Faster Enumeration-based Lattice Reduction: Root Hermite Factor $k^{1/(2k)}$ in Time $k^{k/8+o(k)}$

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Abstract. We give a lattice reduction algorithm that achieves root Hermite factor $k^{1/(2k)}$ in time $k^{k/8+o(k)}$ and polynomial memory. This improves on the previously best known enumeration-based algorithms which achieve the same quality, but in time $k^{k/(2e)+o(k)}$. A cost of $k^{k/8+o(k)}$ was previously mentioned as potentially achievable (Hanrot-Stehlé'10) or as a heuristic lower bound (Nguyen'10) for enumeration algorithms. We prove the complexity and quality of our algorithm under a heuristic assumption and provide empirical evidence from simulation and implementation experiments attesting to its performance for practical and cryptographic parameter sizes. Our work also suggests potential avenues for achieving costs below $k^{k/8+o(k)}$ for the same root Hermite factor, based on the geometry of SDBKZ-reduced bases.

1 Introduction

The cost of (strong) lattice reduction has received renewed attention in recent years due to its relevance in cryptography. Indeed, lattice-based constructions are presumed to achieve security against quantum adversaries and enable powerful functionalities such as computation on encrypted data. Concrete parameters for such schemes are derived from the difficulty of finding relatively short non-zero vectors in a lattice: the parameters are chosen based on extrapolations of the cost of the BKZ algorithm [SE94] and its variants [CN11,AWHT16,MW16]. These algorithms make repeated calls to an oracle that solves the Shortest Vector Problem (SVP), i.e. that finds a shortest non-zero vector in any lattice. Concretely, BKZ with block size k finds relatively short vectors in lattices of dimensions $n \geq k$

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using a k-dimensional SVP solver. The cost of this SVP solver is the dominating component of the cost of BKZ and its variants.

The SVP solver can be instantiated with enumeration-based algorithms, whose asymptotically most efficient variant is Kannan's algorithm [Kan83]. It has a worst-case complexity of $k^{k/(2\,\mathrm{e})+o(k)}$, where k is the dimension of the lattice under consideration [HS07]. This bound is sharp, up to the $o(k)$ term in the exponent [HS08]. If called on an n-dimensional lattice, then BKZ with block size k outputs a vector of norm $\approx \left(k^{1/(2k)}\right)^n \cdot \mathrm{Vol}(\mathcal{L})^{1/n}$ in time $\approx k^{k/(2\,\mathrm{e})}$, when n is sufficiently large compared to k. The $k^{1/(2k)}$ term is called the root Hermite factor and quantifies the strength of BKZ. The trade-off between root Hermite factor and running-time achieved by BKZ has remained the best known for enumeration-based SVP solvers since the seminal work of Schnorr and Euchner almost 30 years ago. (The analysis of Kannan's algorithm and hence BKZ was improved in [HS07], but not the algorithm itself.) Other algorithms, such as [GN08a,MW16,ALNS19], achieve the same asymptotic trade-off with milder conditions on n/k.

We note that while lattice reduction libraries, such as `FPLLL` [dt19a], the Progressive BKZ Library (`PBKZ`) [AWHT18] and `NTL` [Sho18], implement BKZ with an enumeration-based SVP solver, they do not rely on Kannan's algorithm: `NTL` implements enumeration with LLL preprocessing; `FPLLL` and `PBKZ` implement enumeration with stronger preprocessing (typically BKZ with a smaller block size) but not with sufficiently strong preprocessing to satisfy the conditions of [Kan83,HS07]. Hence, the running-times of these implementations is not established by the theorems in these works.

It has been suggested that the running-time achieved by BKZ for the same output quality might potentially be improved. In [HS10], it was argued that the same root Hermite factor would be achieved by BKZ in time $\approx k^{k/8}$ if the Gram–Schmidt norms of so-called HKZ-reduced bases were decreasing geometrically. In [Ngu10], the same quantity was suggested as a cost lower bound for enumeration-based lattice reduction algorithms. On this basis, several works have speculatively assumed this cost [ANS18,ACD+18]. However, so far, no lattice reduction algorithm achieving root Hermite factor $k^{1/(2k)}$ in time $\approx k^{k/8}$ was known.

Contributions. Our main contribution is an enumeration-based lattice reduction algorithm that runs in time $k^{k/8}$ and achieves root Hermite factor $k^{\frac{1}{2k}(1+o(1))}$, where k is a cost parameter akin to the "block size" of the BKZ algorithm (the notion of "block size" for our algorithm is less straightforward, see below). It uses polynomial memory and can be quantumly accelerated to time $k^{k/16}$ using [ANS18]. Our analysis relies on a strengthened version of the Gaussian Heuristic.

To estimate the cost of lattice reduction algorithms, the literature typically relies on concrete experiments and simulations that extrapolate them (see, e.g. [CN11,Che13,MW16,BSW18]). Indeed, the data given in [Che13] is very broadly appealed to. However, this data only covers up to block size of 250 (below cryptographically relevant block sizes) and no source code is available.

As an intermediate contribution, we reproduce and extend the data in [Che13] using publicly available tools such as [dt19a,dt19b] (see Section 2.5). Using this extended dataset, we then argue that BKZ as implemented in public lattice reduction libraries has running-time closely matching $k^{k/(2\,\mathrm{e})}$ (Figure 2). Our cost improvement hence required a different algorithm and not just an improved analysis of the state-of-the-art.

In Section 4 we propose a variant of our improved lattice reduction algorithm that works well in practice. We run simulations and conduct concrete experiments to verify its efficiency, while we leave as a future work to formally analyse it. The simulations suggest that it achieves root Hermite factors $\approx k^{\frac{1}{2k}}$ in time $k^{k/8}$, at least up to $k \approx 1,000$ (which covers cryptographic parameters). Our implementation of this algorithm beats FPLLL's SVP enumeration from dimension ≈ 100 onward. We consider the difference between these two variants as similar to the difference between Kannan's algorithm and what is routinely implemented in practice such as in `FPLLL` and `PBKZ`. We will refer to the former as the "asymptotic variant" and the latter as the "practical variant". Since our results rely on empirical evidence and simulations, we provide the source code used to produce our figures and the data being plotted as an attachment to the electronic version of the full version of this work.

Key idea. Our new algorithms decouple the preprocessing context from the enumeration context: they preprocess a projected sublattice of larger dimension than they aim to enumerate over (as a result, the notion of "block size" is less obvious than in prior works). More concretely, assume that the basis of the preprocessed projected sublattice is SDBKZ-reduced. Then, as shown in [MW16] under the Gaussian Heuristic, the first Gram–Schmidt norms $\|\boldsymbol{b}_i^*\|$ satisfy Schnorr's Geometric Series Assumption (GSA) [Sch03]: $\|\boldsymbol{b}_i^*\|/\|\boldsymbol{b}_{i+1}^*\| \approx r$ for some common r, for all i's corresponding to the start of the basis. On that "GSA part" of the lattice basis, the enumeration runs faster than on a typical preprocessed BKZ block of the same dimension. To achieve SDBKZ-reducedness at a low cost, our algorithms call themselves recursively.

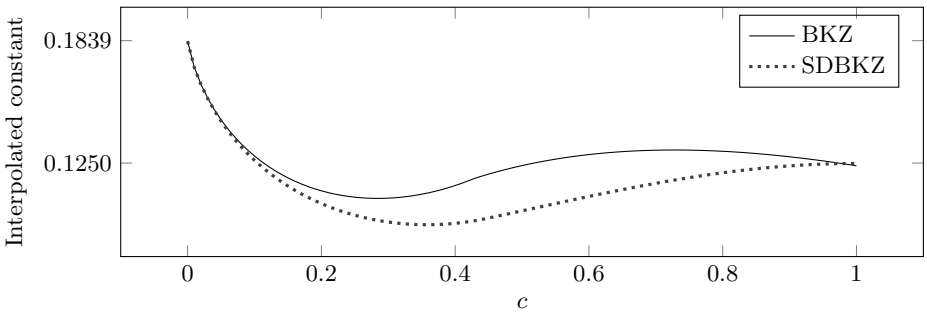
As a side contribution, we show in the full version of this work that the bases output by BKZ do not satisfy the GSA (under the Gaussian Heuristic), contrarily to a common belief (see e.g. [YD17,ANS18,BSW18]). This is why we use SDBKZ in the asymptotic algorithm. Nevertheless, for handlable dimensions, BKZ seems to deviate only slightly from the GSA and as it is a little simpler to implement than SDBKZ, it seems to remain preferable in practice. This is why we use BKZ as preprocessing in the practical algorithm.

To illustrate the idea of our new algorithms, we consider a BKZ-reduced (resp. SDBKZ-reduced) basis with block size k, in a lattice of dimension $n = \lceil (1+c) \cdot k \rfloor$ for various k and c. We choose $n \geq k$ to demonstrate the impact of the GSA region on the enumeration cost. We then estimate the enumeration cost (without pruning) for re-checking that the first basis vector is a shortest non-zero vector in the very first block of size k. For the implementation of this simulation, see `simu_c_cost.py`, attached to the electronic version of the full version of this

work. We consider c for $c = 0$ to 1 with a step size of 0.01. For each c, we take k from $k = 100$ to $50,000$ with a step size of 10. Then for each fixed c, we fit the coefficients a_0, a_1, a_2 of $a_0\, k \log k + a_1\, k + a_2$ over all k on the enumeration cost of the first block of size k. The result is plotted in Figure 1. The x-axis denotes the value of c and the y-axis denotes the interpolated constant in front of the $k \log k$ term.

Let us make several remarks about Figure 1. First, we stress that all leading constants in Figure 1 are hypothetical (they do not correspond to efficient algorithms) as they assume an already (SD)BKZ-reduced basis with block size k, i.e. this ignores the preprocessing cost. With that in mind, for $c = 0$, the simulations show the interpolated constant for both BKZ and SDBKZ is close to $1/(2\,\mathrm{e})$, which corresponds to [HS07]. For $c = 1$, the interpolated constant is close to $1/8$. This illustrates the impact of enumeration in the GSA region (corresponding to Theorem 1). As noted above, in the following section we will describe an algorithm that achieve the corresponding cost of $k^{k/8(1+o(1))}$. It is worth noting that for certain c around 0.3, the a_0 of the re-examination cost can be below 0.125. We stress that we do not know how to construct an algorithm that achieves a cost of $k^{a_0 \cdot k(1+o(1))}$ with $a_0 < 0.125$. However, our practical variant of the algorithm seems to achieve cost $k^{0.125 \cdot k}$ using the region corresponding to those $c \approx 0.3$.

Fig. 1: Interpolated dominating constant a_0 on $k \log k$.



Discussion. At first sight, the endeavour in this work might appear pointless since lattice sieving algorithms asymptotically outperform lattice enumeration. Indeed, the fastest SVP solver currently known [BDGL16] has a cost of $2^{0.292\, n+o(n)}$, where n is the lattice dimension.[6] Furthermore, a sieving implementation [ADH+19] now dominates the Darmstadt SVP Challenge's Hall of Fame, indicating that the crossover between enumeration and sieving is well below

[6] When using this algorithm as the SVP subroutine in BKZ, we thus obtain a running time of $2^{0.292\, k+o(k)}$ for root Hermite factor $k^{1/(2k)}$.

cryptographic parameter sizes. However, the study of enumeration algorithms is still relevant to cryptography.

Sieving algorithms have a memory cost that grows exponentially with the lattice dimension n. For dimensions that are currently handlable, the space requirement remains moderate. The impact of this memory cost is unclear for cryptographically relevant dimensions. For instance, it has yet to be established how well sieving algorithms parallelise in non-uniform memory access architectures. Especially, the exponential memory requirement might present a serious obstacle in some scenarios. In contrast, the memory cost of enumeration grows as a small polynomial in the dimension.

Comparing sieving and enumeration for cryptographically relevant dimensions becomes even more complex in the context of quantum computations. Quantum computations asymptotically enable a quadratic speed-up for enumeration, and much less for sieving [Laa15, Sec. 14.2.10] even assuming free quantum-accessible RAM, which would a priori favour enumeration. However, deciding on how to compare parallelisable classical operations with strictly sequential Grover iterations is unclear and establishing the significant lower-order terms in the quantum costs of these algorithms is an ongoing research programme (see e.g. [AGPS19])

Further, recent advances in sieving algorithms [LM18,Duc18,ADH+19] apply lessons learned from enumeration algorithms to the sieving context: while sieving algorithms are fairly oblivious to the Gram–Schmidt norms of the basis at hand, the cost of enumeration algorithms critically depend on their limited decrease. Current sieving strategies employ a simple form of enumeration (Babai's lifting [Bab86]) to exploit the lattice shape by sieving in a projected sublattice and lifting candidates for short vectors to the full lattice. Here, more sophisticated hybrid algorithms permitting flexible trade-offs between memory consumption and running time seem plausible.

Finally, as illustrated in Figure 1, our work suggests potential avenues for designing faster enumeration algorithms based on further techniques relying on the graph of Gram–Schmidt norms.

Open problems. It would be interesting to remove the heuristics utilised in our analysis to produce a fully proved variant, and to extend the technique to other lattice reduction algorithms such as slide reduction [GN08a]. Further, establishing lower bounds on the root Hermite factor achievable in time $k^{k/8+o(k)}$ for a given dimension of the lattice is an interesting open problem suggested by this work.

2 Preliminaries

Matrices are denoted in bold uppercase and vectors are denoted in bold lowercase. By $\boldsymbol{B}_{[i:j)}$ we refer to the submatrix spanned by the columns $\boldsymbol{b}_i, \ldots, \boldsymbol{b}_{j-1}$ of $\boldsymbol{B}$. We let matrix indices start with index 0. We let $\pi_i(\cdot)$ denote the orthogonal projection onto the linear subspace $(\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{i-1})^{\perp}$ (this depends on a matrix $\boldsymbol{B}$ that will always being clear from context). We let $v_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)} \approx \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi\,\mathrm{e}}{n}\right)^{n/2}$

denote the volume of the n-dimensional unit ball. We let the logarithm to base 2 be denoted by log and the natural logarithm be denoted by ln.

Below, we may refer to the cost or enumeration parameter k of our algorithms as a "block size".

2.1 Lattices

Let $\boldsymbol{B} \in \mathbb{Q}^{m \times n}$ be a full column rank matrix. The lattice $\mathcal{L}$ generated by $\boldsymbol{B}$ is $\mathcal{L}(\boldsymbol{B}) = \{\boldsymbol{B} \cdot \boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{Z}^n\}$ and the matrix $\boldsymbol{B}$ is called a basis of $\mathcal{L}(\boldsymbol{B})$. As soon as $n \geq 2$, any given lattice $\mathcal{L}$ admits infinitely many bases, and full column rank matrices $\boldsymbol{B}, \boldsymbol{B'} \in \mathbb{Q}^{m \times n}$ span the same lattice if and only if there exists $\boldsymbol{U} \in \mathbb{Z}^{n \times n}$ such that $\boldsymbol{B'} = \boldsymbol{B'} \cdot \boldsymbol{U}$ and $|\det(\boldsymbol{U})| = 1$. The Euclidean norm of a shortest non-zero vector in $\mathcal{L}$ is denoted by $\lambda_1(\mathcal{L})$ and called the minimum of $\mathcal{L}$. The task of finding a shortest non-zero vector of $\mathcal{L}$ from an arbitrary basis of $\mathcal{L}$ is called the Shortest Vector Problem (SVP).

We let $\boldsymbol{B}^* = (\boldsymbol{b}_0^*, \ldots, \boldsymbol{b}_{n-1}^*)$ denote the Gram–Schmidt orthogonalisation of $\boldsymbol{B}$ where $\boldsymbol{b}_i^* = \pi_i(\boldsymbol{b}_i)$. We write $\rho_{[a:b)}$ for the slope of the $\log \|\boldsymbol{b}_i^*\|$'s with $i = a, \ldots, b-1$, under a mean-squared linear interpolation. We let $\pi_i(\boldsymbol{B}_{[i:j)})$ denote the local block $(\pi_i(\boldsymbol{b}_i), \ldots, \pi_i(\boldsymbol{b}_{j-1}))$ and let $\pi_i(\mathcal{L}_{[i:j)})$ denote the lattice generated by $\pi_i(\boldsymbol{B}_{[i:j)})$. We will also write $\pi(\mathcal{L})$ if the index i and $\mathcal{L}$ are clear from the context. The volume of a lattice $\mathcal{L}$ with basis $\boldsymbol{B}$ is defined as $\mathrm{Vol}(\mathcal{L}) = \prod_{i<n} \|\boldsymbol{b}_i^*\|$; it does not depend on the choice of basis of $\mathcal{L}$. Minkowski's convex body theorem states that $\lambda_1(\mathcal{L}) \leq 2 \cdot v_n^{-1/n} \cdot \mathrm{Vol}(\mathcal{L})^{1/n}$. We define the root Hermite factor of a basis $\boldsymbol{B}$ of a lattice $\mathcal{L}$ as $\mathtt{rhf}(\boldsymbol{B}) = (\|\boldsymbol{b}_0\|/\mathrm{Vol}(\mathcal{L})^{1/n})^{1/(n-1)}$. The normalization by the $(n-1)$-th root is justified by the fact that the lattice reduction algorithms we consider in this work achieve root Hermite factors that are bounded independently of the lattice dimension n. Given as input an arbitrary basis of $\mathcal{L}$, the task of finding a non-zero vector of $\mathcal{L}$ of norm $\leq \gamma \cdot \mathrm{Vol}(\mathcal{L})^{1/n}$ is called Hermite-SVP with parameter γ (γ-HSVP).

Lattice reduction algorithms and their analyses often rely on heuristic assumptions. Let $\mathcal{L}$ be an n-dimensional lattice and $\mathcal{S}$ a measurable set in the real span of $\mathcal{L}$. The *Gaussian Heuristic* states that the number of lattice points in $\mathcal{S}$ is $|\mathcal{L} \cap \mathcal{S}| \approx \mathrm{Vol}(\mathcal{S})/\mathrm{Vol}(\mathcal{L})$. If $\mathcal{S}$ is an n-ball of radius r, then the latter is $\approx v_n \cdot r^n/\mathrm{Vol}(\mathcal{L})$. By setting $v_n \cdot r^n \approx \mathrm{Vol}(\mathcal{L})$, we see that $\lambda_1(\mathcal{L})$ is close to $\mathrm{GH}(\mathcal{L}) := v_n^{-1/n} \cdot \mathrm{Vol}(\mathcal{L})^{1/n}$. Asymptotically, we have $\mathrm{GH}(\mathcal{L}) \approx \sqrt{\frac{n}{2\pi\,\mathrm{e}}} \cdot \mathrm{Vol}(\mathcal{L})^{1/n}$.

2.2 Enumeration and Kannan's algorithm

The Enum algorithm [Kan83,FP83] is an SVP solver. It takes as input a basis matrix $\boldsymbol{B}$ of a lattice $\mathcal{L}$ and consists in enumerating all $(x_i, \ldots, x_{n-1}) \in \mathbb{Z}^{n-i}$ such that $\|\pi_i(\sum_{j \geq i} x_j \cdot \boldsymbol{b}_j)\| \leq A$ for every $i < n$, where A is an a priori upper bound on or estimate of $\lambda_1(\mathcal{L})$ (such as $\|\boldsymbol{b}_1\|$ and $\mathrm{GH}(\mathcal{L})$, respectively). It may be viewed as a depth-first search of an optimal leaf in a tree indexed by tuples $(x_i, \ldots, x_{n-1})$, where the singletons x_{n-1} lie at the top and the full tuples $(x_0, \ldots, x_{n-1})$ are the leaves. The running-time of Enum is essentially the number of tree nodes

(up to a small polynomial factor), and its space cost is polynomial. As argued in [HS07], the tree size can be estimated as $\max_{i<n}(v_i \cdot A^i / \prod_{j \geq n-i} \|\boldsymbol{b}_j^*\|)$, under the Gaussian Heuristic. In [ANS18], it was showed that a quadratic speedup can be obtained quantumly using Montanaro's quantum backtracking algorithm (and the space cost remains polynomial). We will rely on the following (classical) cost bound, derived from [HS07, Subsection 4.1]. It is obtained by optimising the tree size $\max_{i<n}(v_i \cdot A^i / \prod_{j \geq n-i} \|\boldsymbol{b}_j^*\|)$. We can replace A by twice the Gaussian Heuristic $\mathrm{GH}(\mathcal{L}) = v_n^{-1/n} \cdot \mathrm{Vol}(\mathcal{L})^{1/n}$, where $\mathrm{Vol}(\mathcal{L}) = \prod_{j<n} \|\boldsymbol{b}_j^*\|$. By using the bounds $\|\boldsymbol{b}_i^*\| \in c \cdot \delta^{-i} \cdot [1/2, 2]$, this optimisation problem boils down to maximising $\delta^{ni/2 - i^2/2}$ for $i < n$. The maximum is $\delta^{n^2/8}$ (for $i = n/2$). The other terms are absorbed in the $2^{O(n)}$ factor.

Theorem 1. *Let $\boldsymbol{B}$ be a basis matrix of an n-dimensional rational lattice $\mathcal{L}$. Assume that there exist $c > 0$ and $\delta > 1$ such that $\|\boldsymbol{b}_i^*\| \in c \cdot \delta^{-i} \cdot [1/2, 2]$, for all $i < n$. Then, given $\boldsymbol{B}$ as input (with $A = 2 \cdot v_n^{-1/n} \cdot \mathrm{Vol}(\mathcal{L})^{1/n}$), the* Enum *algorithm returns a shortest non-zero vector of $\mathcal{L}$ within $\delta^{\frac{n^2}{8}} \cdot 2^{O(n)} \cdot \mathrm{poly}(\mathrm{size}(\boldsymbol{B}))$ bit operations. Its space cost is $\mathrm{poly}(\mathrm{size}(\boldsymbol{B}))$.*

Kannan's algorithm [Kan83] relies on recursive calls to Enum to improve the quality of the Gram–Schmidt orthogonalisation of $\boldsymbol{B}$, so that calling Enum on the preprocessed $\boldsymbol{B}$ is less expensive. Its cost bound was lowered in [HS07] and that cost upper bound was later showed to be sharp in the worst case, up to lower-order terms [HS08].

Theorem 2. *Let $\boldsymbol{B}$ be a basis matrix of an n-dimensional rational lattice $\mathcal{L}$. Given $\boldsymbol{B}$ as input, Kannan's algorithm returns a shortest non-zero vector of $\mathcal{L}$ within $n^{\frac{n}{2\mathrm{e}}(1+o(1))} \cdot \mathrm{poly}(\mathrm{size}(\boldsymbol{B}))$ bit operations. Its space cost is $\mathrm{poly}(\mathrm{size}(\boldsymbol{B}))$.*

In practice, enumeration is accelerated using two main techniques. The first one, inspired from Kannan's algorithm, consists in preprocessing the basis with a strong lattice reduction algorithm, such as BKZ (see next subsection). Note that BKZ uses an SVP solver in a lower dimension, so these algorithms can be viewed as calling themselves recursively, in an intertwined manner. The second one is tree pruning [SE94,GNR10]. The justifying observation is that some tree nodes are much more unlikely than others to have leaves in their subtrees, and are hence discarded. More concretely, one considers the strengthened conditioned $\|\pi_i(\sum_{j \geq i} x_j \cdot \boldsymbol{b}_j)\| \leq t_i \cdot A$, for some pruning coefficients $t_i \in (0, 1)$. These coefficients can be used to extract a refined estimated enumeration cost as well as an estimated success probability (see, e.g. [Che13, Sec. 3.3]). By making the probability extremely small, the cost-over-probability ratio can be lowered and the probability can be boosted by re-randomising the basis and repeating the pruned enumeration. This strategy is called extreme pruning [GNR10].

2.3 Lattice reduction

Given a basis matrix $\boldsymbol{B} \in \mathbb{Q}^{m \times n}$ of a lattice $\mathcal{L}$, the LLL algorithm [LLJL82] outputs in polynomial time a basis $\boldsymbol{C}$ of $\mathcal{L}$ whose Gram–Schmidt norms cannot

decrease too fast: $\|\boldsymbol{c}_i^*\| \geq \|\boldsymbol{c}_{i-1}^*\|/2$ for every $i < n$. In particular, we have $\mathtt{rhf}(\boldsymbol{C}) \leq 2$. A lattice basis $\boldsymbol{B}$ is size-reduced if it satisfies $|\mu_{i,j}| \leq 1/2$ for $j < i < n$ where $\mu_{i,j} = \langle \boldsymbol{b}_i, \boldsymbol{b}_j^* \rangle / \langle \boldsymbol{b}_j^*, \boldsymbol{b}_j^* \rangle$. A lattice basis $\boldsymbol{B}$ is HKZ-reduced if it is size-reduced and satisfies $\|\boldsymbol{b}_i^*\| = \lambda_1(\pi_i(\mathcal{L}_{[i:n)}))$, for all $i < n$. A basis $\boldsymbol{B}$ is BKZ-k reduced for block size $k \geq 2$ if it is size-reduced and further satisfies $\|\boldsymbol{b}_i^*\| = \lambda_1(\pi_i(\mathcal{L}_{[i:\min(i+k,n))}))$, for all $i < n$.

The Schnorr-Euchner BKZ algorithm [SE94] is the lattice reduction algorithm that is commonly used in practice, to obtain bases of better quality than those output by LLL (there exist algorithms that admit better analyses, such as [GN08a,MW16,ALNS19], but BKZ remains the best in terms of practical performance reported in the current literature). BKZ inputs a block size k and a basis matrix $\boldsymbol{B}$ of a lattice $\mathcal{L}$, and outputs a basis which is "close" to being BKZ-k reduced, up to algorithm parameters. The BKZ algorithm calls an SVP solver in dimensions $\leq k$ on projected sublattices of the working basis of an n-dimensional input lattice. A BKZ sweep consists in SVP solver calls for $\pi_i(\mathcal{L}_{[i:\min(i+k,n))})$ for i from 0 to $n-2$. BKZ proceeds by repeating such sweeps, and typically a small number of sweeps suffices. At each execution of the SVP solver, if we have $\lambda_1(\pi_i(\mathcal{L}_{[i:\min(i+k,n))})) < \delta \cdot \|\boldsymbol{b}_i^*\|$ where $\delta < 1$ is a relaxing parameter that is close to 1, then BKZ updates the block $\pi_i(\boldsymbol{B}_{[i:\min(i+k,n))})$ by inserting the vector found by the SVP solver at index i. It then removes the created linear dependency, e.g. using a gcd computation (see, e.g. [GN08a]). Whether there was an insertion or not, BKZ finally calls LLL on the local block $\pi_i\left(\boldsymbol{B}_{[i:\min(i+k,n))}\right)$. The procedure terminates when no change occurs at all during a sweep or after certain termination condition is fulfilled. The higher k, the better the BKZ output quality, but the higher the cost: for large n, BKZ achieves root Hermite factor essentially $k^{1/(2k)}$ (see [HPS11]) using an SVP-solver in dimensions $\leq k$ a polynomially bounded number of times.

Schnorr [Sch03] introduced a heuristic on the shape of the Gram–Schmidt norms of BKZ-reduced bases, called the *Geometric Series Assumption* (GSA). The GSA asserts that the Gram–Schmidt norms $\{\|\boldsymbol{b}_i^*\|\}_{i<n}$ of a BKZ-reduced basis behave as a geometric series, i.e., there exists $r > 1$ such that $\|\boldsymbol{b}_i^*\|/\|\boldsymbol{b}_{i+1}^*\| \approx r$ for all $i < n-1$. In this situation, the root Hermite factor is $\sqrt{r}$. It was experimentally observed [CN11] that the GSA is a good first approximation to the shape of the Gram–Schmidt norms of BKZ. However, as observed in [CN11] and studied in [YD17], the GSA does not provide an exact fit to the experiments of BKZ for the last k indices; similarly, as observed in [YD17] and studied in [BSW18], the GSA also does not fit for the very few first indices (the latter phenomenon seems to vanish for large k, as opposed to the former).

We will use the self-dual BKZ algorithm (SDBKZ) from [MW16]. SDBKZ proceeds similarly to BKZ, except that it intertwines forward and backward sweeps (for choosing the inputs to the SVP solver), whereas BKZ uses only forward sweeps. Further, it only invokes the SVP solver in dimension exactly k, so that a forward sweep consists in considering $\pi_i(\mathcal{L}_{[i:i+k)})$ for i from 0 to $n-k$ and a backward sweep consists in considering (the duals of) $\pi_i(\mathcal{L}_{[i:i+k)})$ for i from $n-k$ down to 0. We assume that the final sweep is a forward sweep. We use

SDBKZ in the theoretical analysis rather than BKZ because, under the Gaussian Heuristic and after polynomially many sweeps, the first $n-k$ Gram–Schmidt norms of the basis (almost) decrease geometrically, i.e. satisfy the GSA. This may not be necessary for our result to hold, but this simplifies the computations significantly. We adapt [MW16] by allowing SDBKZ to rely on a γ-HSVP solver $\mathcal{O}$ rather than on an exact SVP solver (which in particular is a $\sqrt{k}$-HSVP solver). We let SDBKZ$^{\mathcal{O}}$ denote the modified algorithm. The analysis of [MW16] can be readily adapted. We will rely on the following heuristic assumption, which extends the Gaussian Heuristic.

Heuristic 1 *Let $\mathcal{O}$ be a γ-*HSVP *solver in dimension k. During the* SDBKZ$^{\mathcal{O}}$ *execution, each call to $\mathcal{O}$ for a projected k-dimensional sublattice $\pi(\mathcal{L})$ of the input lattice $\mathcal{L}$ returns a vector of norm $\approx \gamma \cdot (\mathrm{Vol}(\pi(\mathcal{L})))^{\frac{1}{k}}$.*

The SDBKZ$^{\mathcal{O}}$ algorithm makes the Gram–Schmidt norms converge to a fix-point, very fast in terms of the number of HSVP calls [MW16, Subsection 4.2]. That fix-point is described in [MW16, Corollary 2]. Adapting these results leads to the following.

Theorem 3 (Under Heuristic 1). *Let $\mathcal{O}$ be a γ-*HSVP *solver in dimension k. Given as input a basis of an n-dimensional rational lattice $\mathcal{L}$,* SDBKZ$^{\mathcal{O}}$ *outputs a basis $\boldsymbol{B}$ of $\mathcal{L}$ such that, for all $i < n-k$, we have*

$$\|\boldsymbol{b}_i^*\| \approx \gamma^{\frac{n-1-2i}{k-1}} \cdot (\mathrm{Vol}\,\mathcal{L})^{\frac{1}{n}}.$$

The number of calls to $\mathcal{O}$ is $\leq \mathrm{poly}(n)$ and the bit-size of the output basis is $\leq \mathrm{poly}(\mathrm{size}(\boldsymbol{B}))$.

2.4 Simulating lattice reduction

To understand the behaviour of lattice reduction algorithms in practice, a useful approach is to conduct simulations. The underlying idea is to model the practical behaviour of the evolution of the Gram–Schmidt norms during the algorithm execution, without running a costly lattice reduction. Note that this requires only the Gram–Schmidt norms and not the full basis. Chen and Nguyen first provided a BKZ simulator [CN11] based on the Gaussian Heuristic and with an experiment-driven modification for the blocks at the end of the basis. It relies on the assumption that each SVP solver call in the projected blocks (except the ones at the end of the basis) finds a vector whose norm corresponds to the Gaussian Heuristic applied to that local block. The remaining Gram–Schmidt norms of the block are updated to keep the determinant of the block constant. (Note that in the original [CN11] simulator, these Gram–Schmidt norms are not updated to keep the determinant of the block constant, but are adjusted at the end of the sweep to keep the global determinant constant; our variant helps for taking enumeration costs into account.)

We extend this simulator in two ways: first, we adapt it to estimate the cost and not only the evolution of the Gram–Schmidt norms; second, we adapt it to

other reduction algorithms, such as SDBKZ. To estimate the cost, we use the estimates of the full enumeration cost, or the estimated cost of an enumeration with (extreme) pruning. The full enumeration cost estimate is used in Section 3 to model our first algorithm for which we can heuristically analyse the quality/cost trade-off. The pruned enumeration cost estimate is used in Section 4, which aims to provide a more precise study for practical and cryptographic dimensions. To find the enumeration cost with pruning, we make use of `FPyLLL`'s `pruning` module which numerically optimises pruning parameters for a time/success probability trade-off using a gradient descent.

In small block sizes, the enumeration cost is dominated by calls to LLL. In our code, we simply assume that one LLL call in dimension k costs the equivalent of visiting k^3 nodes. This is an oversimplification but avoids completely ignoring this polynomial factor. We will compare our concrete estimates with empirical evidence from timing experiments with the implementation in `FPLLL`, to measure the effect of this imprecision. This assumption enables us to bootstrap our cost estimates. BKZ in block size up to, say, 40 only requires LLL preprocessing, allowing us to estimate the cost of preprocessing with block size up to 40, which in turn enables us to estimate the cost (including preprocessing) for larger block sizes etc. To extend the simulation to SDBKZ, we simply run the simulation on the Gram–Schmidt norms of the dual basis $1/\|\boldsymbol{b}_n^*\|, \ldots, 1/\|\boldsymbol{b}_1^*\|$. Our simulation source code is available as `simu.py`, as an attachment to the electronic version of the full version of this work.

We give pseudocode for our costed simulation in Algorithm 1. For BKZ simulation, we call Algorithm 1 with $d = k$, $c = 0$ and with $\mathrm{tail}(x, y, z)$ simply outputting x. For our simulations we prepared Gram–Schmidt shapes for LLL-reduced lattices in increasing dimensions d on which we then estimate the cost of running the algorithm in question for increasingly heavy preprocessing parameters k', selecting the least expensive one. In our search, we initialise $c_2 = 2^3$ and then iteratively compute c_{j+1} given $c_2, \ldots, c_j$. When we instantiate Algorithm 1 we either manually pick some small t (Section 4) or pick $t = \infty$ (Section 3.3) which means to run the algorithm until no more changes are made to the basis.

2.5 State-of-the-art enumeration-based SVP solving in practice

To the best of our knowledge, there is no extrapolated running-time for state-of-the-art lattice reduction implementations. Furthermore, the simulation data in [CN11,Che13] is only available up to a block size of 250. The purpose of this section is to fill this gap by providing extended simulations (and the source code used to produce them) and by reporting running times using the state-of-the-art `FPyLLL` [dt19b] and `FPLLL` [dt19a] libraries.

First, in Figure 2 we reproduce the data from [Che13, Table 5.2] for the estimated cost of solving SVP up to dimension 250, using enumeration.

We then also computed the expected cost (expressed as the number of visited enumeration nodes) up to dimension 500 for Figure 2, see `cost.py`, attached to the electronic copy of the full version of this work, and Algorithm 1. We note that the preprocessing strategy adopted in our code is to always run two

Algorithm 1: Costed simulation algorithm

Data: Gram–Schmidt profile $\ell_i = \log \|\boldsymbol{b}_i^*\|$ for $i = 0, \ldots, d-1$.
Data: Block size $k \geq 2$.
Data: Preprocessing block size $k' \geq 2$.
Data: Preprocessing sweep count t.
Data: Overshooting parameter $c \geq 0$.
Data: Configuration `flags`.
Data: Cost estimates c_j for solving (approx-)SVP in dimensions $j = 2, \ldots, k'$, including preprocessing cost estimates.
Result: Cost estimate for (approx-)SVP in dimension k.

if SDBKZ *flag is set in* `flags` **then**
  $(\ell_i)_i \leftarrow$ output of [CN11] style simulator for SDBKZ on $(\ell_i)_i$ for block size k' and $\leq t$ sweeps;
else
  $(\ell_i)_i \leftarrow$ output of [CN11] style simulator for BKZ on $(\ell_i)_i$ for block size k' and $\leq t$ sweeps;
end
`// account for early termination`
$t \leftarrow$ number of preprocessing sweeps actually performed;
$C_p \leftarrow d^3$; `// (estimated) cost of LLL`
for $0 \leq i < d-1$ **do**
  $k^* \leftarrow \text{tail}(\min(k', d-i), c, d-i)$;
  $C_p \leftarrow C_p + t \cdot c_{k^*}$;
end
if *full enumeration cost flag is set in* `flags` **then**
  $C_e \leftarrow$ full enumeration cost for $\ell_0, \ldots, \ell_{k-1}$;
  $p_e \leftarrow 1$;
else
  $(t_i)_{i<k} \leftarrow$ optimised pruning coefficients for $(\ell_i)_{i<k}$ and preprocessing cost C_p;
  $C_e, p_e \leftarrow$ pruned enumeration cost and success probability, given $(t_i)_{i<k}$;
end
$C \leftarrow 1/p_e \cdot (C_p + C_e)$;
return C;

sweeps of preprocessing but that preprocessing proceeds recursively, e.g. preprocessing block size 80 with block size 60 may trigger a preprocessing with block size 40, if previously we found that preprocessing to be most efficient for solving SVP-60, as outlined above. This approach matches that of the `FPLLL/FPyLLL strategizer` [dt17] which selects the default preprocessing and pruning strategies used in `FPLLL/FPyLLL`. Thus, the simulation approach resembles that of the actual implementation.

In Figure 2, we also fitted the coefficients a_1, a_2 of $1/(2\,\mathrm{e})n\,\log n + a_1 \cdot n + a_2$ to dimensions n from 150 to 249.[7]

Furthermore, we plot the chosen preprocessing block sizes and success probability of a single enumeration (`FPLLL` uses extreme pruning) in Figure 3. This highlights that, even in dimension 500, preprocessing is still well below the $n - o(n)$ required for Kannan's algorithm [Kan83,MW15].

Fig. 2: Expected number of nodes visited during enumeration in dimension n.

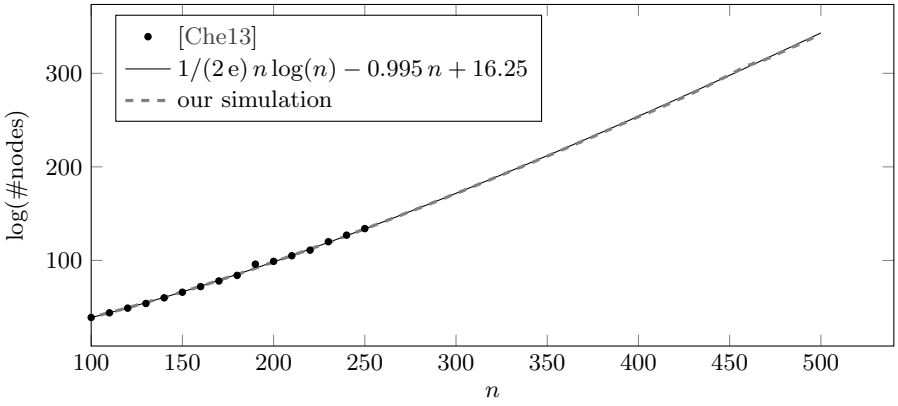


Figure 4 plots the running-times of `FPLLL` in terms of enumeration nodes, timed using `call.py`, available as an attachment to the electronic version of the full version of this work. Concretely, running-time in seconds is first converted to CPU cycles by multiplying with the clock speed 2.6 GHz[8] and we then convert from cycles to nodes by assuming visiting a node takes about 64 clock cycles.[9] Figure 4 illustrates that our simulation is reasonably accurate. We note that for

[7] Throughout this work, we fit curves to simulation data. For this, we use SciPy's `scipy.optimize.curve_fit` function [VGO+20] which implements a non-linear least-square fit. To prevent overfitting, we err on the side of fewer parameters and fit on a subset of the available data, using the remaining data to check the accuracy of the fit.

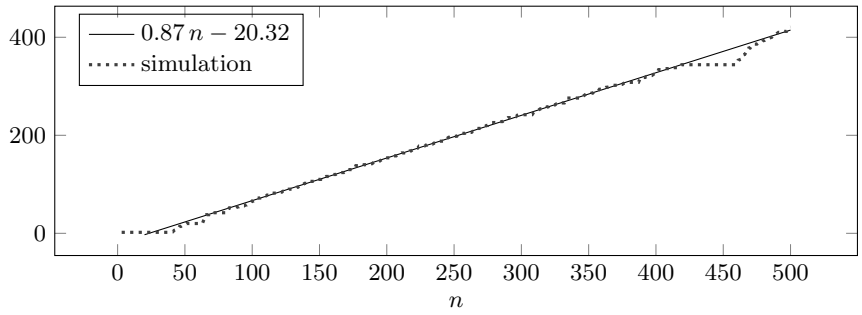
[8] CPU: Intel(R) Xeon(R) CPU E5-2690 v4 @ 2.60GHz, machine: "atomkohle".

[9] We note that [CN11] mentions 200 cycles per node, whereas [dt17]'s `set_mdc.py` reports 64 cycles per node on our test machine in dimension 55.

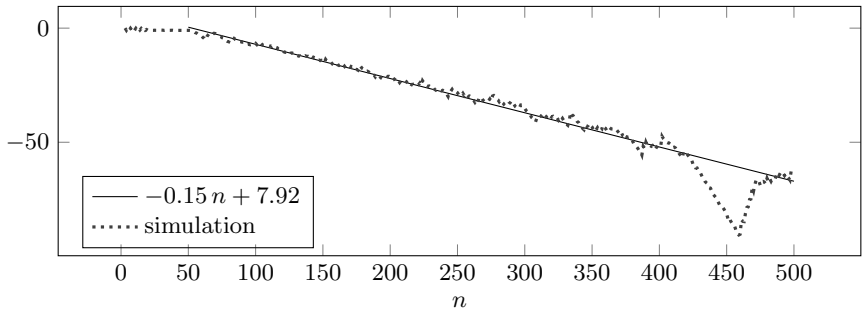
running the timing experiments with FPLLL we relied on FPLLL's own (recursive call and pruning) strategies, not those produced by our simulator.

Fig. 3: Reduction strategies used for Figure 2.

(a) Preprocessing block sizes used in our simulations.



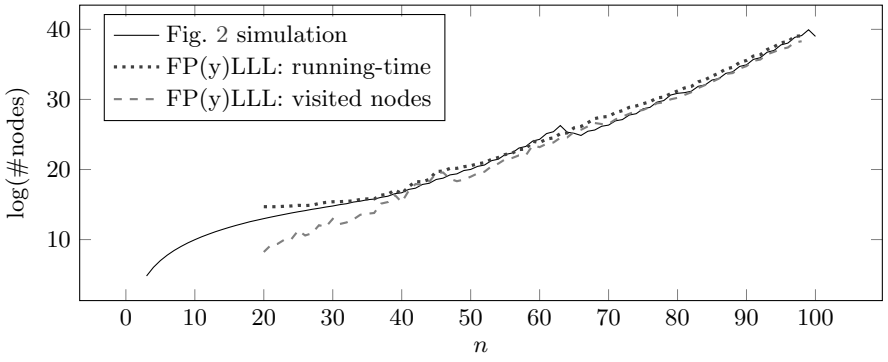
(b) Success probability of a single enumeration (in log scale).



The largest computational results known for finding short vectors in unstructured lattices is the Darmstadt SVP Challenge [SG10]. This challenge asks contestants to find a vector at most 1.05 times larger than the Gaussian Heuristic. Thus, the challenge does not require to solve SVP exactly but the easier $(0.254\sqrt{n})$-HSVP problem. The strategy we used for SVP can be adapted to this problem as well, see `chal.py`, attached to the electronic version of the full version of document. To validate our simulation methodology against this data, we compare our estimates with various entries from the Hall of Fame for [SG10] and the literature in Figure 5.

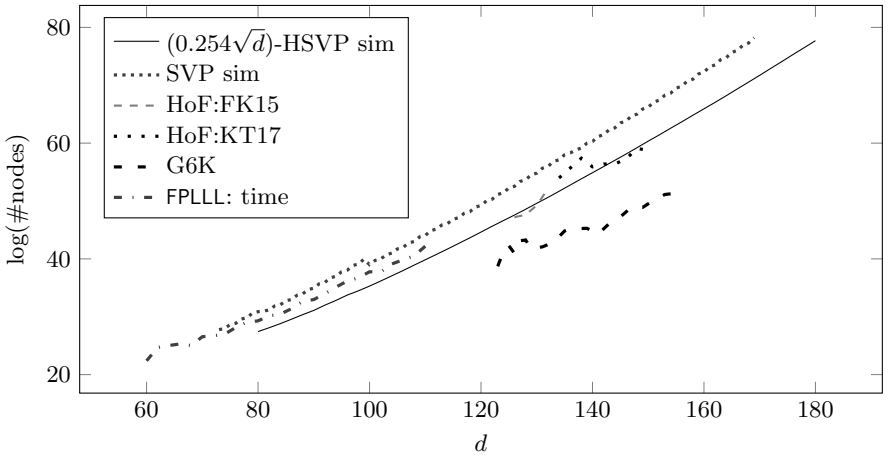
We conclude this section by interpreting our simulation results in the context of BKZ. The quality output by BKZ in practice has been studied in the literature [GN08b,Che13,AGVW17,YD17,BSW18]. Thus, our simulations imply that the running time of BKZ as implemented in [dt19a] achieves root Hermite factor $k^{1/(2k)}$ is bounded by $k^{k/(2\,\mathrm{e})+o(k)}$. Indeed, this bound is tight, i.e. BKZ does not achieve a lower running time. To see this, consider the sandpile model of BKZ's behaviour [HPS11]. It implies that even if we start with a GSA line, this line from

Fig. 4: Number of nodes visited during enumeration in dimension n.



In our simulations, the estimate for the number of visited nodes includes the cost of LLL (expressed as a number of nodes), whereas actually "visited nodes" does not. The "running-time" (converted from seconds to a number of nodes), on the other hand, contains all operations as it literally is the cputime.

Fig. 5: Darmstadt SVP Challenge.



"HoF" stands for Hall of Fame [SG10]. Core hours are translated to #nodes by multiplying by $3600 \cdot 2 \cdot 10^7$, which assumes each core has a 2Ghz CPU and that one enumeration node costs 64 clock cycles to process. Except for G6K [ADH+19] which is a sieving implementation, all entries are for variants of lattice-point enumeration. We translate G6K timings to #nodes in the same way as for other timings, ignoring that it is not an enumeration implementation. In other words, #nodes is merely a unit of time here.

index i onward deteriorates as we perform updates on indices $< i$. Furthermore, extreme pruning, which involves rerandomising local blocks, destroys the GSA shape. Thus, we can conclude that in practice BKZ

- achieves root Hermite factor $\approx (\frac{k}{2\pi\,\mathrm{e}} \cdot (\pi\,k)^{\frac{1}{k}})^{\frac{1}{2(k-1)}}$ [Che13]
- in time $\mathrm{poly}(d) \cdot 2^{1/(2\,\mathrm{e})\,k\log k - 0.995\,k + 16.25} \approx \mathrm{poly}(d) \cdot 2^{1/(2\,\mathrm{e})\,k\log k - k + 16}$

where the unit of time is the number of nodes visited during enumeration. We note that a similar conclusion was already drawn in [APS15] and discussed in [ABD+16]. However, that conclusion was drawn for the unpublished implementation and limited data in [Che13].

3 Reaching root Hermite factor $k^{\frac{1}{2k}(1+o(1))}$ in time $k^{\frac{k}{8}}$

This section contains our main contribution: a lattice reduction algorithm that achieves root Hermite factor $k^{\frac{1}{2k}(1+o(1))}$ in time $k^{\frac{k}{8}}$. We start by a quality running-time trade-off boosting theorem, based on SDBKZ. We then give and analyze the main algorithm, FastEnum, and finally propose a simulator for that algorithm.

3.1 A boosting theorem

We first show that SDBKZ allows to obtain a reduction from a γ'-HSVP solver in dimension n' to a γ-HSVP solver in dimension n achieving a larger root Hermite factor. This reduction is not polynomial-time, but we will later aim at making it no more costly than the cost of our γ-HSVP solver.

Theorem 4 (Under Heuristic 1). *Let $\mathcal{O}$ be a γ-HSVP solver in dimension n. Assume we are given as input a basis $\mathbf{B}$ of an n'-dimensional lattice $\mathcal{L}$, with $n' > n$. We first call* SDBKZ$^{\mathcal{O}}$ *on $\mathbf{B}$: let $\mathbf{C}$ denote the output basis. Then we call the* Enum *algorithm on the sublattice basis made of the first $n'-n$ vectors of $\mathbf{C}$. This provides a γ'-HSVP solver in dimension n', with*

$$\gamma' \leq \sqrt{n'-n}\,\gamma^{\frac{n}{n-1}}.$$

The total cost is bounded by $\mathrm{poly}(n')$ *calls to $\mathcal{O}$ and* $\gamma^{\frac{(n'-n)^2}{4(n-1)}} \cdot 2^{O(n'-n)} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$ *bit operations.*

Proof. By Theorem 3, we have $\|\mathbf{c}_i^*\| \in \gamma^{\frac{n'-1-2i}{n-1}} \cdot (\mathrm{Vol}(\mathcal{L}))^{\frac{1}{n'}} \cdot [1/2, 2]$, for all $i < n'-n$. Also, the number of calls to $\mathcal{O}$ is $\leq \mathrm{poly}(n')$ and the bit-size of $\mathbf{C}$ is $\leq \mathrm{poly}(\mathrm{size}(\mathbf{B}))$.

By Theorem 1 (with "$\delta = \gamma^{\frac{2}{n-1}}$"), the cost of the call to Enum is bounded as $\gamma^{\frac{(n'-n)^2}{4(n-1)}} \cdot 2^{O(n'-n)} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{C}))$, which, by the above is $\leq \gamma^{\frac{(n'-n)^2}{4(n-1)}} \cdot 2^{O(n'-n)} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$. Further, by Minkowski's theorem, the vector output by Enum has norm bounded from above by:

$$\sqrt{n'-n} \cdot \prod_{i=0}^{n'-n-1} \left(\gamma^{\frac{n'-1-2i}{n-1}} (\mathrm{Vol}(\mathcal{L}))^{\frac{1}{n'}}\right)^{\frac{1}{n'-n}} = \sqrt{n'-n} \cdot \gamma^{\frac{n}{n-1}} \cdot (\mathrm{Vol}(\mathcal{L}))^{\frac{1}{n'}}.$$

This completes the proof of the theorem. □

Note that the result is not interesting if $n' - n$ is chosen too small, as such a choice results in an increased root Hermite factor. Also, if $n' - n$ is chosen too large, then the cost grows very fast. We consider the following instructive application of Theorem 4. By Theorem 2, Kannan's algorithm finds a shortest non-zero of $\mathcal{L}$ in time $n^{\frac{n}{2\mathrm{e}}(1+o(1))} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$, when given as input a basis $\mathbf{B}$ of an n-dimensional lattice $\mathcal{L}$. In particular, it solves γ-HSVP with $\gamma = \sqrt{n}$ and provides a root Hermite factor $\leq n^{\frac{1}{2n}}$. We want to achieve a similar root Hermite factor, but for a lower cost. Now, for a cost parameter k, we would like to restrict the cost to $k^{\frac{k}{8}} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$ (ideally, while still achieving root Hermite factor $k^{\frac{1}{2k}}$). We hence choose an integer $k_0 := \frac{\mathrm{e}}{4}(1 + o(1))k$. This indeed provides a cost bounded as $k^{\frac{k}{8}} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$, but this only solves γ_0-HSVP with $\gamma_0 = \Theta(\sqrt{k_0})$ in dimension k_0, i.e. only provides a root Hermite factor $\approx \sqrt{k_0}^{\frac{1}{k_0}} = k^{\frac{2}{k\,\mathrm{e}}(1+o(1))} \approx k^{\frac{0.74}{k}}$, which is much more than $k^{\frac{1}{2k}}$. So far, we have not done anything but a change of variable. Now, let us see how Theorem 4 can help. We use it with $\mathcal{O}$ being Kannan's algorithm in dimension "$n = k_0$". We set "$n' = k_1$" with $k_1 = k_0 + \lceil\sqrt{k_0 k}\rceil$. This value is chosen so that the total cost bound of Theorem 4 remains $k^{\frac{k}{8}} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$. The achieved root Hermite factor is $\leq k^{\frac{1}{k(\mathrm{e}/4+\sqrt{\mathrm{e}/4})}(1+o(1))} \approx k^{\frac{0.66}{k}}$. Overall, for a similar cost bound, we have decreased the achieved root Hermite factor.

3.2 The FastEnum Algorithm

We iterate the process above to obtain the FastEnum algorithm, described in Algorithm 2. For this reason, we define $k_0 = x_0 \cdot k$ with $x_0 = \frac{\mathrm{e}}{4}(1 + o(1))$ and, for all $i \geq 1$:

$$k_i = \lceil x_i \cdot k \rceil \quad \text{with} \quad x_i = x_{i-1} + \sqrt{\frac{x_{i-1}}{i}}. \tag{1}$$

We first study the sequence of x_i's.

Lemma 1. *We have $i + 1 - \sqrt{i} < x_i < i + 1$ for all $i \geq 1$.*

Proof. The upper bound can be readily proved using an induction based on (1). It may be numerically checked that the lower bound holds for $i \in \{1, 2, 3\}$. We show by induction to prove that $1 - \frac{x_i}{i} < \frac{1}{\sqrt{i}} - \frac{2}{i}$ for $i \geq 4$, which is a stronger statement. It may be numerically checked that the latter holds for $i = 4$. Now, assume it holds for some $i - 1 \geq 4$ and that we aim at proving it for i. We have

$$\begin{aligned} 1 - \frac{x_i}{i} =& \frac{1}{i}\left((i-1)\Big(1 - \frac{x_{i-1}}{i-1}\Big) + \Big(1 - \sqrt{\frac{x_{i-1}}{i}}\Big)\right) \\ =& \frac{1}{i}\left((i-1)\Big(1 - \frac{x_{i-1}}{i-1}\Big) + \sqrt{\frac{i-1}{i}}\Big(1 - \sqrt{\frac{x_{i-1}}{i-1}}\Big) + 1 - \sqrt{\frac{i-1}{i}}\right). \end{aligned}$$

Now, note that $\sqrt{\frac{x_{i-1}}{i-1}} > 0.2$ (using our induction hypothesis). Using the bound $1-\sqrt{t} < \frac{1}{2}(1-t)+\frac{1}{4}(1-t)^2$ which holds for all $t > 0.2$, we can bound $1-\frac{x_i}{i}$ from above by:

$$\frac{1}{i}\left((i-1)\Big(1-\frac{x_{i-1}}{i-1}\Big)+1+\sqrt{\frac{i-1}{i}}\Big(-1+\frac{1}{2}\Big(1-\frac{x_{i-1}}{i-1}\Big)+\frac{1}{4}\Big(1-\frac{x_{i-1}}{i-1}\Big)^2\Big)\right).$$

It now suffices to observe that the right hand side is smaller than $\frac{1}{\sqrt{i}}-\frac{2}{i}$, when $1-\frac{x_{i-1}}{i-1}$ is replaced by $\frac{1}{\sqrt{i-1}}-\frac{2}{i-1}$. This may be checked with a computer algebra software. □

The FastEnum algorithm (Algorithm 2) consists in calling the process described in Theorem 4 several times, to improve the root Hermite factor while staying within a $k^{\frac{k}{8}}$ cost bound.

Algorithm 2: The FastEnum algorithm.

Data: A cost parameter k and a level $i \geq 0$.
Data: A basis matrix $\mathbf{B} \in \mathbb{Q}^{k_i \times k_i}$, with k_i defined as in (1).
Result: A short non-zero vector of $\mathcal{L}(\boldsymbol{B})$.

```
if i = 0 then
    b ← output of Kannan's enumeration algorithm on B;
else
    C ← output of SDBKZ^O on B with O being FastEnum for i − 1;
    b ← Enum(C_[0:k_i − k_{i−1}]) with k_{i−1} defined as in (1);
end
return b;
```

Theorem 5 (Under Heuristic 1). *Let $k \geq 4$ tending to infinity, and $i \leq 2^{o(k)}$.*[10] *The* FastEnum *algorithm with parameters k and i solves γ_i-HSVP in dimension k_i, with $\gamma_i \leq k^{\frac{i+1}{2}(1+o(1))}$. For $i \geq 1$, the corresponding root Hermite factor is below $k^{\frac{i+1}{2(i+1-\sqrt{i})k}(1+o(1))}$. Further,* FastEnum *runs in time $k^{\frac{k}{8}(1+o(1))+i\cdot O(1)} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$.*

For constant values of i (as a function of k), the root Hermite factor is not quite $k^{\frac{1}{2k}(1+o(1))}$, but it is so for any choice of $i = \omega(1)$. For i satisfying both $i = \omega(1)$ and $i = o(k)$, FastEnum reaches a root Hermite factor $k^{\frac{1}{2k}(1+o(1))}$ in time $k^{\frac{k}{8}(1+o(1))} \cdot \mathrm{poly}(\mathrm{size}(\mathbf{B}))$.

Proof. For $\gamma_0 = \Theta(\sqrt{k})$ and, by Theorem 4, we have $\gamma_i \leq \sqrt{k_i - k_{i-1}} \cdot \gamma_{i-1}^{\frac{k_{i-1}}{k_{i-1}-1}}$ for all $i \geq 1$. Using the definition of the k_i's and the bounds of Lemma 1, we

[10] We stress that in this theorem, all asymptotic notations are with respect to k only.

obtain, for $i \geq 1$:

$$\gamma_i \leq \left(1 + \frac{x_{i-1}}{i}k^2\right)^{1/4} \cdot \gamma_{i-1}^{\frac{k(i-\sqrt{i-1})+1}{k(i-\sqrt{i-1})-1}} \leq \sqrt{2k} \cdot \gamma_{i-1}^{1+\frac{2}{ki/2-1}}.$$

Using $k \geq 4$, we see that the latter is $\leq \sqrt{2k}\gamma_{i-1}^{1+\frac{8}{ki}}$. By unfolding the recursion, we get, for $i \geq 1$:

$$\gamma_i \leq \sqrt{2k}^{1+\sum_{j=0}^{i-1}\prod_{\ell=j}^{i-1}(1+\frac{8}{k(\ell+1)})} \cdot \gamma_0^{\prod_{\ell=0}^{i-1}(1+\frac{8}{k(\ell+1)})}.$$

Now, note that we have (using the bound $\sum_{\ell=0}^{i-1}\frac{1}{\ell+1} \leq \ln(i)+1$, and the inequalities $1+x \leq \exp(x) \leq 1+2x$ for $x \in [0,1]$)

$$\prod_{\ell=j}^{i-1}(1+\frac{8}{k(\ell+1)}) \leq \exp\left(\sum_{\ell=j}^{i-1}\frac{8}{k(\ell+1)}\right) \leq \exp\left(\frac{8}{k}(\ln(i)+1)\right) \leq 1+\frac{16}{k}(\ln(i)+1).$$

As $i \leq 2^{o(k)}$, the latter is $\leq 1+o(1)$. Overall, this gives $\gamma_i \leq k^{\frac{i+1}{2}(1+o(1))}$. The claim on the root Hermite factor follows from the lower bound of Lemma 1.

We now consider the run-time of the algorithm, and in particular the term $\gamma_{i-1}^{\frac{(k_i-k_{i-1})^2}{4(k_{i-1}-1)}} \cdot 2^{O(k_i-k_{i-1})}$ from Theorem 4. Recall that by definition of the k_i's, we have $k_i - k_{i-1} \leq 1 + \sqrt{\frac{k_{i-1}k}{i}}$. Using the upper bound of Lemma 1, we obtain that $k_i - k_{i-1} \leq O(k)$, and hence that $2^{O(k_i-k_{i-1})} \leq 2^{O(k)}$. We also have

$$\gamma_{i-1}^{\frac{(k_i-k_{i-1})^2}{4(k_{i-1}-1)}} \leq k^{\frac{i}{2}(1+o(1))\frac{k_{i-1}k}{4i(k_{i-1}-1)}} \leq k^{\frac{k}{8}(1+o(1))}.$$

Further, the number of recursive calls is bounded as $\text{poly}(\prod_{j\leq i}k_j)$. By Lemma 1, this is $\leq k^{i\cdot O(1)}$. To complete the proof, it may be shown using standard techniques that all bases occurring during the algorithm have bit-sizes bounded as $\text{poly}(\text{size}(\mathbf{B}))$ (where the bound is independent from i). □

3.3 Simulation of asymptotic behaviour

In this subsection, we instantiate the FastEnum algorithm as described in Algorithm 2 and confirm its asymptotic behaviour via simulations. Note that the FastEnum algorithm requires SDBKZ subroutines. To simulate this subroutine, we use the costed simulation of Algorithm 1 with flags: SDBKZ and full enumeration cost. We also omit the cost of LLL in the simulation as the enumeration cost dominates in the parameter range considered in this subsection.

To compare the simulation with the theorems, we consider two scenarios. In the first one, called "Theoretical" we numerically compute the k_i's, γ_i's and the slope of the Gram–Schmidt log-norms of the enumeration block (i.e. the first $k_i - k_{i-1}$ vectors) according to Theorem 5. Here the index i denotes the recursion level. Similarly, k_i and γ_i are defined in the same way as in (1) and Theorem 5,

respectively. In the second one, called "Simulated" we still set the k_i's according to (1). However, at the i-th level, we first run an SDBKZ simulation on a lattice of dimension k_i, using the γ_{i-1}-HSVP (simulated) oracle from the previous level. Here, the Hermite factor γ_{i-1} is computed from the simulated basis at the $(i-1)$-th level. The initial γ_0 is computed from a simulated HKZ-reduced basis of dimension k_0. During the SDBKZ simulation, for each HSVP call, we assume that the same Hermite factor γ_{i-1} is achieved. We let the simulated SDBKZ run until no change occurs to the basis or if it has already achieved the theoretical root Hermite factor at the same level, as guided by the proof of Theorem 5. After the simulated SDBKZ preprocessing, we simulate an enumeration in the first block of dimension $k_i - k_{i-1}$. The enumeration cost is estimated using the full enumeration cost model (see Section 2.4), since here we are only interested in the asymptotic behaviour (we defer to Section 4 for the concrete behaviour). For a fixed cost parameter k, we consider $\lceil \ln k \rfloor$ recursion levels $i = 0, \ldots, (\lceil \ln k \rfloor - 1)$. For the implementation used for these experiments, we refer to `simu_asym.py` attached to the electronic version of the full version of this work. This simulation algorithm is an instantiation of Algorithm 1.

Using the simulator described above, we computed the achieved simulated root Hermite factors for various cost parameters k from 100 to 2,999. The results are plotted in Figure 6. We also computed the theoretical root Hermite factors as established by Theorem 5. More precisely, we used the proof of Theorem 5 to update the root Hermite factors recursively, replacing the term $\sqrt{n'-n}$ of Theorem 4 by $v_{n'-n}^{-1/(n'-n)}$ (which corresponds to using the Gaussian Heuristic). It can be observed that the theoretical and simulated root Hermite factors agree closely.

Fig. 6: Simulated and theoretical root Hermite factors for $k = 100$ to $2,999$ after $\ln k$ levels of recursion.

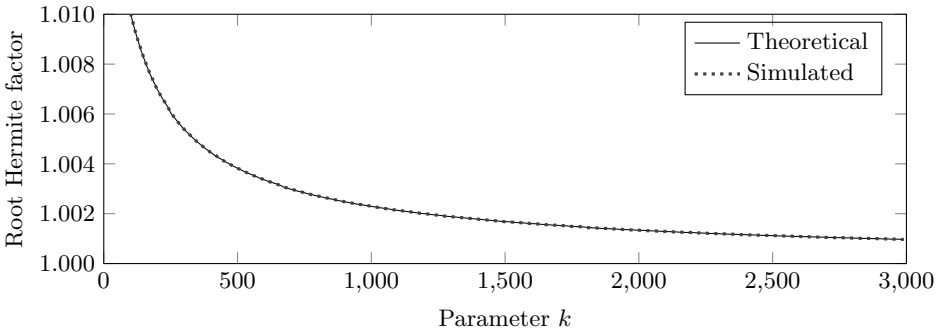
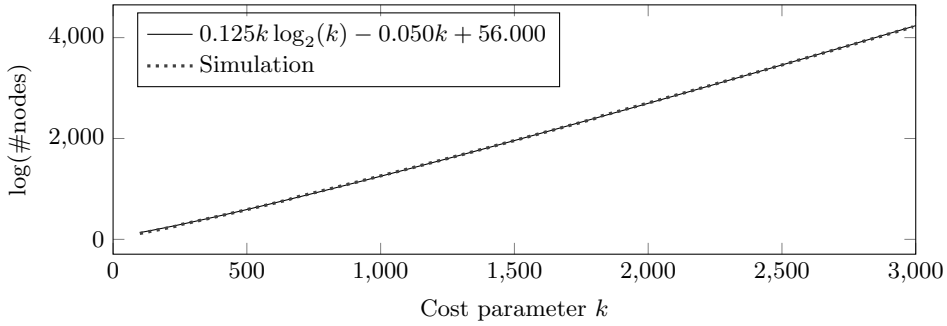


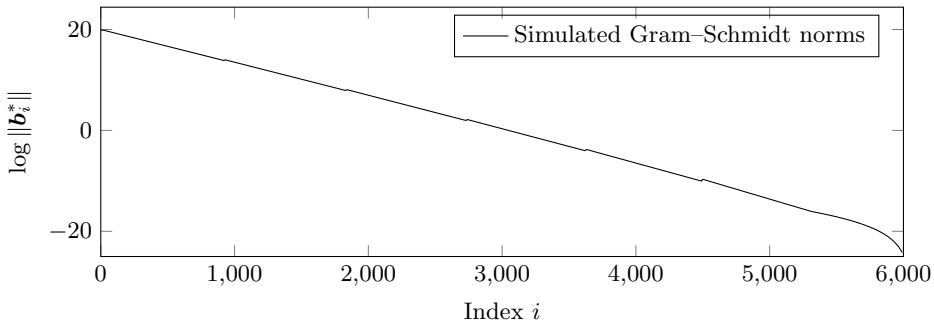
Figure 7 shows the number of nodes visited during the simulation from $k = 100$ to $2,999$, as well as a curve fit. As an example of the output, Figure 8 plots the Gram–Schmidt log-norms of the (simulated) reduced basis for $k = 1,000$ right

Fig. 7: Number of nodes in full enumeration visited during simulation, and a fit.



after 7 levels of recursion. Note that last Gram–Schmidt norms of the basis have the shape of those of an HKZ-reduced basis, since we use Kannan's algorithm at level 0. Also, the successive segments correspond to levels of recursion, their lengths decrease and their respective (negative) slopes decrease with the indices of the Gram–Schmidt norms.

Fig. 8: Gram–Schmidt log-norms of simulated experiments with $k = 1,000$ after $7 \approx \ln k$ recursion levels.

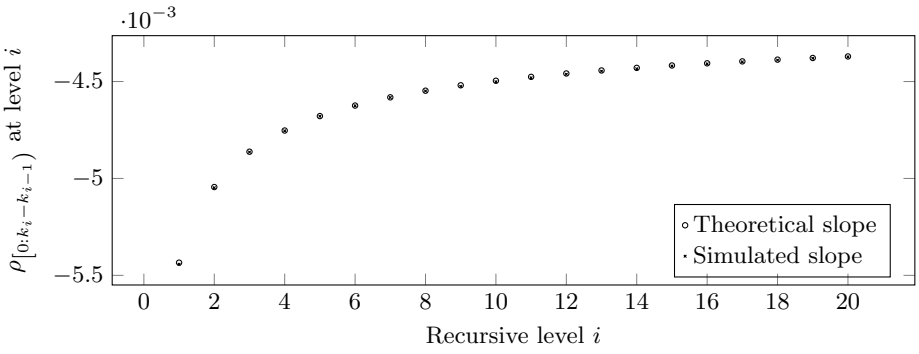


Finally, we plot the Gram–Schmidt log-norms slope for $k = 1,000$ during the first 20 recursion levels. At level i, we compute the slope for the enumeration region (i.e. the first block of size $k_i - k_{i-1}$). It can be observed that the simulated slope is indeed increasing.

4 A practical variant

It can be observed that, in our analysis of Algorithm 2, the dimension of the lattice is relatively large. It is thus interesting to investigate algorithms that require smaller dimensions. In this subsection, we describe a practical strategy that works

Fig. 9: Simulated and theoretical Gram–Schmidt log-norms slope of enumeration region, for $k = 1,000$ and during the first 20 iterations.



with dimensions $d = O(k)$ where the hidden constants are small. As mentioned in the introduction, practical implementations of lattice reduction algorithms often deviate from the asymptotically efficient variants, e.g. by applying much weaker preprocessing than required asymptotically. In this section, we use numerically optimised preprocessing and enumeration strategies to parameterise Algorithm 3, which we view as a practical variant of Algorithm 2, working with dimensions $d = \lceil (1+c) \cdot k \rceil$ for some small constant $c \geq 0$. It differs from Algorithm 2 in two respects. First, it applies BKZ preprocessing instead of SDBKZ preprocessing. This is merely an artefact of the latter seemingly not providing an advantage in the parameter ranges we considered. Second, the algorithm adapts the enumeration dimension based on the "space available" for preprocessing. This is to enforce that it stays within d dimensions, instead of requiring $\approx ik$ dimensions where i is the number of recursion levels.

We use the following functions in Algorithm 3:

- The function $\mathrm{pre}(k)$ returns a preprocessing cost parameter for a given k.
- The function $\mathrm{tail}(k, c, d)$ returns a new cost parameter $k^\star$ such that enumeration in dimension $k^\star$ after preprocessing with $\mathrm{pre}(k^\star)$ in dimension d costs at most as much as enumeration in dimension k after preprocessing in dimension $\lceil (1+c) \cdot k \rceil$. In particular, if $d \geq \lceil (1+c) \cdot k \rceil$ then $k^\star = k$.
- Preprocessing (Step 4) calls Algorithm 4, perhaps restricted to a small number of while loops. Algorithm 4 is simply the BKZ algorithm where the SVP oracle is replaced by Algorithm 3.

We plot the output of our simulations for Algorithm 3 in Figure 10. These simulations are instantiations of Algorithm 1 with $d > k$, $c > 0$ and $\mathrm{tail}(x, y, z)$ matching those used in Algorithm 3. These were produced using `blck.py`, attached to the electronic version of the full version of this work. Our strategy finding strategy follows the same blueprint as described in Section 2.4. Through such simulation experiments we manually established that $c = 0.25$, four sweeps of preprocessing and using BKZ over SDBKZ seems to provide the best performance,

Algorithm 3: Solving Approx-HSVP with preprocessing dimension larger than enumeration dimension.

Data: A basis matrix $\boldsymbol{B} \in \mathbb{R}^{d \times d}$.
Data: Cost parameter $k \geq 2$ and an overshooting parameter $c > 0$.
Result: A short vector $\boldsymbol{b}$.

```
k* ← tail(k, c, d);
k' ← pre(k*);
if k' > 2 then
    run Algorithm 4 on B with parameter k';
else
    run LLL on B;
end
b ← Enum(B_[0:k*));
return b;
```

which is why we report data on these choices.[11] We also fitted the coefficients a_0, a_1, a_2 of $a_0 \cdot k \log k + a_1 \cdot k + a_2$ to points from 100 to 249. Furthermore, we plot the data from Figure 2 to provide a reference point for the performance of the new algorithm and also provide some data on the hypothetical performance of Algorithm 3 assuming the cost of all preprocessing costs is only as much as LLL regardless of the choice of k'. This can be considered the best case scenario for Algorithm 3 and thus a rough lower bound on its running time.[12]

In Figure 11 we give the preprocessing cost parameters and probabilities of success of a single enumeration selected by our optimisation. In particular, these figures suggest that the success probability per enumeration does not drop exponentially fast in Figure 11b. This is consistent with the second order term in the time complexity which is closer to 1/2 (corresponding to standard pruning) than 1 (corresponding to extreme pruning). Similarly, in contrast to Figure 3a the preprocessing cost parameter (or "block size") k' in Figure 11a does not seem to follow an affine function of k, i.e. it seems to grow faster for larger dimensions.

We also give experimental data comparing our implementation of Algorithm 3, `impl.py`, attached to the electronic version of the full version of document, with our simulations in Figure 12. We note that our implementation of Algorithm 3 is faster than `FPyLLL`'s SVP solver from dimension 82 onward. As in Section 2.5, we

[11] The choice $c = 0.25$ may be interpreted a posteriori as consistent with Figure 1 where the minimum for BKZ is attained at $c \approx 0.30$. We note, however, that Figure 1 considers BKZ-reduced bases for block size k, whereas here the algorithm encounters BKZ-reduced bases for block sizes $k' < k$.

[12] We note that this data only extends until $k = 323$. Computing pruning parameters requires increasing precision in increasing dimension and become more "brittle" the cheaper the preprocessing is compared to the enumeration cost. In other words, our simulation code simply crashed with a floating-point error in dimension 324. Since the trend is clear in the data already, we did not push it further using higher precision.

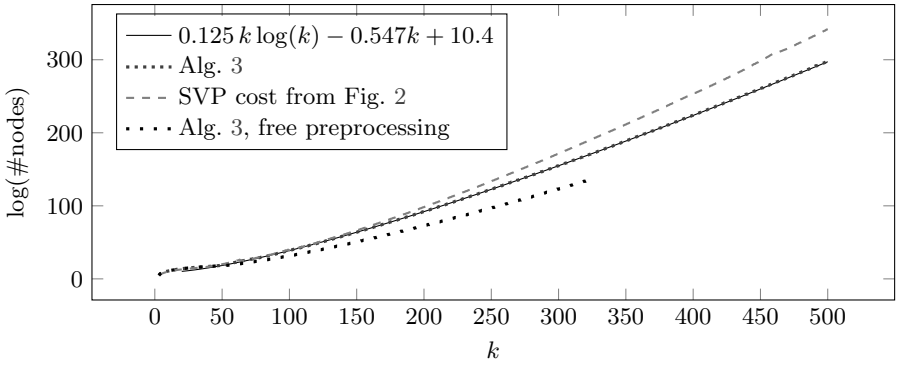
Algorithm 4: BKZ with Algorithm 3 as Approx-HSVP oracle.

Data: A basis matrix $\boldsymbol{B} \in \mathbb{R}^{d \times d}$.
Data: Cost parameter $k \geq 2$ and an overshooting parameter $c \geq 0$.
Result: A reduced basis of $L(\boldsymbol{B})$.

$\boldsymbol{B} \leftarrow \mathrm{LLL}(\boldsymbol{B})$;
while *change was made in previous iteration* **do**
  for $0 \leq \kappa < d - 1$ **do**
    $e \leftarrow \min(d, \kappa + \lceil (1+c) \cdot k \rceil)$;
    $\boldsymbol{v} \leftarrow$ output of Algorithm 3 on $\left(\pi(\boldsymbol{B}_{[\kappa:e)}), k, c\right)$;
    if $\|\boldsymbol{v}\| < \|\boldsymbol{b}^*_\kappa\|$ **then**
      insert $\boldsymbol{v}$ at index κ;
      call LLL to remove linear dependencies;
      record that a change was made;
    end
  end
end
return $\boldsymbol{B}$;

do not use the strategies produced by our simulation to run the implementation but rely on a variant of `FPLLL`'s `strategizer` [dt17] to optimise these strategies.

Fig. 10: Cost of one call to Alg. 3 with enumeration dimension k, $c = 1/4$, $d = \lceil (1+c) \cdot k \rceil$ and four preprocessing sweeps.

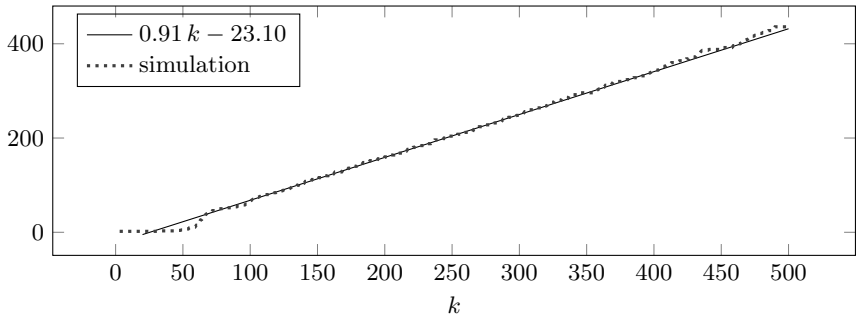


Comparing Figures 2 and 10 is meaningless without taking the obtained root Hermite factors into account. First, Algorithm 3 is not an SVP solver but an Approx-HSVP solver. Second, if $d < \lceil (1+c) \cdot k \rceil$ then Algorithm 3 will reduce the enumeration dimension, further decreasing the quality of the output.

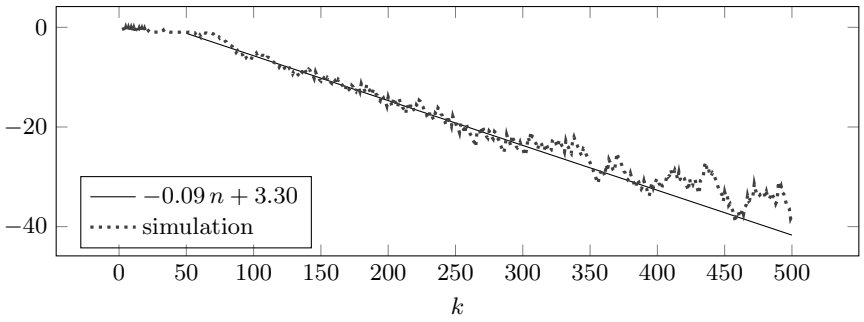
Since we are interested in running Algorithm 3 as a subroutine of Algorithm 4, we compare the latter against plain BKZ. For this comparison we consider the

Fig. 11: Reduction strategies used for Figure 10.

(a) Preprocessing cost parameters used in our simulations.



(b) Success probability of a single enumeration (in log) scale.



case $d = 2 \cdot k$, which corresponds to a typical setting encountered in cryptographic applications.

In Figure 13, we plot the slope of the Gram–Schmidt log-norms as predicted by our simulations for BKZ on the one hand, and a self-dual variant of Algorithm 4. This variant first runs Algorithm 4 on the dual basis, followed by running Algorithm 4 on the original basis. Each run is capped at half the number of sweeps as used for BKZ. The rationale for this strategy is that it handles the quality degradation as the BKZ index i surpasses $d - \lceil (1 + c) \cdot k \rceil$ where $k^\star < k$. As Figure 13 illustrates, the obtained quality of the two algorithms is very close. Indeed our SD variant slightly outperforms BKZ, but we note that the ratio of the two is increasing, i.e. the quality advantage will invert as d increases.

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Fig. 12: Number of nodes visited during one Approx-HSVP call with enumeration dimension k, $c = 1/4$, $d = \lceil (1+c) \cdot k \rceil$ and four sweeps of preprocessing.

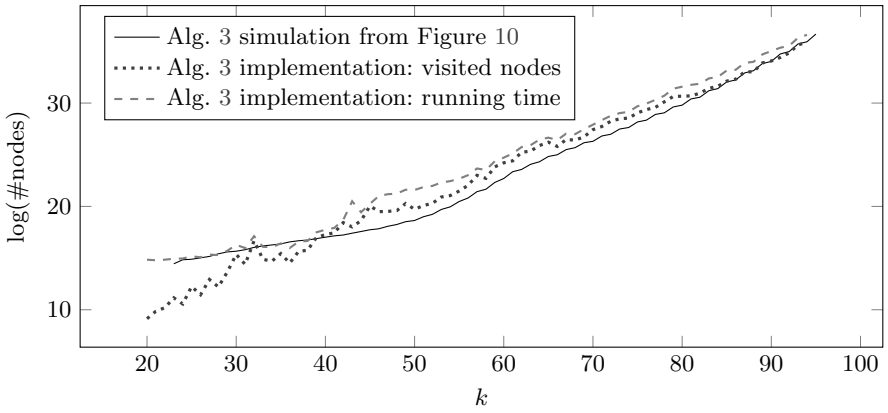
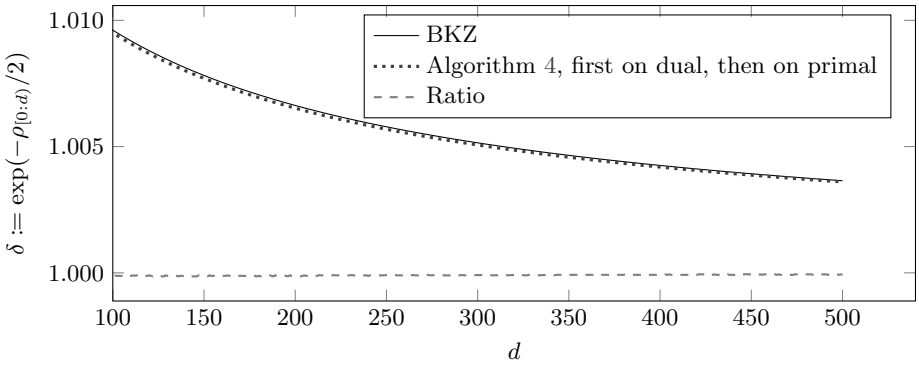


Fig. 13: Basis quality (BKZ vs SD-Algorithm 4)



Four sweeps of Algorithm 4 on the dual, followed by four sweeps of Algorithm 4 on the primal lattice in dimension $d = 2\,k$, using $c = 0.25$ and four preprocessing sweeps.

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