Quadratic Secret Sharing and Conditional Disclosure of Secrets*

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Abstract. There is a huge gap between the upper and lower bounds on the share size of secret-sharing schemes for arbitrary n-party access structures, and consistent with our current knowledge the optimal share size can be anywhere between polynomial in n and exponential in n. For linear secret-sharing schemes, we know that the share size for almost all n-party access structures must be exponential in n. Furthermore, most constructions of efficient secret-sharing schemes are linear. We would like to study larger classes of secret-sharing schemes with two goals. On one hand, we want to prove lower bounds for larger classes of secret-sharing schemes, possibly shedding some light on the share size of general secret-sharing schemes. On the other hand, we want to construct efficient secret-sharing schemes for access structures that do not have efficient linear secret-sharing schemes. Given this motivation, Paskin-Cherniavsky and Radune (ITC'20) defined and studied a new class of secret-sharing schemes in which the shares are generated by applying degree-d polynomials to the secret and some random field elements. The special case $d = 1$ corresponds to linear and multi-linear secret-sharing schemes.
We define and study two additional classes of polynomial secret-sharing schemes: (1) schemes in which for every authorized set the reconstruction of the secret is done using polynomials and (2) schemes in which both sharing and reconstruction are done by polynomials. For linear secret-sharing schemes, schemes with linear sharing and schemes with linear reconstruction are equivalent. We give evidence that for polynomial secret-sharing schemes, schemes with polynomial sharing are probably stronger than schemes with polynomial reconstruction. We also prove lower bounds on the share size for schemes with polynomial reconstruction. On the positive side, we provide constructions of secret-sharing schemes and conditional disclosure of secrets (CDS) protocols

* The work of the authors was partially supported by Israel Science Foundation grant no. 152/17 and a grant from the Cyber Security Research Center at Ben-Gurion University. Part of this work was done while the first author was visiting Georgetwon University, supported by NSF grant no. 1565387, TWC: Large: Collaborative: Computing Over Distributed Sensitvie Data. The first author was also supported by ERC grant 742754 (project NTSC). The second author was also supported by a scholarship from the Israeli Council For Higher Education. The third author was also supported by the European Union's Horizon 2020 Programme (ERC-StG-2014-2020) under grant agreement no. 639813 ERC-CLC, and by the Rector's Office at Tel-Aviv University.

with quadratic sharing and reconstruction. We extend a construction of Liu et al. (CRYPTO'17) and construct optimal quadratic k-server CDS protocols for functions $f : [N]^k \to \{0, 1\}$ with message size $O(N^{(k-1)/3})$. We show how to transform our quadratic k-server CDS protocol to a robust CDS protocol, and use the robust CDS protocol to construct quadratic secret-sharing schemes for arbitrary access structures with share size $O(2^{0.705n})$; this is better than the best known share size of $O(2^{0.7576n})$ for linear secret-sharing schemes and worse than the best known share size of $O(2^{0.585n})$ for general secret-sharing schemes.

1 Introduction

A secret-sharing scheme is a cryptographic tool that enables a dealer holding a secret to share it among a set of parties such that only some predefined subsets of the parties (called authorized sets) can learn the secret and all the other subsets cannot get any information about the secret. The collection of authorized sets is called an access structure. These schemes were presented by Shamir [43], Blakley [21], and Ito, Saito, and Nishizeky [31] for secure storage. Nowadays, secret-sharing schemes are used in many cryptographic tasks, see, e.g., [13] for a list of applications. There are many constructions of secret-sharing schemes for specific families of access structures that have short shares, e.g., [31,19,22,32,20,16,44]. However, in the best known secret-sharing schemes for general n-party access structures, the share size is exponential in n [35,5,8], resulting in impractical secret-sharing schemes. In contrast, the best known lower bound on the share size of a party for some n-party access structure is $\Omega(n/\log n)$ [24,23]. There is a huge gap between the upper bounds and lower bounds, and in spite of active research for more than 30 years, we lack understanding of the share size.

One of the directions to gain some understanding on the share size is to study sub-classes of secret-sharing schemes. Specifically, the class of *linear* secret-sharing schemes was studied in many papers, e.g., [22,32,15,12,11,26,27,41]. In these schemes the sharing algorithm applies a linear mapping on the secret and some random field elements to generate the shares. For linear secret-sharing schemes there are strong lower bounds, i.e., in linear secret-sharing schemes almost all n-party access structures require shares of size at least $2^{0.5n-o(n)}$ [11] and there exists explicit n-party access structures require shares of size at least $2^{\Omega(n)}$ [42,40,41]. It is an important question to extend these lower bounds to other classes of secret-sharing schemes. Furthermore, we would like to construct efficient secret-sharing schemes (i.e., schemes with small share size) for a richer class of access structures than the access structures that have efficient linear secret-sharing schemes (which by [32] coincide with the access structures that have a small monotone span program). Currently, only few such constructions are known [16,44].[3] Studying broader classes of secret-sharing schemes will hopefully result in efficient schemes for more access structures and will develop new

[3] In [44] they construct efficient secret-sharing schemes for access structures that correspond to languages that have statistical zero-knowledge proofs with log-space verifiers and simulators.

techniques for constructing non-linear secret-sharing schemes. In a recent work, Paskin-Cherniavsky and Radune [38] perused these directions – they defined and studied a new class of secret-sharing schemes, called polynomial secret-sharing schemes, in which the sharing algorithm applies (low-degree) polynomials on the secret and some random field elements to generate the shares.

In this paper, we broaden the study of polynomial secret-sharing schemes and define and study two additional classes of polynomial secret-sharing schemes – (1) schemes in which the reconstruction algorithm, which computes the secret from the shares of parties of an authorized set, is done by polynomials, and (2) schemes in which both sharing and reconstruction algorithms are done by applying polynomials. We prove lower bounds for schemes of the first type (hence also for schemes of the second type). We then focus on *quadratic* secret-sharing schemes – schemes in which the sharing and/or reconstruction are done by polynomials of *degree-2*, and provide constructions of such schemes that are more efficient than linear secret-sharing schemes. Thus, we show that considering the wider class of polynomial secret-sharing schemes gives rise to better schemes than linear schemes.

As part of our results, we construct conditional disclosure of secrets (CDS) protocols, a primitive that was introduced in [29]. In a k-server CDS protocol for a Boolean function $f : [N]^k \to \{0, 1\}$, there is a set of k servers that hold a secret s and have a common random string. In addition, each server Q_i holds a private input $x_i \in [N]$. Each server sends one message such that a referee, who knows the private inputs of the servers but nothing more, learns the secret s if $f(x_1, \ldots, x_k) = 1$ and learns nothing otherwise. CDS protocols have been used recently in [35,4,5,8] to construct the best known secret-sharing schemes for arbitrary access structures. Continuing this line of research, we construct quadratic k-server CDS protocols that are provably more efficient than linear CDS protocols. We use them to construct quadratic secret-sharing schemes for arbitrary access structures; these schemes are more efficient than the best known linear secret-sharing schemes.

1.1 Our Contributions and Techniques

Polynomial Sharing vs. Polynomial Reconstruction. Our conceptional contribution is the distinction between three types of polynomial secret-sharing schemes: schemes with polynomial sharing (defined in [38]), schemes with polynomial reconstruction, and schemes in which both sharing and reconstruction are done by polynomials. For linear secret-sharing schemes (in which the secret contains one field element) these notions are equivalent [32,12]. In the full version of the paper [17], we extend this equivalence to multi-linear secret-sharing schemes (i.e., schemes in which the secret can contain more than one filed element). In Section 3.1, we give evidence that such equivalence does not hold for polynomial secret-sharing schemes. We show that a small variation of a secret-sharing scheme of [16] for the quadratic non-residuosity modulo a prime access struc-

ture has an efficient secret-sharing scheme with degree-3 sharing.[4] Following [16], we conjecture that the quadratic non-residuosity modulo a prime is not in NC (the class of problems that have a sequence of circuits of polynomial size and poly-logarithmic depth). By our discussion in Remark 4.6, every sequence of access structures that has efficient secret-sharing schemes with polynomial reconstruction is in NC. Thus, under the conjecture about quadratic non-residuosity modulo a prime problem, we get the desired separation.

Lower bounds for Secret-Sharing Schemes with Degree-d Reconstruction. In Section 4, we show lower bounds for secret-sharing schemes with degree-d reconstruction. Using a result of [34], we show a lower bound of $\Omega(2^{n/(d+1)})$ for sharing one-bit secrets. We also show that every secret-sharing scheme with degree-d reconstruction and share size c can be converted to a multi-linear secret-sharing scheme with share size $O(c^d)$ (with the same domain of secrets). Using a lower bound on the share size of linear secret-sharing schemes over any finite field from [41], we obtain that there exists an explicit access structure such that for every finite field $\mathbb{F}$ it requires shares of size $2^{\Omega(n/d)} \log |\mathbb{F}|$ in every secret-sharing schemes over $\mathbb{F}$ with degree-d reconstruction. Furthermore, this transformation implies that every sequence of access structures that have efficient secret-sharing schemes with degree-d reconstruction for a constant d is in NC.

Quadratic Multi-Server Conditional Disclosure of Secrets Protocols. Liu et al. [36] constructed a quadratic two-server CDS protocol for any function $f : [N]^2 \to \{0,1\}$ with message size $O(N^{1/3})$. In Section 5, we construct quadratic k-server CDS protocols with message size $O(N^{(k-1)/3})$. By our lower bounds from Section 4, this is the optimal message size for quadratic CDS protocols. Our construction uses the two-server CDS protocol of [36] (denoted $\mathcal{P}_{\text{LVW}}$) to construct the k-server CDS protocol. Specifically, the k servers $Q_1, \ldots, Q_k$ simulate the two servers in the CDS protocol $\mathcal{P}_{\text{LVW}}$, where Q_1 simulates the first server in $\mathcal{P}_{\text{LVW}}$ and servers $Q_2, \ldots, Q_k$ simulate the second server in $\mathcal{P}_{\text{LVW}}$.

Quadratic Multi-Server Robust Conditional Disclosure of Secrets Protocols. In a *t-robust* CDS protocol (denoted t-RCDS protocol), each server can send up to t messages for different inputs using the same shared randomness such that the security is not violated if the value of the function f is 0 for all combinations of inputs. RCDS protocols were defined in [5] and were used to construct secret-sharing schemes for arbitrary access structures. Furthermore, Applebaum et al. [5] showed a general transformation from CDS protocol to RCDS protocol. Using their transformation as is, we get a quadratic RCDS protocol with message size $\tilde{O}(N^{(k-1)/3} t^{k-1})$, which is not useful for constructing improved secret-sharing schemes (compared to the best known linear secret-sharing schemes). In Section 6, we show that with a careful analysis that exploits the structure of our quadratic k-server CDS protocol, we can get an improved message size of $\tilde{O}(N^{(k-1)/3} t^{2(k-1)/3+1})$.

[4] We present it as a CDS protocol for the quadratic non-residuosity function. Using known equivalence, this implies a secret-sharing scheme, as in [16].

Quadratic Secret-Sharing Schemes for Arbitrary Access Structures and Almost All Access Structures. Applebaum et al. [5] and Applebaum and Nir [8] showed transformations from k-server RCDS protocols to secret-sharing schemes for arbitrary access structures. In [8], they achieved a general secret-sharing scheme for arbitrary access structures with share size $2^{0.585n+o(n)}$. In Section 7, we plug our quadratic k-server RCDS protocol in the transformation of [8] and get a quadratic secret-sharing scheme for arbitrary access structures with share size $2^{0.705+o(n)}$. This should be compared to the best known linear secret-sharing scheme for arbitrary access structures, given in [8], that has share size $2^{0.7576n+o(n)}$.

Beimel and Farràs [14] proved that for almost all access structures, there is a secret-sharing scheme for one-bit secrets with shares of size $2^{\tilde{O}(\sqrt{n})}$ and a linear secret-sharing scheme with shares of size $2^{n/2+o(n)}$. By a lower bound of [11] this share size is tight for linear secret-sharing schemes. In Section 7, we construct quadratic secret-sharing schemes for almost all access structures. Plugging our quadratic k-server CDS protocol in the construction of [14], we get that for almost all access structures there is a quadratic secret-sharing scheme for sharing one-bit secrets with shares of size $2^{n/3+o(n)}$. This proves a separation between quadratic secret-sharing schemes and linear secret-sharing schemes for almost all access structures.

Quadratic Two-Server Robust CDS Protocols. Motivated by the interesting application of robust CDS (RCDS) protocols for constructing secret-sharing schemes, we further investigate quadratic two-server RCDS protocols. In the full version of the paper [17], we show how to transform the quadratic two-server CDS protocol of [36] to an RCDS protocol that is $N^{1/3}$-robust for one server while maintaining the $\tilde{O}(N^{1/3})$ message size. In comparison, the quadratic two-server $N^{1/3}$-RCDS protocol of Section 6 has message size $\tilde{O}(N^{8/9})$, however, it is robust for both servers. This transformation is non-blackbox, and uses polynomials of degree t to mask messages, where the masks of every messages of t inputs are uniformly distributed. Non-blackbox constructions of RCDS protocols may avoid limitations of constructing using CDS protocols as a blackbox.

1.2 Open Questions

Next, we mention a few open problems arising from this paper. We show non-trivial lower bounds for secret-sharing schemes with degree-d reconstruction. In [38], they ask the analogous question:

Question 1.1. Prove lower bounds on the share size of secret-sharing schemes with degree-d sharing.

We show a construction with degree-3 sharing that under a plausible conjecture does not have degree-3 reconstruction. We would like to prove such a separation without any assumptions.

Question 1.2. Prove (unconditionally) that there is some access structure that has an efficient secret-sharing scheme with polynomial sharing but does not have an efficient secret-sharing scheme with polynomial reconstruction.

Question 1.3. Are there access structures that have an efficient secret-sharing scheme with polynomial reconstruction (of non-constant degree) but do not have an efficient secret-sharing scheme with polynomial sharing?

We construct quadratic CDS protocols and secret-sharing schemes for arbitrary access structures. For quadratic CDS protocols we prove a matching lower bound on the message size. However, for larger values of d, the lower bound on the message size of degree-d CDS protocols is smaller.

Question 1.4. Are there degree-d CDS protocols with smaller message size than the message size of quadratic CDS protocols? Are there degree-d secret-sharing schemes that are more efficient than quadratic secret-sharing schemes?

Perhaps the most important question is to construct efficient secret-sharing schemes for a wide class of access structures.

Question 1.5. Construct efficient degree-d secret-sharing schemes for a larger class of access structures than the access structures that have efficient linear secret-sharing schemes.

1.3 Additional Related Works

Conditional Disclosure of Secrets (CDS) Protocols. Conditional disclosure of secrets (CDS) protocols were first defined by Gertner et al. [29]. The motivation for this definition was to construct symmetric private information retrieval protocols. CDS protocols were used in many cryptographic applications, such as attribute based encryption [28,10,45], priced oblivious transfer [1], and secret-sharing schemes [35,18,4,5,14,8].

Liu et al. [36] showed two constructions of two-server CDS protocols. In their first construction, which is most relevant to our work, they constructed a quadratic two-server CDS protocol for any Boolean function $f : [N]^2 \to \{0,1\}$ with message size $O(N^{1/3})$. In their second construction, which is non-polynomial, they constructed a two-server CDS protocol with message size $2^{O(\sqrt{\log N \log\log N})}$. Applebaum and Arkis [2] (improving on [3]) have shown that for long secrets, i.e., secrets of size $\Theta(2^{N^2})$, there is a two-server CDS protocol in which the message size is 3 times the size of the secret. There are also several constructions of multi-server CDS protocols. Liu et al. [37] constructed a k-server CDS protocol (for one-bit secrets) with message size $2^{\tilde{O}(\sqrt{k \log N})}$. Beimel and Peter [18] and Liu et al. [37] constructed a linear k-server CDS protocol (for one-bit secrets) with message size $O(N^{(k-1)/2})$; by [18], this bound is optimal (up to a factor of k). When we have long secrets, i.e., secrets of size $\Theta(2^{N^k})$, Applebaum and Arkis [2] showed that there is a k-server CDS protocol in which the message size is 4 times the size of the secret. Gay et al. [28] proved a lower bound of

$\Omega(\log\log N)$ on the message size of two-server CDS protocols for some function and a lower bound of $\Omega(\sqrt{\log N})$ on the message size of linear two-server CDS protocols. Later, Applebaum et al. [3], Applebaum et al. [7], and Applebaum and Vasudevan [9] proved a lower bound of $\Omega(\log N)$ on the message size of two-server CDS protocols.

Polynomial Secret-Sharing Schemes. Paskin-Cherniavsky and Radune [38] presented the model of secret-sharing schemes with polynomial sharing, in which the sharing is a polynomial of low (constant) degree and the reconstruction can be any function. They showed limitations of various sub-classes of secret-sharing schemes with polynomial sharing. Specifically, they showed that the subclass of schemes for which the sharing is linear in the randomness (and the secret can be with any degree) is equivalent to multi-linear schemes up to a multiplicate factor of $O(n)$ in the share size. This implies that schemes in this subclass cannot significantly reduce the known share size of multi-linear schemes. In addition, they showed that the subclass of schemes over finite fields with odd characteristic such that the degree of the randomness in the sharing function is exactly 2 or 0 in any monomial of the polynomial can efficiently realize only access structures whose all minimal authorized sets are singletons. They also studied the randomness complexity of schemes with polynomial sharing. They showed an exponential upper bound on the randomness complexity (as a function of the share size). For linear and multi-linear schemes, we have a tight linear upper bound on the randomness complexity.

2 Preliminaries

In this section we define secret-sharing schemes, conditional disclosure of secrets protocols, and robust conditional disclosure of secrets protocols.

Notations. We say that two probability distributions $\mathcal{Y}_1, \mathcal{Y}_2$ over domain $\mathcal{X}$ are identical, and denote $\mathcal{Y}_1 \equiv \mathcal{Y}_2$, if $\mathcal{Y}_1(x) = \mathcal{Y}_2(x)$ for every $x \in \mathcal{X}$. We denote by $\binom{N}{[m]}$ the set of all subsets of N of size m. We say that $g(n) = \tilde{O}(f(n))$ if $g(n) = O(f(n)\log^c n)$ for some constant c, i.e., the $\tilde{O}$ notation ignores poly-logarithmic factors.

Secret-Sharing. We start by presenting the definition of secret-sharing schemes.

Definition 2.1 (Access Structures). *Let $P = \{P_1, \ldots, P_n\}$ be a set of parties. A collection $\Gamma \subseteq 2^P$ is* monotone *if $B \in \Gamma$ and $B \subseteq C$ imply that $C \in \Gamma$. An* access structure *is a monotone collection $\Gamma \subseteq 2^P$ of non-empty subsets of P. Sets in Γ are called* authorized*, and sets not in Γ are called* unauthorized*.*

Definition 2.2 (Secret-Sharing Schemes). *A* secret-sharing scheme *Π with domain of secrets S is a mapping from $S \times R$, where R is some finite set called the set of random strings, to a set of n-tuples $S_1 \times S_2 \times \cdots \times S_n$, where S_j is called the* domain of shares *of party P_j. A dealer distributes a secret $s \in S$ according to Π by first sampling a random string $r \in R$ with uniform distribution, computing a*

vector of shares $\Pi(s,r) = (s_1, \ldots, s_n)$*, and privately communicating each share* s_j *to party* P_j*. For a set* $A \subseteq P$*, we denote* $\Pi_A(s,r)$ *as the restriction of* $\Pi(s,r)$ *to its* A*-entries (i.e., the shares of the parties in* A*).*

Given a secret-sharing scheme Π*, define the* size *of the secret as* $\log|S|$*, the* share size *of party* P_j *as* $\log|S_j|$*, and the* total share size *as* $\sum_{j=1}^{n} \log|S_j|$*.*

Let S *be a finite set of secrets, where* $|S| \geq 2$*. A secret-sharing scheme* Π *with domain of secrets* S *realizes an access structure* Γ *if the following two requirements hold:*

CORRECTNESS. *The secret can be reconstructed by any authorized set of parties. That is, for any set* $B = \{P_{i_1}, \ldots, P_{i_{|B|}}\} \in \Gamma$ *there exists a* reconstruction function $\text{Recon}_B : S_{i_1} \times \cdots \times S_{i_{|B|}} \to S$ *such that for every secret* $s \in S$ *and every random string* $r \in R$*,* $\text{Recon}_B(\Pi_B(s,r)) = s$*.*

SECURITY. *Every unauthorized set cannot learn anything about the secret from its shares. Formally, for any set* $T = \{P_{i_1}, \ldots, P_{i_{|T|}}\} \notin \Gamma$*, every pair of secrets* $s, s' \in S$*, and every vector of shares* $(s_{i_1}, \ldots, s_{i_{|T|}}) \in S_{i_1} \times \cdots \times S_{i_{|T|}}$*, it holds that* $\Pi_T(s,r) \equiv \Pi_T(s',r)$*, where the probability distributions are over the choice of* r *from* R *with uniform distribution.*

Definition 2.3 (Threshold Secret-Sharing Schemes). *Let* Π *be a secret-sharing scheme on a set of* n *parties* P*. We say that* Π *is a* t-out-of-n secret-sharing scheme *if it realizes the access structure* $\Gamma_{t,n} = \{A \subseteq P : |A| \geq t\}$*.*

Conditional Disclosure of Secrets. Next, we define k-server conditional disclosure of secrets (CDS) protocols, first presented in [29]. We consider a model where k servers[5] $Q_1, \ldots, Q_k$ hold a secret s and a common random string r; every server Q_i holds an input x_i for some k-input function f. In addition, there is a referee that holds $x_1, \ldots, x_k$ but, prior to the execution of the protocol, does not know s and r. In a CDS protocol for f, for every $i \in [k]$, server Q_i sends a single message to the referee, based on r, s, and x_i; the server does not see neither the inputs of the other servers nor their messages when computing its message. The requirements are that the referee can reconstruct the secret s if $f(x_1, \ldots, x_k) = 1$, and it cannot learn any information about the secret s if $f(x_1, \ldots, x_k) = 0$.

Definition 2.4 (Conditional Disclosure of Secrets Protocols). *Let* $f : X_1 \times \cdots \times X_k \to \{0,1\}$ *be a* k*-input function. A* k-server CDS protocol $\mathcal{P}$ *for* f*, with domain of secrets* S*, domain of common random strings* R*, and finite message domains* $M_1, \ldots, M_k$*, consists of* k *message computation functions* $\text{ENC}_1, \ldots, \text{ENC}_k$*, where* $\text{ENC}_i : X_i \times S \times R \to M_i$ *for every* $i \in [k]$*. For an input* $x = (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$*, secret* $s \in S$*, and randomness* $r \in R$*, we let* $\text{ENC}(x,s,r) = (\text{ENC}_1(x_1,s,r), \ldots, \text{ENC}_k(x_k,s,r))$*. We say that a protocol* $\mathcal{P}$ *is a CDS protocol for* f *if it satisfies the following properties: (1) Correctness: There is a deterministic reconstruction function* $\text{DEC} : X_1 \times \cdots \times X_k \times M_1 \times$

[5] For clarity of the presentation (especially when using CDS protocols to construct secret-sharing schemes) we denote the entities in a CDS protocol by servers and the entities in a secret-sharing scheme by parties.

$\cdots \times M_k \to S$ *such that for every input* $x = (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ *for which* $f(x_1, \ldots, x_k) = 1$*, every secret* $s \in S$*, and every common random string* $r \in R$*, it holds that* $\text{DEC}(x, \text{ENC}(x, s, r)) = s$*. (2) Security: For every input* $x = (x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ *for which* $f(x_1, \ldots, x_k) = 0$ *and every pair of secrets* $s, s' \in S$ *it holds that* $\text{ENC}(x, s, r) \equiv \text{ENC}(x, s', r)$*, where* r *is sampled uniformly from* R*.*

The message size *of a CDS protocol* $\mathcal{P}$ *is defined as the size of the largest message sent by the servers, i.e.,* $\max_{1 \le i \le k} \log |M_i|$*. In two-server CDS protocols, we sometimes refer to the servers as Alice and Bob (instead of* Q_1 *and* Q_2*, respectively).*

Definition 2.5 (The Predicate INDEX_N^k**).** *We define the* k*-input function* $\text{INDEX}_N^k : \{0,1\}^{N^{k-1}} \times [N]^{k-1} \to \{0,1\}$ *where for every* $D \in \{0,1\}^{N^{k-1}}$ *(a* $(k-1)$ *dimensional array called the database) and every* $(i_2, \ldots, i_k) \in [N]^{k-1}$ *(called the index),* $\text{INDEX}_N^k(D, i_2, \ldots, i_k) = D_{i_2, \ldots, i_k}$*.*

Observation 2.6 ([28]). *If there is a* k*-server CDS protocol for* INDEX_N^k *with message size* M*, then for every* $f : [N]^k \to \{0,1\}$ *there is a* k*-server CDS protocol with message size* M*.*

We obtain the above CDS protocol for f in the following way: Server Q_1 constructs a database $D_{i_2,\ldots,i_k} = f(x_1, i_2, \ldots, i_k)$ for every $i_2, \ldots, i_k \in [N]$ and servers $Q_2, \ldots, Q_{k-1}$ treat their inputs $(x_2, \ldots, x_k) \in [N]^{k-1}$ as the index, and execute the CDS protocol for $\text{INDEX}_N^k(D, x_2, \ldots, x_k) = f(x_1, x_2, \ldots, x_k)$.

Robust Conditional Disclosure of Secrets. In the definition of CDS protocols (Definition 2.4), if a server sends messages for different inputs with the same randomness, then the security is not guaranteed and the referee can possibly learn information on the secret. In [5], the notion of robust CDS (RCDS) protocols was presented. In RCDS protocols, the security is guaranteed even if the referee receives messages of different inputs with the same randomness. Next we define the notion of t-RCDS protocols.

Definition 2.7 (Zero Sets). *Let* $f : X_1 \times X_2 \times \cdots \times X_k \to \{0,1\}$ *be a* k*-input function. We say that a set of inputs* $Z \subseteq X_1 \times X_2 \cdots \times X_k$ *is a zero set of* f *if* $f(x) = 0$ *for every* $x \in Z$*. For sets* $Z_1, \ldots, Z_k$*, we denote* $\text{ENC}_i(Z_i, s, r) = (\text{ENC}_i(x_i, s, r))_{x_i \in Z_i}$ *and*

$$\text{ENC}(Z_1 \times Z_2 \cdots \times Z_k, s, r) = (\text{ENC}_1(Z_1, s, r), \ldots, \text{ENC}_k(Z_k, s, r)).$$

Definition 2.8 (t**-RCDS Protocols).** *Let* $\mathcal{P}$ *be a* k*-server CDS protocol for a* k*-input function* $f : X_1 \times X_2 \times \cdots \times X_k \to \{0,1\}$ *and* $Z = Z_1 \times Z_2 \times \cdots \times Z_k \subseteq X_1 \times X_2 \times \cdots \times X_k$ *be a zero set of* f*. We say that* $\mathcal{P}$ *is robust for the set* Z *if for every pair of secrets* $s, s' \in S$*, it holds that* $\text{ENC}(Z, s, r)$ *and* $\text{ENC}(Z, s', r)$ *are identically distributed. For every integers* $t_1, \ldots, t_k$*, we say that* $\mathcal{P}$ *is a* $(t_1, \ldots, t_k)$*-RCDS protocol if it is robust for every zero set* $Z_1 \times Z_2 \times \cdots \times Z_k$ *such that* $|Z_i| \le t_i$ *for every* $i \in [k]$*. Finally, for every integer* t*, we say that* $\mathcal{P}$ *is a* t*-RCDS protocol if it is a* $(t, \ldots, t)$*-RCDS protocol.*

3 Degree-d Secret Sharing and Degree-d CDS Protocols

In [38], polynomial secret-sharing schemes are defined as secret-sharing schemes in which the sharing function can be computed by polynomial of low degree. In this paper, we define secret-sharing schemes with polynomial reconstruction and secret-sharing schemes with both polynomial sharing and reconstruction.

Definition 3.1 (Degree of Polynomial). *The degree of each multivariate monomial is the sum of the degree of all its variables; the degree of a polynomial is the maximal degree of its monomials.*

Definition 3.2 (Degree-d Mapping over $\mathbb{F}$). *A function $f : \mathbb{F}^\ell \to \mathbb{F}^m$ can be computed by degree-d polynomials over $\mathbb{F}$ if there are m polynomials $Q_1, \ldots, Q_m : \mathbb{F}^\ell \to \mathbb{F}$ of degree at most d s.t. $f(x_1, \ldots, x_\ell) = (Q_1(x_1, \ldots, x_\ell), \ldots, Q_m(x_1, \ldots, x_\ell))$.*

A secret-sharing scheme has a polynomial sharing if the mapping that the dealer uses to generate the shares given to the parties can be computed by polynomials, as we formalize at the following definition.

Definition 3.3 (Secret-Sharing Schemes with Degree-d Sharing [38]). *Let Π be a secret-sharing scheme with domain of secrets S. We say that the scheme Π has* degree-d sharing over a finite field $\mathbb{F}$ *if there are integers $\ell, \ell_r, \ell_1, \ldots, \ell_n$ such that $S \subseteq \mathbb{F}^\ell, R = \mathbb{F}^{\ell_r}$, and $S_i = \mathbb{F}^{\ell_i}$ for every $i \in [n]$, and Π can be computed by degree-d polynomials over $\mathbb{F}$.*

In Definition 3.3, we allow S to be a subset of $\mathbb{F}^\ell$ (in [38], $S = \mathbb{F}^\ell$). In particular, we will study the case where $\ell = 1$ and $S = \{0, 1\} \subseteq \mathbb{F}$.

A secret-sharing scheme has a polynomial reconstruction if for every authorized set the mapping that the set uses to reconstruct the secret from its shares can be computed by polynomials.

Definition 3.4 (Secret-Sharing Schemes with Degree-d Reconstruction). *Let Π be a secret-sharing scheme with domain of secrets S. We say that the scheme Π has a* degree-d reconstruction over a finite field $\mathbb{F}$ *if there are integers $\ell, \ell_r, \ell_1, \ldots, \ell_n$ such that $S \subseteq \mathbb{F}^\ell, R = \mathbb{F}^{\ell_r}$, and $S_i = \mathbb{F}^{\ell_i}$ for every $i \in [n]$, and Recon_B, the reconstruction function of the secret, can be computed by degree-d polynomials over $\mathbb{F}$ for every $B \in \Gamma$.*

Definition 3.5 (Degree-d Secret-Sharing Schemes). *A secret-sharing scheme Π is a degree-d secret-sharing scheme over $\mathbb{F}$ if it has degree-d sharing and degree-d reconstruction over $\mathbb{F}$.*

Definition 3.6 (CDS Protocols with Degree-d Encoding). *A CDS protocol $\mathcal{P}$ has a degree-d encoding over a finite field $\mathbb{F}$ if there are integers $\ell, \ell_r, \ell_1, \ldots, \ell_k \geq 1$ such that $S \subseteq \mathbb{F}^\ell$, $R = \mathbb{F}^{\ell_r}, M_i = \mathbb{F}^{\ell_i}$ for every $1 \leq i \leq k$, and for every $i \in [k]$ and every $x \in X_i$ the function $\mathrm{ENC}_{i,x} : \mathbb{F}^{\ell+\ell_r} \to M_i$ can be computed by degree-d polynomials over $\mathbb{F}$, where $\mathrm{ENC}_{i,x}(s, r) = \mathrm{ENC}_i(x, r, s)$.*

Definition 3.7 (CDS Protocols with Degree-d Decoding). *A CDS protocol $\mathcal{P}$ has a degree-d decoding over a finite field $\mathbb{F}$ if there are integers $\ell, \ell_r, \ell_1, \ldots, \ell_k \geq 1$ such that $S \subseteq \mathbb{F}^\ell$, $R = \mathbb{F}^{\ell_r}$, $M_i = \mathbb{F}^{\ell_i}$ for every $1 \leq \ell \leq k$, and for every inputs $x_1, \ldots, x_k$ the function $\text{DEC}_{x_1,\ldots,x_k} : \mathbb{F}^{\ell_1+\cdots+\ell_k} \to S$ can be computed by degree-d polynomials over $\mathbb{F}$, where $\text{DEC}_{x_1,\ldots,x_k}(m_1, \ldots, m_k) = \text{DEC}(x_1, \ldots, x_k, m_1, \ldots, m_k)$.*

Note that in Definition 3.7, the polynomials computing the decoding can be different for every input x.

Definition 3.8 (Degree-d CDS Protocols). *A CDS protocol $\mathcal{P}$ is a degree-d CDS protocol over $\mathbb{F}$ if it has degree-d encoding and degree-d decoding over $\mathbb{F}$.*

Definition 3.9 (Linear Secret-Sharing Schemes and CDS Protocols). *A linear polynomial is a degree-1 polynomial. A linear secret-sharing scheme is a degree-1 secret-sharing scheme and $\ell = 1$ (i.e., the secret contains one field element). A secret-sharing scheme has a linear sharing (resp., reconstruction) if it has degree-1 sharing (resp., reconstruction). Similar notations hold for CDS protocols.*

Secret-sharing schemes with linear sharing are equivalent to secret-sharing schemes with linear reconstruction as shown by [32,12].

Claim 3.10 ([32,12]). *A secret-sharing scheme Π is linear if and only if for every authorized set B the reconstruction function Recon_B is a linear mapping.*

In the full version of this paper [17], we generalize Claim 3.10 and show that secret-sharing schemes with degree-1 sharing (i.e., multi-linear schemes) are equivalent to secret-sharing schemes with degree-1 reconstruction.

Definition 3.11 (Quadratic Secret-Sharing Schemes and CDS Protocols). *A quadratic polynomial is a degree-2 polynomial. A quadratic secret-sharing scheme is a degree-2 secret-sharing scheme. A secret-sharing scheme has a quadratic sharing (resp., reconstruction) if it has degree-2 sharing (resp., reconstruction). Similar notations hold for CDS protocols.*

Let $\mathcal{A} = \{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be a family of access structures, where $\mathcal{A}_n$ is an n-party access structure. We informally say that $\mathcal{A}$ can be realized by polynomial secret-sharing schemes if it can be realized by degree-$f(n)$ secret-sharing schemes where $f(n)$ is a constant or relatively small function, i.e., $\log n$.

Remark 3.12. Observe that for every finite field, every function can be computed by a polynomial (with high degree). Therefore, every access structure can be realized by a secret-sharing scheme with polynomial reconstruction of high degree. This is not true for sharing since we require that the polynomial sharing uses uniformly distributed random elements of the field. However, by relaxing correctness and security, we can also get a statistical secret-sharing scheme with polynomial sharing of high degree (by sampling many field elements and constructing a distribution that is close to uniform on the set R of the random strings of the secret-sharing scheme).

3.1 CDS with Degree-3 Encoding for the Non-Quadratic Residues Function

In this section we show an example of a function that can be realized by an efficient CDS protocol with degree-3 encoding, but, under the assumption that the quadratic residue modulo a prime problem is not in NC, it does not have an efficient CDS protocol with degree-d decoding (for any constant d). Our construction is built upon [16] where they construct an efficient non-linear secret-sharing scheme for an access structure that corresponds to the quadratic residue function. In the construction of [16], the random string is not uniformly distributed in the field (as we require from CDS protocols with polynomial encoding). In the following construction, in order to get a degree-d encoding, we choose the random string uniformly, resulting in a small error in the correctness.

The Quadratic Residue Modulo a Prime Problem. For a prime p, let $\mathrm{QR}_p = \{a \in \{1,\ldots,p-1\} : \exists b \in \{1,\ldots,p-1\}\, a \equiv b^2 \pmod p\}$. The quadratic residue modulo a prime problem is given p and a, where p is a prime, and outputs 1 if and only if $a \in \mathrm{QR}_p$. All the *known* algorithms for the quadratic residue modulo a prime problem are sequential and it is not not known if efficient parallel algorithms for this problem exist. The known algorithms are of two types; the first type requires computing a modular exponentiation and the second requires computing the gcd. Therefore, the problem is related to modular exponentiation and gcd problems, and thus according to the current state of the art, it is reasonable to assume that the problem is not in NC (see [16] for more details).

CDS protocol for f_{NQRP_p}

- The secret: A bit $s \in \{0,1\}$.
- Q_i for every $1 \le i \le k$ holds $x_i \in \{0,1\}$.
- Common randomness: $r, z_1, \ldots, z_{k-1} \in \mathbb{F}_p$.
- **The protocol**
  - Calculate $z_k = -\sum_{j=1}^{k-1} z_j$.
  - Server Q_1 sends $(z_1 + s \cdot 2^1 x_1 r^2 + r^2) \bmod p$.
  - Server Q_i for every $2 \le i \le k$ sends $(z_i + s \cdot 2^i x_i r^2) \bmod p$.

Fig. 1. A k-server CDS protocol with Degree-3 Encoding for f_{NQRP_p}.

We define, for a prime p and $k = \lfloor \log p \rfloor - 1$, the function $f_{\mathrm{NQRP}_p} : \{0,1\}^k \to \{0,1\}$ such that $f_{\mathrm{NQRP}_p}(x_1,\ldots,x_k) = 1$ if $(1 + \sum_{i=1}^{k} 2^i x_i) \bmod p \notin \mathrm{QR}_p$ and $f_{\mathrm{NQRP}_p}(x_1,\ldots,x_k) = 0$ otherwise.[6] The function f_{NQRP_p} is realized by the CDS

[6] We add 1 to the input to avoid the input 0, which is neither a quadratic residue nor a quadratic non residue.

protocol depicted in Fig. 1. This protocol has perfect security, however, it has a one-side error $1/p$ in the correctness. Repeating this protocol t times will result in a protocol with error $O(1/p^t)$.

Lemma 3.13. *For every t, there is a k-server CDS protocol with degree-3 encoding over $\mathbb{F}_p$ for the function f_{NQRP_p} with $S = \{0,1\}$ and an error in correctness of $1/p^t$ and message size of $O(t \log p)$.*

Proof. In Fig. 1, we describe a k-server CDS protocol for f_{NQRP_p}. We next prove its correctness and security.

For correctness, assuming $r \neq 0$, when $s = 0$ the sum of the messages the referee gets is $\sum_{i=1}^{k} z_i + r^2 \equiv r^2 \bmod p$, and when $s = 1$ the sum is $r^2(1 + \sum_{i=1}^{k} 2^i x_i) \bmod p$. Recall that $r^2 \cdot a \in \mathrm{QR}_p$ iff $a \in \mathrm{QR}_p$. Therefore, when $f_{\mathrm{NQRP}_p}(x_1, \ldots, x_k) = 1$, $s = 1$ iff the sum of the messages is not in QR_p. The referee can reconstruct the secret when the random element r is in $\mathbb{F}_p \setminus \{0\}$, thus the referee can reconstruct the secret with probability $1 - 1/p$. To amplify the correctness, we repeat the protocol t times and get correctness with probability of $1 - 1/p^t$.

In order to prove security, we prove that every k-tuples of messages for an input $x_1, \ldots, x_k$ such that $f_{\mathrm{NQRP}_p}(x_1, \ldots, x_k) = 0$ the messages are identically distributed when $s = 0$ and when $s = 1$. When $r = 0$ the messages are uniform random elements whose sum is 0 regardless of the secret. Otherwise, regardless of the secret, the sum of the messages is a uniformly random distributed quadratic residue: for $s = 0$ the sum is $r^2 \bmod p$ and for $s = 1$ the sum is $b = r^2(1 + \sum_{i=1}^{k} 2^i x_i) \bmod p \in \mathrm{QR}_p$ which is also a uniformly distributed quadratic residue. Thus, in both cases the messages are random elements in $\mathbb{F}_p$ with the restriction that their sum is a random quadratic residue.

Each message contains only one field element of size $\log p$. As we repeat the protocol t times, the message size is $t \log p$. The encoding function is $z_i + (2^i x_i) \cdot sr^2 \bmod p$ which is a degree-3 polynomial in the secret and the randomness (for every x_i). □

In Lemma 4.4 we show that for any constant d, any CDS protocol with degree-d decoding and message size M can be transformed to a linear CDS protocol in which the message size is M^d. Recall that any sequence of functions $\{f_i\}_{i \in \mathbb{N}}$ that can be realized by a linear CDS protocol with polynomial message size (in the number of servers) is in NC, i.e., it has a family of circuits of poly-logarithmic depth and polynomial size (see discussion in Remark 4.6). The above is true even if there is an exponentially small error in the correctness (this is discussed in the full version of the paper [17]). Thus, we obtain the following corollary.

Corollary 3.14. *Under the assumption that $\{\mathrm{NQRP}_p\}_{p \text{ is a prime}} \notin \mathrm{NC}$, there is a sequence of functions that can be realized by an efficient CDS protocol with degree-3 encoding, but for any constant d, cannot be realized by an efficient CDS protocol with degree-d decoding.*

4 Lower Bounds for Secret Sharing with Degree-d Reconstruction

In this section, we show lower bounds for secret-sharing schemes with degree-d reconstruction.

4.1 Lower Bounds for 1-Bit Secrets for Implicit Access Structures

The following theorem was showed in [34].

Theorem 4.1 (Implied by [34]). *Let $\mathcal{F}_{\text{rec}}$ be the family of possible reconstruction functions, c be the sum of the share sizes of all the parties (i.e., the total share size), and $\mathcal{F}_{\mathcal{A}}$ be a family of n-party access structures. For all but at most $\sqrt{|\mathcal{F}_{\mathcal{A}}|}$ access structures $\Gamma \in \mathcal{F}_{\mathcal{A}}$, for any secret-sharing scheme with domain of secrets $\{0,1\}$ and reconstruction function from $\mathcal{F}_{\text{rec}}$, it holds that*

$$\log|\mathcal{F}_{\text{rec}}| \cdot c = \Omega(\log|\mathcal{F}_{\mathcal{A}}|).$$

We obtain the following two corollaries.

Corollary 4.2. *For almost all n-party access structures, any secret-sharing scheme realizing them over any finite field with domain of secrets $\{0,1\}$ and degree-d reconstruction requires total share size of $2^{n/(d+1)-o(n)}$.*

Proof. Let $\mathcal{F}_{\mathcal{A}}$ be the family of all n-party access structures. Thus, $|\mathcal{F}_{\mathcal{A}}| = 2^{\Theta(2^n/\sqrt{n})}$. We next consider the family of degree-d polynomials as the family of reconstruction functions.

Fix a finite field $\mathbb{F}$, and consider shares of total size c, hence they contain $v = c/\log|\mathbb{F}|$ field elements. In this case the reconstruction function is a polynomial of degree $\leq d$ in v variables. There are at most $(v+1)^d$ monomials of degree $\leq d$ (for each of the d variables we choose either an element from the v shares or 1 for degree smaller than d), thus less than $|\mathbb{F}|^{(v+1)^d} = 2^{\log|\mathbb{F}|\cdot(c/\log|\mathbb{F}|+1)^d} \leq 2^{(c+1)^d}$ polynomials of degree $\leq d$ (as the reconstruction function can choose any coefficient in $\mathbb{F}$ for every monomial). If $|\mathbb{F}| > 2^{2^{n/(d+1)}}$, then the share size of every secret-sharing scheme over $\mathbb{F}$ is $> 2^{n/(d+1)}$ (since $\log|\mathbb{F}| \geq 2^{n/(d+1)}$). Thus, we only need to consider at most $2^{2^{n/(d+1)}}$ fields, and consider $\mathcal{F}_{\text{rec}}$ of size $2^{2^{n/(d+1)}} \cdot 2^{(c+1)^d}$. Thus, by Theorem 4.1, $(2^{n/(d+1)} + (c+1)^d) \cdot c \geq \Omega(2^n/\sqrt{n})$, so $c^{d+1} \geq 2^{n-o(n)}$ and $c \geq 2^{n/(d+1)-o(n)}$. □

Corollary 4.3. *For almost all k-input functions $f : [N]^k \to \{0,1\}$, the message size in any degree-d CDS protocol for them over any finite field with domain of secrets $\{0,1\}$ is $\Omega(N^{(k-1)/(d+1)}/k)$.*

The proof of Corollary 4.3 is similar to the proof of Corollary 4.2 when we use the fact that CDS protocol for a function $f : [N]^k \to \{0,1\}$ is equivalent to secret-sharing scheme for an access structure with kN parties (see e.g. [18,4]). The formal proof of Corollary 4.3 is given in the full version of this paper [17].

4.2 A Transformation from Secret Sharing with Degree-d Reconstruction into a Linear Secret Sharing

We start with a transformation from secret-sharing schemes with polynomial reconstruction to linear schemes. The idea of the transformation is to add random field elements to the randomness of the original polynomial scheme and generate new shares using these random elements, such that the reconstruction of the secret in the resulting scheme is a linear combination of the elements in the shares of the resulting scheme. In particular, for every monomial of degree at least two in a polynomial used for the reconstruction, we share the value of the monomial among the parties that have elements in the monomial. That is, the sharing function computes the polynomials instead of the reconstruction algorithm. As a corollary, we obtain a lower bound on the share size for schemes with polynomial reconstruction.

Lemma 4.4. *Let Γ be an n-party access structure, and assume that there exists a secret-sharing scheme Π_P realizing Γ over $\mathbb{F}$ with ℓ-elements secrets and degree-d reconstruction, in which the shares contain together c field elements. Then, there is a multi-linear secret-sharing scheme Π_L realizing Γ over $\mathbb{F}$ with ℓ-elements secrets, in which the share of each party contains $O(c^d)$ field elements. In particular, if the secret in Π_P contains one field element then Π_L is linear.*

Proof. To construct the desired scheme Π_L, the dealer first shares the secret according to scheme Π_P. Then, for every possible monomial $x_{i_1}^{\ell_1} \cdot \ldots \cdot x_{i_{d'}}^{\ell_{d'}}$ in the reconstruction of some authorized set such that $2 \leq \sum_{i=1}^{d'} \ell_i \leq d$, where x_{i_j} is a field element in the share of a party P_{i_j} for every $j \in [d']$, the dealer computes the value v of the monomial (using the shares that it creates) and shares v using a d'-out-of-d' secret-sharing scheme among the parties $P_{i_1}, \ldots, P_{i_{d'}}$ (i.e., the dealer chooses d' random field elements $r_{i_1}^v, \ldots, r_{i_{d'}}^v$ such that $v = r_{i_1}^v + \cdots + r_{i_{d'}}^v$).[7] Note that the randomness of scheme Π_L contains the random elements of scheme Π_P and the random elements $r_{i_1}^v, \ldots, r_{i_{d'-1}}^v$ for every possible monomial $x_{i_1}^{\ell_1} \cdot \ldots \cdot x_{i_{d'}}^{\ell_{d'}}$ of value v such that $2 \leq \sum_{i=1}^{d'} \ell_i \leq d$ as above (the dealer computes $r_{i_{d'}}^v = x_{i_1}^{\ell_1} \cdot \ldots \cdot x_{i_{d'}}^{\ell_{d'}} - r_{i_1}^v - \cdots - r_{i_{d'-1}}^v$).

We prove that the construction of Π_L realizes Γ and has linear reconstruction. By the equivalence between linear reconstruction and linear sharing (even for multi-element secrets), which is shown in the full version of this paper [17], Π_L can be converted to a secret-sharing scheme with linear sharing and reconstruction while preserving the share size.

We now prove the correctness of Π_L. For an authorized set $B \in \Gamma$, denote S_B as the field elements in the shares of B, and let

$$\mathrm{Recon}_{B,j}(S_B) = \sum_{x_i \in S_B} \alpha_{x_i} x_i + \sum_{\substack{x_{i_1}, \ldots, x_{i_{d'}} \in S_B, d' \leq d, \\ 2 \leq \ell_1 + \cdots + \ell_{d'} \leq d}} \alpha_{x_{i_1}^{\ell_1}, \ldots, x_{i_{d'}}^{\ell_{d'}}} x_{i_1}^{\ell_1} \cdot \ldots \cdot x_{i_{d'}}^{\ell_{d'}}$$

[7] If there is more than one element of some party in the monomial, the dealer can share the monomial among the parties that have elements in it, or give to such a party the sum of the shares that corresponding to its elements.

be the reconstruction function of B of the j-th element of the secret in scheme Π_P. Then, the set B can reconstruct the secret in scheme Π_L by applying the linear combination of the field elements in the shares of the parties as follows:

$$\sum_{x_i \in S_B} \alpha_{x_i} x_i + \sum_{\substack{x_{i_1},\ldots,x_{i_{d'}} \in S_B, d' \le d, \\ 2 \le \ell_1 + \cdots + \ell_{d'} \le d}} \alpha_{x_{i_1}^{\ell_1},\ldots,x_{i_{d'}}^{\ell_{d'}}} \sum_{j=1}^{d'} r_{i_j}^v$$
$$= \sum_{x_i \in S_B} \alpha_{x_i} x_i + \sum_{\substack{x_{i_1},\ldots,x_{d'\,i_{d'}} \in S_B, d' \le d, \\ 2 \le \ell_1 + \cdots + \ell_{d'} \le d}} \alpha_{x_{i_1}^{\ell_1},\ldots,x_{i_{d'}}^{\ell_{d'}}} x_{i_1}^{\ell_1} \cdot \ldots \cdot x_{i_{d'}}^{\ell_{d'}}.$$

We next prove the security of Π_L. Let T be an unauthorized set. For every authorized subset T' it must be that $T' \not\subseteq T$, thus, the set T misses at least one random field element $r_{i_j}^v$ from any monomial for the set T', so it cannot learn information on the value of these monomials, and hence cannot learn information on the secret from these values. In the scheme Π_L, the set T can only learn its shares in scheme Π_P, and every possible monomial of at most d variables that contains elements of those shares; these additional values can be computed from the original shares of T. Thus, in scheme Π_L, the set T learns only the information it can learn in scheme Π_P, and, hence, by the security of scheme Π_P, the set T cannot learn any information about the secret.

Finally, in scheme Π_L, each party gets at most c field elements from the share of scheme Π_P, and an element from the d'-out-of-d' secret-sharing scheme, for every monomial as above $x_{i_1}^{\ell_1} \cdot \ldots \cdot x_{i_{d'}}^{\ell_{d'}}$ such that $2 \le \sum_{i=1}^{d'} \ell_i \le d$; there are at most $\sum_{d'=2}^{d} c^{d'}$ such monomials. Overall, each party gets $c + \sum_{d'=2}^{d} c^{d'} = O(c^d)$ field elements. □

The above transformation gives us a lower bound on the share size of secret-sharing schemes with polynomial reconstruction, using any lower bound on the share size of linear secret-sharing schemes, as described next.

Corollary 4.5. *Assume that there exist an n-party access structure Γ such that the share size of at least one party in every linear secret-sharing scheme realizing Γ is c. Then, the share size of at least one party in every secret-sharing scheme realizing Γ with degree-d reconstruction is $\Omega(c^{1/d})$.*

Remark 4.6. Recall that the class NC^i contains all Boolean functions (or problems) that can be computed by polynomial-size Boolean circuits with gates with fan-in at most two and depth $O(\log^i n)$. Following the discussion in [16], the class of access structures that have a linear secret-sharing scheme with polynomial share size contains monotone NC^1 and is contained in algebraic NC^2 and in NC^3 for small enough fields (at most exponential in polynomial of the number of parties n). Lemma 4.4 implies that the class of access structures that have a secret-sharing scheme with polynomial reconstruction and polynomial share size is also contained in NC^3.

4.3 Lower Bounds for 1-Element Secrets for Explicit Access Structures

Now, let us recall the explicit lower bound of Pitassi and Robere [41] on the share size of linear secret-sharing schemes.

Theorem 4.7 ([41]). *There is a constant $\beta > 0$ such that for every n, there is an explicit n-party access structure Γ such that for every finite field $\mathbb{F}$, any linear secret-sharing scheme realizing Γ over $\mathbb{F}$ requires total share size of $\Omega(2^{\beta n} \log |\mathbb{F}|)$.*

The next explicit lower bound for secret-sharing schemes with polynomial reconstruction and one-element secrets follows directly from Corollary 4.5 when using Theorem 4.7.

Corollary 4.8. *There is a constant $\beta > 0$ such that for every n, there is an explicit n-party access structure Γ such that for every d and every finite field $\mathbb{F}$, any secret-sharing scheme realizing Γ over $\mathbb{F}$ with degree-d reconstruction and one-element secrets requires total share size of $\Omega(2^{\beta n/d} \log |\mathbb{F}|)$.*

Recall that the information ratio (or the normalized share size) is the ratio between the share size and the secret size. Corollary 4.8 provides a lower bound on the information ratio of an explicit access structure even for large finite fields. Corollary 4.2 provides a lower bound with a better constant in the exponent, however, it only applies to implicit access structures and does not give a non-trivial lower bound on the information ratio for large finite fields.

5 Quadratic CDS Protocols

In this section, we construct a quadratic k-server CDS protocol, i.e., a CDS protocol in which the encoding and decoding are computed by degree-2 polynomials. We start by describing a quadratic two-server CDS protocol (a variant of the quadratic two-server CDS protocol of [36]) and then construct a quadratic k-server CDS protocol that "simulates" the two-server CDS protocol.

A Quadratic Two-Server CDS Protocol. As a warm-up, we describe in Fig. 2 a two-server CDS protocol in which the encoding and the decoding are computed by polynomials of degree 2 over $\mathbb{F}_2$. This protocol is a variant of the protocol of [36] using a different notation (i.e., using cubes instead of polynomials).

Lemma 5.1. *Protocol Π_2, described in Fig. 2, is a quadratic two-server CDS protocol over $\mathbb{F}_2$ for the function* INDEX_N^2 *with message size $O(N^{1/3})$.*

Proof. We start with analyzing the value of the expression in (1). When $s = 0$, Bob sends $A_1 = S_1, A_2 = S_2$, and $A_3 = S_3$ to the referee. Thus, when $s = 0$, we get that $m^1_{i_1} = m_1 \oplus r_{1,i_1} \oplus r_1$, $m^2_{i_2} = m_2 \oplus r_{2,i_2} \oplus r_2$, and $m^3_{i_3} = m_3 \oplus r_{3,i_3} \oplus r_3$, and the value of the expression in (1) is

$$m_1 \oplus m_2 \oplus m_3 \oplus m^1_{i_1} \oplus r_{1,i_1} \oplus m^2_{i_2} \oplus r_{2,i_2} \oplus m^3_{i_3} \oplus r_{3,i_3} = r_1 \oplus r_2 \oplus r_3 = 0. \quad (2)$$

Protocol Π_2

- The secret: A bit $s \in \{0,1\}$.
- Alice holds a database $D \in \{0,1\}^N$ and Bob holds an index $i \in [N]$ viewed as (i_1, i_2, i_3) such that $i_1, i_2, i_3 \in [N^{1/3}]$.
- Common randomness: $S_1, S_2, S_3 \subseteq [N^{1/3}]$, $r_1, r_2 \in \{0,1\}$, and $3N^{1/3}$ bits $r_{1,j_1}, r_{2,j_2}, r_{3,j_3} \in \{0,1\}$ for every $j_1, j_2, j_3 \in [N^{1/3}]$.
- **The protocol**
  - Compute $r_3 = r_1 \oplus r_2$.
  - Alice computes $3N^{1/3}$ bits:
    - $m^1_{j_1} = \bigoplus_{j_2 \in S_2, j_3 \in S_3} D_{j_1,j_2,j_3} \oplus r_{1,j_1} \oplus r_1$ for every $j_1 \in [N^{1/3}]$.
    - $m^2_{j_2} = \bigoplus_{j_1 \in S_1, j_3 \in S_3} D_{j_1,j_2,j_3} \oplus r_{2,j_2} \oplus r_2$ for every $j_2 \in [N^{1/3}]$.
    - $m^3_{j_3} = \bigoplus_{j_1 \in S_1, j_2 \in S_2} D_{j_1,j_2,j_3} \oplus r_{3,j_3} \oplus r_3$ for every $j_3 \in [N^{1/3}]$.
  - Alice sends $(m^1_{j_1})_{j_1 \in [N^{1/3}]}$, $(m^2_{j_2})_{j_2 \in [N^{1/3}]}$, $(m^3_{j_3})_{j_3 \in [N^{1/3}]}$ to the referee.
  - Bob computes 3 strings $A_h = (A_h[1], \ldots, A_h[N^{1/3}])$ for $h \in \{1,2,3\}$ (each string of length $N^{1/3}$), where
    - $A_h[j_h] = S_h[j_h]$ for every $j_h \neq i_h$.
    - $A_h[i_h] = S_h[i_h] \oplus s$
      (that is, if $s = 0$ then $A_h = S_h$, otherwise $A_h = S_h \oplus \{i_h\}$).
  - Bob sends $r_{1,i_1}, r_{2,i_2}, r_{3,i_3}$, and A_1, A_2, A_3 to the referee.
  - The referee computes:
    $m_1 = \bigoplus_{j_2 \in A_2, j_3 \in A_3} D_{i_1,j_2,j_3}$, $\quad m_2 = \bigoplus_{j_1 \in A_1, j_3 \in A_3} D_{j_1,i_2,j_3}$,
    $m_3 = \bigoplus_{j_1 \in A_1, j_2 \in A_2} D_{j_1,j_2,i_3}$
    and outputs

$$m_1 \oplus m_2 \oplus m_3 \oplus m^1_{i_1} \oplus r_{1,i_1} \oplus m^2_{i_2} \oplus r_{2,i_2} \oplus m^3_{i_3} \oplus r_{3,i_3}. \qquad (1)$$

Fig. 2. A quadratic two-server CDS protocol Π_2 for the function INDEX^2_N.

When $s = 1$, Bob sends $A_1 = S_1 \oplus \{i_1\}, A_2 = S_2 \oplus \{i_2\}$, and $A_3 = S_3 \oplus \{i_3\}$ to the referee. We observe the following:

$$\begin{aligned} m_1 &= \left(\bigoplus_{j_2 \in S_2 \oplus \{i_2\}, j_3 \in S_3 \oplus \{i_3\}} D_{i_1,j_2,j_3} \right) \\ &= \left(\bigoplus_{j_2 \in S_2, j_3 \in S_3 \oplus \{i_3\}} D_{i_1,j_2,j_3} \right) \oplus \left(\bigoplus_{j_3 \in S_3 \oplus \{i_3\}} D_{i_1,i_2,j_3} \right) \\ &= \left(\bigoplus_{j_2 \in S_2, j_3 \in S_3} D_{i_1,j_2,j_3} \right) \oplus \left(\bigoplus_{j_2 \in S_2} D_{i_1,j_2,i_3} \right) \oplus \left(\bigoplus_{j_3 \in S_3} D_{i_1,i_2,j_3} \right) \oplus D_{i_1,i_2,i_3}. \end{aligned} \qquad (3)$$

Similarly,

$$m_2 = \left(\bigoplus_{j_1\in S_1, j_3\in S_3} D_{j_1,i_2,j_3}\right) \oplus \left(\bigoplus_{j_1\in S_1} D_{j_1,i_2,i_3}\right) \oplus \left(\bigoplus_{j_3\in S_3} D_{i_1,i_2,j_3}\right) \oplus D_{i_1,i_2,i_3}.$$

$$m_3 = \left(\bigoplus_{j_1\in S_1, j_2\in S_2} D_{j_1,j_2,i_3}\right) \oplus \left(\bigoplus_{j_1\in S_1} D_{j_1,i_2,i_3}\right) \oplus \left(\bigoplus_{j_2\in S_2} D_{i_1,j_2,i_3}\right) \oplus D_{i_1,i_2,i_3}.$$

Therefore,

$$m_1 \oplus m_2 \oplus m_3 = \left(\bigoplus_{j_2\in S_2, j_3\in S_3} D_{i_1,j_2,j_3}\right) \oplus \left(\bigoplus_{j_1\in S_1, j_3\in S_3} D_{j_1,i_2,j_3}\right) \oplus \left(\bigoplus_{j_1\in S_1, j_2\in S_2} D_{j_1,j_2,i_3}\right) \oplus D_{i_1,i_2,i_3}.$$

Thus, when $s = 1$, the value of the expression in (1) is

$$m_1 \oplus m_2 \oplus m_3 \oplus m^1_{i_1} \oplus r_{1,i_1} \oplus m^2_{i_2} \oplus r_{2,i_2} \oplus m^3_{i_3} \oplus r_{3,i_3} \oplus r_1 \oplus r_2 \oplus r_3 = D_{i_1,i_2,i_3}. \quad (4)$$

Correctness. We next prove the correctness of the protocol, that is, when $D_{i_1,i_2,i_3} = 1$ the referee correctly reconstructs s. Recall that the output of the referee is the expression in (1). As explained above, when $s = 0$ the referee outputs 0 and when $s = 1$ the referee outputs $D_{i_1,i_2,i_3} = 1$.

Security. Fix inputs D and $i = (i_1, i_2, i_3)$ such that $D_{i_1,i_2,i_3} = 0$, a message of Alice $(m^1_{j_1})_{j_1\in[N^{1/3}]}$, $(m^2_{j_2})_{j_2\in[N^{1/3}]}$, $(m^3_{j_3})_{j_3\in[N^{1/3}]}$, and a message of Bob $A_1, A_2, A_3, r_{1,i_1}, r_{2,i_2}, r_{3,i_3}$ such that

$$\bigoplus_{j_2\in A_2, j_3\in A_3} D_{i_1,j_2,j_3} \oplus \bigoplus_{j_1\in A_1, j_3\in A_3} D_{j_1,i_2,j_3} \oplus \bigoplus_{j_1\in A_1, j_2\in A_2} D_{j_1,j_2,i_3} \oplus m^1_{i_1} \oplus r_{1,i_1} \oplus m^2_{i_2} \oplus r_{2,i_2} \oplus m^3_{i_3} \oplus r_{3,i_3} = 0 \quad (5)$$

(no other restrictions are made on the messages). By (2) and (4), when $D_{i_1,i_2,i_3} = 0$ only such messages are possible. We next argue that the referee cannot learn any information about the secret given these inputs and messages, i.e., these messages have the same probability when $s = 0$ and when $s = 1$. In particular, we show that for every secret $s \in \{0, 1\}$ there is a unique common random string r such that Alice and Bob send these messages with the secret s. We define the common random string r as follows:

- For $h \in \{1, 2, 3\}$, define $S_h = A_h$ if $s = 0$ and $S_h = A_h \oplus \{i_h\}$ if $s = 1$. These S_1, S_2, S_3 are consistent with the message of Bob and s and are the

only consistent choice. Both when $s = 0$ and $s = 1$, as $D_{i_1,i_2,i_3} = 0$, it holds that

$$\bigoplus_{j_2 \in A_2, j_3 \in A_3} D_{i_1,j_2,j_3} \oplus \bigoplus_{j_1 \in A_1, j_3 \in A_3} D_{j_1,i_2,j_3} \oplus \bigoplus_{j_1 \in A_1, j_2 \in A_2} D_{j_1,j_2,i_3}$$
$$= \bigoplus_{j_2 \in S_2, j_3 \in S_3} D_{i_1,j_2,j_3} \oplus \bigoplus_{j_1 \in S_1, j_3 \in S_3} D_{j_1,i_2,j_3} \oplus \bigoplus_{j_1 \in S_1, j_2 \in S_2} D_{j_1,j_2,i_3}. \quad (6)$$

This is true since when $s = 0$ the sets A_1, A_2, A_3 are the same as the sets S_1, S_2, S_3, and when $s = 1$, by (4), the two sides of the expression are differ by D_{i_1,i_2,i_3} which is 0.

- The message of Bob determines r_{1,i_1}, r_{2,i_2}, and r_{3,i_3}.
- Define

$$r_1 = m^1_{i_1} \oplus \bigoplus_{j_2 \in S_2, j_3 \in S_3} D_{i_1,j_2,j_3} \oplus r_{1,i_1} \quad (7)$$

$$r_2 = m^2_{i_2} \oplus \bigoplus_{j_1 \in S_1, j_3 \in S_3} D_{j_1,i_2,j_3} \oplus r_{2,i_2}. \quad (8)$$

  Given the secret s, the inputs, and the messages of Alice and Bob, these values are possible and unique.
- Define $r_3 = r_1 \oplus r_2$. By (5), (6), (7), and (8), this value is possible, i.e., it satisfies

$$m^3_{i_3} = \bigoplus_{j_1 \in S_1, j_2 \in S_2} D_{j_1,j_2,i_3} \oplus r_{3,i_3} \oplus r_3.$$

- For every $j_1 \neq i_1, j_2 \neq i_2$, and $j_3 \neq i_3$ define

$$r_{1,j_1} = m^1_{j_1} \oplus \bigoplus_{j_2 \in S_2, j_3 \in S_3} D_{i_1,j_2,j_3} \oplus r_1,$$

$$r_{2,j_2} = m^2_{j_2} \oplus \bigoplus_{j_1 \in S_1, j_3 \in S_3} D_{j_1,i_2,j_3} \oplus r_2,$$

$$r_{3,j_3} = m^3_{j_3} \oplus \bigoplus_{j_1 \in S_1, j_2 \in S_2} D_{j_1,j_2,i_3} \oplus r_3.$$

  Given the secret s, the inputs, and the messages of Alice and Bob, these values are possible and unique.

Recall that the common random string is uniformly distributed (i.e., the probability of each such string is $1/2^{6N^{1/3}+2}$, as it contains $6N^{1/3} + 2$ bits). Since for every pair of messages of Alice and Bob when $D_{i_1,i_2,i_3} = 0$ we have that every secret s has exactly one consistent random string, this pair has the same probability when $s = 0$ and when $s = 1$ and the security follows.

Message Size. Alice sends $3N^{1/3}$ bits and Bob sends 3 strings each of size $N^{1/3}$ and 3 random bits, so the message size is $O(N^{1/3})$.

Degree of the Protocol. The message of Alice contains an exclusive or of bits of a 3-dimension cubes, where two dimensions are determined by the

common randomness (the sets S_1, S_2, S_3). That is, when we represent a set $S \subseteq [N^{1/3}]$ by $N^{1/3}$ bits $S = (S[1], \ldots, S[N^{1/3}])$, then for every $j_1 \in [N^{1/3}]$

$$m^1_{j_1} = \bigoplus_{j_2 \in [N^{1/3}], j_3 \in [N^{1/3}]} S_2[j_2] \cdot S_3[j_3] \cdot D_{j_1,j_2,j_3} \oplus r_{1,j_1} \oplus r_1.$$

Thus, $m^1_{j_1}$, for every input D, is a polynomial of degree 2 over $\mathbb{F}_2$ whose variables are the bits of the random string. Similarly, $m^2_{j_2}$, $m^3_{j_3}$ are polynomials of degree 2 over $\mathbb{F}_2$. The message of Bob for every $j_h \neq i_h$ contains a polynomial of degree 1 over $\mathbb{F}_2$, since it sends $S_h[j_h]$. For the index $i_h \in [N^{1/3}]$, Bob sends $S_h[i_h] \oplus s$, which is a polynomial of degree 1 over $\mathbb{F}_2$. The decoding is also a computation of a 3-dimension cube such that only two dimensions are determined by the common randomness, i.e., the decoding is a degree-2 polynomial over $\mathbb{F}_2$. □

An Auxiliary Protocol $\boldsymbol{\Pi_{\mathbf{XOR}}}$. In Fig. 4, we will describe a k-server CDS protocol, where servers $Q_2, \ldots, Q_k$ simulate Bob in the two-server CDS protocol. To construct this protocol, we design a k-server protocol Π_{XOR} that simulates Bob, i.e., sends a set A, where $A = S$ if $s = 0$ and $A = S \oplus \{i\}$ if $s = 1$. In Π_{XOR}, each server Q_ℓ holds an index i_ℓ, which together determine an index $i = (i_1, i_2, \ldots, i_k)$, and they need to send messages to the referee such that the referee will learn A without learning any information on s. Let $N_1, \ldots, N_k$ be integers and $N = N_1 \cdot \ldots \cdot N_k$. We construct the following protocol in which server Q_1 holds a set $S \subseteq [N]$ represented by a k-dimensional Boolean array $(Sj_1, \ldots, j_k)_{j_1 \in [N_1], \ldots, j_k \in [N_k]}$, the secret s, and an index $i_1 \in [N_1]$. Server Q_ℓ for $2 \leq \ell \leq k$ holds an index $i_\ell \in [N_\ell]$. If $s = 1$, the referee outputs $S \oplus \{(i_1, i_2, \ldots, i_k)\}$ and if $s = 0$ it outputs S (without learning any information on s). Define the function[8]

$$f_{\mathrm{XOR}}(S, s, i_1, \ldots, i_k) = \begin{cases} i_1, i_2, \ldots, i_k, S & \text{If } s = 0, \\ i_1, i_2, \ldots, i_k, S \oplus \{(i_1, i_2, \ldots, i_k)\} & \text{If } s = 1. \end{cases}$$

We next define when a protocol for f_{XOR} is secure. This is a special case of security of private simultaneous messages (PSM) protocols [25,30], that is, we require that for every two inputs for which f_{XOR} outputs the same value, the distribution of messages is the same. Observe that every possible output of f_{XOR} results from exactly two inputs.

Definition 5.2. *We say that a protocol for f_{XOR} is secure if for every $i_1 \in [N_1], \ldots, i_k \in [N_k]$, and every S, the distributions of messages of the protocol on inputs $S, s = 0, i_1, \ldots, i_k$ and inputs $S \oplus \{(i_1, i_2, \ldots, i_k)\}, s = 1, i_1, \ldots, i_k$ are the same.*

The protocol Π_{XOR} for f_{XOR} is described in Fig. 3. Next we present a high level description of the protocol. Server Q_1 sends to the referee three arrays: A, A^0, A^1. The array A contains all the indices for which Q_1 knows that S and

[8] We include $i_1, \ldots, i_k$ in the output of f_{XOR} to be consistent with PSM protocols, in which the referee does not know the input.

The protocol Π_{XOR}

- Input: Q_1 holds an array $S = (S_{j_1,\ldots,j_k})_{j_1\in[N_1],\ldots,j_k\in[N_k]}$, a bit $s \in \{0,1\}$, and $i_1 \in [N_1]$, and Q_ℓ, for every $2 \le \ell \le k$, holds an index $i_\ell \in [N_\ell]$. The referee holds $i_1, \ldots, i_k$.
- Output: An array $A = (A_{j_1,\ldots,j_k})_{j_1\in[N_1],\ldots,j_k\in[N_k]}$ s.t. $A_{j_1,\ldots,j_k} = S_{j_1,\ldots,j_k}$ for every $(j_1,\ldots,j_k) \neq (i_1,\ldots,i_k)$ and $A_{i_1,\ldots,i_k} = S_{i_1,\ldots,i_k} \oplus s$.
- Common randomness: $r_{j_2,\ldots,j_k,\ell} \in \{0,1\}$ for every $j_2 \in [N_2], \ldots, j_k \in [N_k]$ and every $\ell \in \{1,\ldots,k\}$.
- **The protocol**
  - Q_1 computes an $(N_1 - 1) \times N_2 \times \ldots \times N_k$ array A and two $1 \times N_2 \times \ldots \times N_k$ arrays A^0 and A^1.
    - $A_{j_1,\ldots,j_k} = S_{j_1,\ldots,j_k}$ for every $j_1 \in [N_1] \setminus \{i_1\}, j_2 \in [N_2], \ldots, j_k \in [N_k]$.
    - $A^0_{i_1,j_2,\ldots,j_k} = S_{i_1,j_2,\ldots,j_k} \oplus r_{j_2,\ldots,j_k,1}$ for every $j_2 \in [N_2], \ldots, j_k \in [N_k]$.
    - $A^1_{i_1,j_2,\ldots,j_k} = S_{i_1,j_2,\ldots,j_k} \oplus r_{j_2,\ldots,j_k,2} \oplus \cdots \oplus r_{j_2,\ldots,j_k,k} \oplus s$ for every $j_2 \in [N_2], \ldots, j_k \in [N_k]$.
  - Q_1 sends A, A^0, A^1.
  - Q_ℓ, for every $2 \le \ell \le k$, sends $r_{j_2,\ldots,j_k,1}$ for every $(j_2,\ldots,j_k) \in [N_2]\times\cdots\times[N_k]$ such that $j_\ell \neq i_\ell$, and $r_{j_2,\ldots,j_k,\ell}$ for every $(j_2,\ldots,j_k) \in [N_2] \times \cdots \times [N_k]$ such that $j_\ell = i_\ell$.
  - The referee completes A to an $N_1 \times N_2 \times \ldots \times N_k$ array as follows
    - $A_{i_1,i_2,\ldots,i_k} = A^1_{i_1,i_2,\ldots,i_k} \oplus r_{i_2,\ldots,i_k,2} \oplus \cdots \oplus r_{i_2,\ldots,i_k,k}$.
    - $A_{i_1,j_2,\ldots,j_k} = A^0_{i_1,j_2,\ldots,j_k} \oplus r_{j_2,\ldots,j_k,1}$ for every $(j_2,\ldots,j_k) \neq (i_2,\ldots,i_k)$.
  - The referee returns A.

Fig. 3. The protocol Π_{XOR} for the function f_{XOR}.

A are equal (i.e., indices $j_1,\ldots,j_k$ where $j_1 \neq i_1$, so $A_{j_1,\ldots,j_k} = S_{j_1,\ldots,j_k}$), the array A^0 enables the referee to compute $A_{i_1,j_2,\ldots,j_k}$ for all the indices for which there is at least one $j_\ell \neq i_\ell$ for some $2 \le \ell \le k$, and the array A^1 enables the referee to compute $A_{i_1,\ldots,i_k}$.

Lemma 5.3. *Protocol Π_{XOR} is a correct and secure protocol for f_{XOR} with message size $O(N_1 \cdot \ldots \cdot N_k)$. The degree of the message generation and output reconstruction in the protocol (as a function of the randomness and the input S) is 1 over $\mathbb{F}_2$.*

The proof of Lemma 5.3 appears in full version of this paper [17].

The k-Server CDS Protocol. Now we present our k-server CDS protocol for the function INDEX^k_N, assuming that $k \equiv 1 \pmod 3$. The case of $k \not\equiv 1 \pmod 3$ is somewhat more messy and can be handled as done in [18].

We next present an overview of our construction. The input of the protocol is a database $D \in \{0,1\}^{N^{k-1}}$ held by Q_1 and an index $i \in [N]^{k-1}$

The protocol Π_k

- The secret: A bit $s \in \{0,1\}$.
- Q_1 holds a database $D \in \{0,1\}^{N^{k-1}}$, and $Q_2, \ldots, Q_k$ hold $x_2, \ldots, x_k \in [N]$, respectively.
- Common randomness: $S_1, S_2, S_3 \subseteq [N^{(k-1)/3}]$, $r_1, r_2 \in \{0,1\}$, $r_{h,j,1}, \ldots, r_{h,j,(k-1)/3} \in \{0,1\}$ for every $h \in \{1,2,3\}$ and every $j \in [N^{(k-1)/3}]$, and the common randomness of three independent executions of protocol Π_{XOR}.
- **The protocol**
  - Let:
    - $i_h^\ell = x_{1+(h-1)(k-1)+\ell}$ for every $h \in \{1,2,3\}$ and every $1 \le \ell \le (k-1)/3$.
    - $r_3 = r_1 \oplus r_2$.
  - Q_1 computes $3N^{(k-1)/3}$ bits:
    - $m^1_{j_1} = \bigoplus_{j_2 \in S_2, j_3 \in S_3} D_{j_1,j_2,j_3} \oplus r_{1,j_1,1} \oplus \cdots \oplus r_{1,j_1,(k-1)/3} \oplus r_1$ for every $j_1 \in [N^{(k-1)/3}]$.
    - $m^2_{j_2} = \bigoplus_{j_1 \in S_1, j_3 \in S_3} D_{j_1,j_2,j_3} \oplus r_{2,j_2,1} \oplus \cdots \oplus r_{2,j_2,(k-1)/3} \oplus r_2$ for every $j_2 \in [N^{(k-1)/3}]$.
    - $m^3_{j_3} = \bigoplus_{j_1 \in S_1, j_2 \in S_2} D_{j_1,j_2,j_3} \oplus r_{3,j_3,1} \oplus \cdots \oplus r_{3,j_3,(k-1)/3} \oplus r_3$ for every $j_3 \in [N^{(k-1)/3}]$.
  - Q_1 sends $(m^1_{j_1})_{j_1 \in [N^{(k-1)/3}]}$, $(m^2_{j_2})_{j_2 \in [N^{(k-1)/3}]}$, $(m^3_{j_3})_{j_3 \in [N^{(k-1)/3}]}$ to the referee.
  - $Q_{2+(h-1)(k-1)/3}, \ldots, Q_{1+h(k-1)/3}$, for every $h \in \{1,2,3\}$, execute Π_{XOR} with the set S_h held by $Q_{2+(h-1)(k-1)/3}$, the secret s, and i_h^ℓ held by $Q_{1+(h-1)(k-1)/3+\ell}$. Let $m^h_{\mathrm{xor},1}, \ldots, m^h_{\mathrm{xor},(k-1)/3}$ be the messages sent in this execution of Π_{XOR}.
  - Q_ℓ, for every $2 \le \ell \le k$:
    - Computes $h = \lfloor 3\ell/(k-1) \rfloor$ and $\alpha = \ell - 1 - (h-1)(k-1)/3$.
    - Sends $m^h_{\mathrm{xor},\alpha}$, and for every $j = (j_1, \ldots, j_{(k-1)/3}) \in [N^{(k-1)/3}]$ such that $j_\alpha = i_h^\alpha$, sends $r_{h,j,\alpha}$.
  - The referee computes:
    - A_h, for every $h \in \{1,2,3\}$, from the messages $m^h_{\mathrm{xor},1}, \ldots, m^h_{\mathrm{xor},(k-1)/3}$ of Π_{XOR}.
    - $r_{h,i_h} = r_{h,i_h,1} \oplus r_{h,i_h,2} \oplus \cdots \oplus r_{h,i_h,(k-1)/3}$, for every $h \in \{1,2,3\}$.
    - $m_1 = \bigoplus_{j_2 \in A_2, j_3 \in A_3} D_{i_1,j_2,j_3}$, $m_2 = \bigoplus_{j_1 \in A_1, j_3 \in A_3} D_{j_1,i_2,j_3}$, $m_3 = \bigoplus_{j_1 \in A_1, j_2 \in A_2} D_{j_1,j_2,i_3}$

    and outputs

$$m_1 \oplus m_2 \oplus m_3 \oplus m^1_{i_1} \oplus r_{1,i_1} \oplus m^2_{i_2} \oplus r_{2,i_2} \oplus m^3_{i_3} \oplus r_{3,i_3}. \quad (9)$$

Fig. 4. A quadratic k-server CDS protocol Π_k for the function INDEX^k_N.

jointly held by $Q_2, \ldots, Q_k$. The input $i \in [N]^{k-1}$ is viewed as (i_1, i_2, i_3) where $i_1, i_2, i_3 \in [N^{(k-1)/3}]$, where i_h, for $h \in \{1,2,3\}$, contains the inputs of servers

$Q_{2+(h-1)(k-1)/3}, \ldots, Q_{1+h(k-1)/3}$. The common randomness contains three random subsets, one for each dimension, i.e., $S_1, S_2, S_3 \subseteq [N^{(k-1)/3}]$. In the protocol, we want that the referee will be able to compute $S_1 \oplus \{i_1\}, S_2 \oplus \{i_2\}$, and $S_3 \oplus \{i_3\}$ when $s = 1$, and S_1, S_2, S_3 when $s = 0$ (as in the protocol Π_2 described in Fig. 2). For this task, we use the protocol Π_{XOR}. Servers $Q_2, \ldots, Q_{1+(k-1)/3}$ execute the protocol Π_{XOR} in order to generate messages that enable the referee to learn $S_1 \oplus \{i_1\}$ when $s = 1$ and S_1 when $s = 0$. Similarly, servers $Q_{2+(k-1)/3}, \ldots, Q_{1+2(k-1)/3}$ and servers $Q_{2+2(k-1)/3}, \ldots, Q_k$ independently execute the protocol Π_{XOR} in order to generate messages that enable the referee to learn $S_2 \oplus \{i_2\}$ when $s = 1$ and S_2 when $s = 0$ and $S_3 \oplus \{i_3\}$ when $s = 1$ and S_3 when $s = 0$, respectively. In addition, we want the referee to learn the bits $r_{1,i_1}, r_{2,i_2}, r_{3,i_3}$ as in Π_2. To achieve this goal, we define $r_{h,j,1} \ldots, r_{h,j,(k-1)/3}$ for every $j \in [N^{(k-1)/3}]$ and every $h \in \{1, 2, 3\}$, such that $r_{h,j,1} \oplus \cdots \oplus r_{h,j,(k-1)/3} = r_{h,j}$.

Theorem 5.4. *Protocol Π_k, described in Fig. 4, is a quadratic k-server CDS protocol over $\mathbb{F}_2$ for the function* INDEX_N^k *with message size $O(N^{(k-1)/3})$.*

The proof of Theorem 5.4 appears in full version of this paper [17].

Corollary 5.5. *Every function $f : [N]^k \to \{0, 1\}$ has a quadratic k-server CDS protocol over $\mathbb{F}_2$ with message size $O(N^{(k-1)/3})$.*

6 A Quadratic Robust CDS Protocol

In this section, we construct a quadratic k-server t-RCDS protocol, which is a CDS protocol in which the referee gets no information on the secret even if each server sends messages on multiple inputs with the same common randomness.

6.1 An Improved Analysis of the Transformation of [5]

We first show an improved analysis of the transformation of [5] from t'-RCDS protocols to t-RCDS protocols for $t' < t$; in particular, from CDS protocols (i.e., $t' = 1$) to t-RCDS protocols. In the transformation of [5], the servers independently execute $O(t^{k-1})$ copies of the underlying RCDS protocol for $f : [N]^k \to \{0, 1\}$. This is done in a way that ensures that even if a server sends messages of many inputs, in at least some of the executions of the underlying RCDS protocol the referee gets messages of few inputs. We observe that the input domain in each execution of the underling RCDS is $[N/t]$ (as opposed to $[N]$), and this will reduce the total message size. In Lemma 6.2, we present the improved analysis.

We start with an overview of the ideas behind our analysis. Following the construction of the linear two-server RCDS protocol in [6] (the full version of [5]), when making a server Q_i robust, we divide the domain of inputs of Q_i using a hash function $h : [N] \to [v]$ (actually we do this for several hash functions, as will be explained later); for every $b \in [v]$, the servers execute the underlying

CDS protocol where the input of Q_i is restricted to the inputs $\{x_i : h(x_i) = b\}$. We next define families of hash functions that we use in the transformation.

Definition 6.1 (Families of m'-Collision-Free Hash Functions). *A set of functions $\mathcal{H}_{N,m,m',v} = \{h_d : [N] \to [v] : d \in [\ell]\}$ (where ℓ is the number of functions in the family) is a* family of m'-collision-free hash functions *if for every set $T \in \binom{N}{[m]}$ there exists at least one function $h \in \mathcal{H}_{N,m,m',v}$ for which for every $b \in [v]$ it holds that $|\{x \in T : h(x) = b\}| \leq m'$, that is, h restricted to T is at most m'-to-one. A family $\mathcal{H}_{N,m,1,v}$ is a* family of perfect hash functions *if it is a family of 1-collision-free hash functions. A family $\mathcal{H}_{N,m,m',v}$ is* output-balanced *if $|\{x \in [N] : h(x) = a\}| \leq \lceil N/v \rceil$ for every $a \in [v]$ and $h \in \mathcal{H}_{N,m,m',v}$, i.e., each h divides $[N]$ to v sets of almost the same size.*

Lemma 6.2. *Let $f : [N]^k \to \{0,1\}$ be a k-input function and t and t' be integers such that $t' < t \leq N$. Assume that there is a k-server t'-RCDS protocol $\mathcal{P}'$ for f, in which for every $N' \leq N$ and for every restriction of f with input domain $A_1 \times \ldots, \times A_k$, where $A_i \subseteq [N]$ is of size N' for $1 \leq i \leq k$, the message size is $c(N')$. In addition, assume that there is a family of an output-balanced t'-collision-free hash functions $\mathcal{H}_{N,kt,t',v} = \{h_1, \ldots, h_\ell\}$ of size ℓ. Then, there is a k-server t-RCDS protocol $\mathcal{P}$ for f with message size $O(\ell v^{k-1} \cdot c(N/v))$. This transformation preserves the degree of the encoding and the decoding of the underlying RCDS protocol.*

Proof. The desired protocol $\mathcal{P}$ is described in Fig. 5. This is actually the transformation of [5] with the following difference. Instead of executing $\mathcal{P}'$ with domain of inputs of size N per server, we execute it with a restriction of f with domain of inputs of size $\lceil N/v \rceil$ per server.[9] The correctness and robustness of the protocol follows from the proof of the transformation of [5].

Next, we analyze the message size. Observe that for each $h \in \mathcal{H}_{N,kt,t',v}$, each server sends messages in v^{k-1} copies of $\mathcal{P}'$, where each copy is for a restriction of f with input domain of size $\max_{a \in [v]} |S_a|$ per server, where $S_a = \{x \in [N] : h(x) = a\}$. Since $\mathcal{H}_{N,kt,t',v}$ is output balanced, it holds that $\max_{a \in [v]} |S_a| \leq \lceil N/v \rceil$ and since $|\mathcal{H}_{N,kt,t',v}| = \ell$, the message size is $O(\ell v^{k-1} \cdot c(\lceil N/v \rceil))$. We next argue that the degree of the encoding and decoding in the transformation does not change when S is the additive group of the field in the protocol $\mathcal{P}'$ (see Fig. 5). In the encoding, the servers execute a linear operation on the secret and the field elements $s_1, \ldots, s_{\ell-1}$ in order to generate s_ℓ. Then, they encode each s_d by executing the underlying RCDS protocol. That is, the encoding is computed by the degree-d polynomials that compute the encoding in the underlying RCDS protocol. For the decoding, the referee first executes the decoding procedure of the underlying RCDS protocol in order to learn $s_1, \ldots, s_\ell$ and then by summing them up the referee learns the secret. That is, the decoding is actually summing up the degree-d polynomials that compute the decoding of the ℓ copies of the underlying RCDS protocol. Therefore, the degree of the encoding and the decoding of the transformation are the same as for the underlying RCDS protocol. □

[9] in [5], they do not deal with restrictions of the domain of inputs since it does not improve the asymptotic message size of their protocols.

A t-RCDS protocol

The secret: $s \in S$, where, w.l.o.g., S is a group (e.g., $S = \mathbb{Z}_m$ for some m).
The protocol

- Choose $\ell - 1$ random elements $s_1, \ldots, s_{\ell-1} \in S$ and let $s_\ell = s - (s_1 + \cdots + s_{\ell-1})$ (addition is in the group).
- For every $d \in [\ell]$:
  - Let $S_a = \{x \in [N] : h_d(x) = a\}$ for every $a \in [v]$.
  - For every $a_1, \ldots, a_k \in [v]$, independently execute the k-server t'-RCDS protocol $\mathcal{P}'$ for the restriction of f to $S_{a_1} \times \cdots \times S_{a_k}$ with the secret s_d, that is, for every $i \in [k]$, server Q_i with input x_i sends a message for the restriction of f to $S_{a_1} \times \cdots \times S_{a_{i-1}} \times S_{h_d(x_i)} \times S_{a_{i+1}} \times \cdots \times S_{a_k}$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \in [v]$.

Fig. 5. A transformation of a t'-RCDS protocol to a t-RCDS protocol for $t' < t$.

6.2 The Construction of the Quadratic t-RCDS Protocol

We next construct a quadratic k-server t-RCDS protocol. Our construction uses the improved analysis in Lemma 6.2 of the transformation of [5] for converting a t'-RCDS protocol into a t-RCDS protocol for $t' < t$. Applying the transformation of [5] without our improved analysis starting from our quadratic k-server CDS protocol in Theorem 5.4 will result in a quadratic k-server t-RCDS protocol with message size $\tilde{O}(N^{(k-1)/3}t^{k-1})$. Using our improved analysis, we get better message size of $\tilde{O}(N^{(k-1)/3}t^{2(k-1)/3+1})$.

We start by quoting the following two lemmas that we use in order to instantiate Lemma 6.2. Both lemmas can be proved by a simple probabilistic argument. Their proofs can be found in [39].

Lemma 6.3. *Let N be an integer and $m \in [\sqrt{N}]$. Then, there exists an output-balanced family of perfect hash functions $\mathcal{H}_{N,m,1,m^2} = \{h_i : [N] \to [m^2] : i \in [\ell]\}$, where $\ell = 16m \ln N$.*

Lemma 6.4. *Let N be an integer and $m \in \{15, \ldots, N/2\}$. Then, there exists an output-balanced family of $\log m$-collision-free hash functions $\mathcal{H}_{N,m,\log m,2m} = \{h_i : [N] \to [2m] : i \in [\ell]\}$, where $\ell = 16m \ln N$.*

Theorem 6.5. *Let $t < \min\left\{N/2k, 2^{\sqrt{N}/k}\right\}$. Then, there is a quadratic k-server t-RCDS protocol over $\mathbb{F}_2$ with message size*

$$O(N^{(k-1)/3}t^{2(k-1)/3+1} \cdot k^{2k} \cdot \log^2 N \cdot \log^{(4k-1)/3} t) = \tilde{O}(N^{(k-1)/3}t^{2(k-1)/3+1} \cdot k^{2k}).$$

Proof. Similarly to [5], we construct the protocol in two stages. In the first stage, we transform our quadratic k-server CDS protocol from Fig. 4 into a quadratic

k-server $\log t$-RCDS protocol, and then, in the second stage, we transform this protocol into a quadratic k-server t-RCDS protocol.

For the first stage, we use the output-balanced family $\mathcal{H}_{N,k\log t,1,k^2\log^2 t}$ of perfect hash functions with $O(k\log t\log N)$ hash functions promised by Lemma 6.3. Applying the transformation of Lemma 6.2 with $\mathcal{H}_{N,k\log t,1,k^2\log^2 t}$ and our quadratic (non-robust) k-server CDS protocol described in Theorem 5.4 as the underlying protocol (this protocol has message size $O(N^{(k-1)/3})$) results in a quadratic k-server $\log t$-RCDS protocol, which we denote by $\mathcal{P}'$, with message size $c'(N) = O(N^{(k-1)/3}\cdot(k\log t)^{(4k-1)/3}\cdot\log N)$.

For the second stage, we apply Lemma 6.2 with the $\log t$-RCDS protocol $\mathcal{P}'$ and the output-balanced family of $(\log t)$-collision-free hash functions, denoted by $\mathcal{H}_{N,kt,\log t,2kt}$ with $O(kt\log N)$ hash functions promised by Lemma 6.4; therefore, we get message size of

$$O(kt\log N\cdot(2kt)^{k-1}\cdot c'(N/2kt)) = O(N^{(k-1)/3}t^{\frac{2(k-1)}{3}+1}\cdot k^{2k}\cdot\log^2 N\cdot\log^{\frac{4k-1}{3}} t).$$

□

7 A Quadratic Secret Sharing for General Access Structures

In this section, we use our results described in Section 5 and Section 6.2 to construct improved quadratic secret-sharing schemes. Our upper bounds are better than the best known upper bounds for linear schemes. In addition, our upper bounds imply a separation between quadratic and linear secret-sharing schemes for almost all access structures.

A Construction for All Access Structures. Next we use our quadratic k-server RCDS protocol in the construction of general secret sharing of [8].

Theorem 7.1 (Implied by [8]). *Assume that for every function $f:[N]^k\to\{0,1\}$ there is a k-server t-RCDS protocol with message size $c(k,N,t)$, then there is a secret-sharing scheme realizing an arbitrary n-party access structure with share size*

$$\max\left\{\max_{0<\beta\le 0.5} c(\sqrt{n},2^{\sqrt{n}},2^{\beta\sqrt{n}}),\right.$$
$$\left.\max_{0.5<\beta\le 1} c\left(\sqrt{2n(1-\beta)},2^{\sqrt{2n(1-\beta)}},2^{\sqrt{n(1-\beta)/2}}\right)\cdot 2^{H_2(\beta)n-2(1-\beta)n}\right\}\cdot 2^{o(n)}.$$

Furthermore, the degree of sharing and reconstruction of this secret-sharing scheme is the degree of encoding and decoding, respectively, of the underlying RCDS protocol.

In the construction of [8], they use a t-RCDS protocol that is robust only for some of the subsets of size t (rather than all subsets). In our construction, we can avoid the more complex definition of robustness and use a t-RCDS protocol that is robust against all subsets of size at most t.

Theorem 7.2. *Every n-party access structure can be realized by a quadratic secret-sharing scheme over $\mathbb{F}_2$ with share size $2^{0.705n+o(n)}$.*

Proof. The theorem follows from Theorem 7.1 using our quadratic t-RCDS protocol with message size $\tilde{O}(N^{(k-1)/3}t^{2(k-1)/3+1} \cdot k^{2k})$ from Theorem 6.5. We get share size

$$\max\left\{\max_{0<\beta\leq 0.5} 2^{n(2\beta+1)/3}, \max_{0.5<\beta\leq 1} 2^{H_2(\beta)n-2/3(1-\beta)n}\right\} \cdot 2^{o(n)}.$$

The maximum value of this expression is at $\beta \approx 0.613512$ and it is $2^{0.705n}$. □

In comparison, Applebaum and Nir [8] construct a linear secret-sharing scheme over $\mathbb{F}_2$ with share size $2^{0.7576n+o(n)}$ and a general (non-polynomial) secret-sharing scheme with share size $2^{0.585n+o(n)}$.

A Construction for Almost All Access Structures. It was shown in [14] that almost all access structures can be realized by a general secret-sharing scheme with shares of size $2^{o(n)}$ and by a linear secret-sharing scheme with share size $2^{n/2+o(n)}$. Furthermore, it was shown in [11] that almost all access structures require share size $2^{n/2-o(n)}$ in any linear secret-sharing scheme even with 1-bit secrets over all finite fields $\mathbb{F}_q$. Following [14], we show that almost all access structures can be realized by a quadratic secret-sharing scheme with 1-bit secrets over $\mathbb{F}_2$ with share size $2^{n/3+o(n)}$, proving a separation between quadratic and linear schemes for almost all access structures.

Theorem 7.3. *Almost all access structures can be realized by a quadratic secret-sharing scheme with 1-bit secrets over $\mathbb{F}_2$ and with share size $2^{n/3+o(n)}$.*

Proof. We say that Γ is an $[a, b]$-slice access structure if for every set of parties A it holds that if $|A| < a$, then $A \notin \Gamma$ and if $|A| > b$, then $A \in \Gamma$.

By [33], almost all access structures are $[n/2-1, n/2+2]$-slice access structure, thus it suffices to construct secret-sharing schemes for them. Let $c(k, N)$ be the message size in a quadratic k-server protocol for any function $f : [N]^k \to \{0, 1\}$. By [35], for every k there is a secret-sharing scheme for $[a, b]$-slice access structure with share size $\dfrac{c(k,N) \cdot 2^{(b-a+1)n/k} O(n) \binom{n}{a}}{\binom{n/k}{a/k}^k}$. In our case, $a = \lfloor n/2 \rfloor - 1$ and $b = \lfloor n/2 \rfloor + 2$, and by taking $k = \sqrt{n/\log n}$ we get share size $c(k, N) \cdot 2^{O(\sqrt{n \log n})}$. Using our quadratic k-server CDS protocol described in Theorem 5.4 with $c(k, N) = N^{(k-1)/3}$ and $N = \binom{n/k}{a/k} < 2^{n/k}$, the share size is $2^{n/3+o(n)}$. □

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