedings, Part I*, pages 3–32, 2016. https://doi.org/10.1007/978-3-662-53018-4_1.

LPS12. Rodolphe Lampe, Jacques Patarin, and Yannick Seurin. An asymptotically tight security analysis of the iterated even-mansour cipher. In *Advances in Cryptology - ASIACRYPT 2012 - 18th International Conference on the Theory and Application of Cryptology and Information Security, Beijing, China, December 2-6, 2012. Proceedings*, pages 278–295, 2012. https://doi.org/10.1007/978-3-642-34961-4_18.

Pat08. Jacques Patarin. The "coefficients h" technique. In *Selected Areas in Cryptography, 15th International Workshop, SAC 2008, Sackville, New Brunswick, Canada, August 14-15, Revised Selected Papers*, pages 328–345, 2008. https://doi.org/10.1007/978-3-642-04159-4_21.

Ste12. John P. Steinberger. Improved security bounds for key-alternating ciphers via hellinger distance. *IACR Cryptology ePrint Archive*, 2012:481, 2012. http://eprint.iacr.org/2012/481.

WYCD20. Yusai Wu, Liqing Yu, Zhenfu Cao, and Xiaolei Dong. Tight security analysis of 3-round key-alternating cipher with a single permutation. Cryptology ePrint Archive, Report 2020/1073, 2020. https://eprint.iacr.org/2020/1073.