Quadratic Time, Linear Space Algorithms for Gram-Schmidt Orthogonalization and Gaussian Sampling in Structured Lattices

Vadim Lyubashevsky and $\boxed{\text{Thomas Prest}}$

LETTRES SCIENCES
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THALES

1 Introduction: Key Sizes in Lattice-Based Cryptography

2 Faster Gram-Schmidt Orthogonalization in Structured Lattices

3 Storage-Efficient Gaussian Sampler in Structured Lattices

4 Conclusion

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Key Sizes in Cryptography

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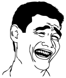
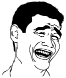
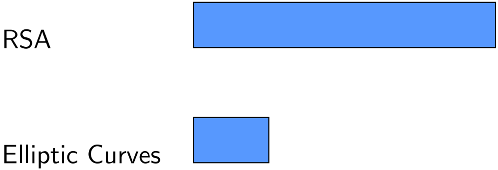


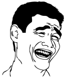
RSA

Key Sizes in Cryptography





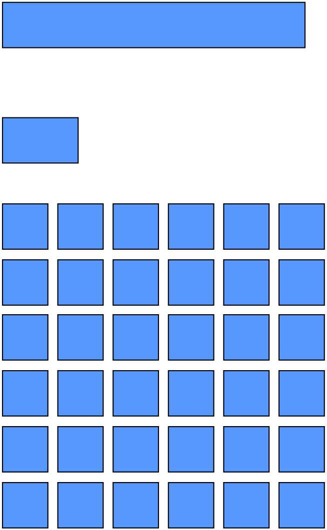
Key Sizes in Cryptography



RSA

Elliptic Curves

Lattices



Ideal lattices [LM06]

Figure : Size of the keys in lattice-based cryptography

1	4	-9	-8	9	7
-12	19	23	5	-3	52
22	27	-6	2	45	9
5	15	-7	11	-1	-65
7	-8	13	5	-72	-14
10	-3	4	8	21	9

Generic lattices

1	2	3	4	5	6
-6	1	2	3	4	5
-5	-6	1	2	3	4
-4	-5	-6	1	2	3
-3	-4	-5	-6	1	2
-2	-3	-4	-5	-6	1

Ideal lattices

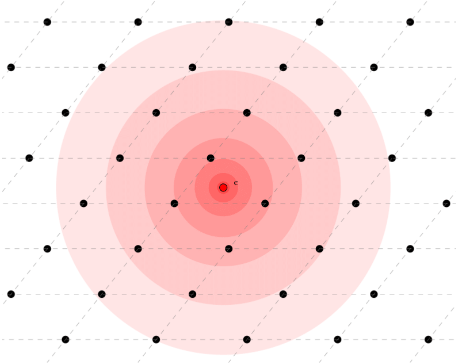
Space requirement goes from $O(n^2)$ to $O(n)$, where n is the dimension.

Gaussian Sampling [Kle00, GPV08]

Very important primitive in lattice-based cryptography:

- Hash-and-sign signatures [GPV08]
- (H)IBE [GPV08, CHKP10, ABB10]
- Standard-model signatures [ABB10]
- Attribute-based encryption [BGG+14]
- ...

Current "best" one: variation [BLP+13] of Gaussian Sampler [GPV08]



The Gaussian Sampler of [Kle00, GPV08]

What is the data required by the Gaussian Sampler for ideal lattices?

Basis **B**

1	2	3	4	5	6
-6	1	2	3	4	5
-5	-6	1	2	3	4
-4	-5	-6	1	2	3
-3	-4	-5	-6	1	2
-2	-3	-4	-5	-6	1

Highly structured!

The Gaussian Sampler of [Kle00, GPV08]

What is the data required by the Gaussian Sampler for ideal lattices?

Basis $\mathbf{B}$ | $\tilde{\mathbf{B}} = \texttt{GramSchmidt}(\mathbf{B})$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -6 & 1 & 2 & 3 & 4 & 5 \\ -5 & -6 & 1 & 2 & 3 & 4 \\ -4 & -5 & -6 & 1 & 2 & 3 \\ -3 & -4 & -5 & -6 & 1 & 2 \\ -2 & -3 & -4 & -5 & -6 & 1 \end{matrix} \quad + \quad \begin{matrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{matrix}$$

Highly structured! | No space-saving structure!

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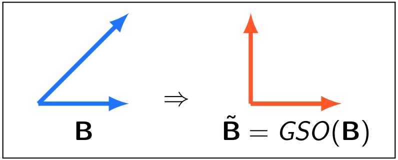
Gram-Schmidt Orthogonalization (GSO)

What is the Gram-Schmidt Orthogonalization (GSO)?



The GSO for two vectors

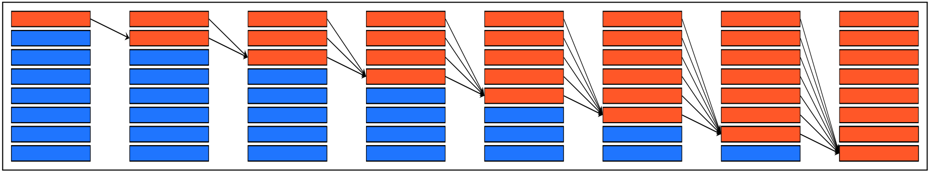


Figure : Computing the GSO $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2, ..., \tilde{\mathbf{b}}_n\}$ for n vectors.
One $\longrightarrow$ arrow takes time $O(n)$. Total time complexity $= O(n^3)$

The Geometry of Ideal Lattices

For the bases we study, we have $\forall k, \mathbf{b}_k = r(\mathbf{b}_{k-1})$:

$$\mathbf{b}_1 \xrightarrow{r} \mathbf{b}_2 \xrightarrow{r} \mathbf{b}_3 \xrightarrow{r} \cdots \xrightarrow{r} \mathbf{b}_{n-2} \xrightarrow{r} \mathbf{b}_{n-1} \xrightarrow{r} \mathbf{b}_n$$

Where r is an easily computable isometry: $\|r(\mathbf{v})\| = \|\mathbf{v}\|$.

Example:

1	2	3	4	$f \mod x^4 + 1$
-4	1	2	2	$xf \mod x^4 + 1$
-3	-4	1	2	$x^2 f \mod x^4 + 1$
-2	-3	-4	1	$x^3 f \mod x^4 + 1$

(It still is the case when replacing $x^4 + 1$ with any cyclotomic polynomial)

Exploiting the Relationship

$\mathbf{b}_1 \xrightarrow{r} \mathbf{b}_2 \xrightarrow{r} \mathbf{b}_3 \xrightarrow{r} \cdots \xrightarrow{r} \mathbf{b}_{n-2} \xrightarrow{r} \mathbf{b}_{n-1} \xrightarrow{r} \mathbf{b}_n$

The idea: compute $\tilde{\mathbf{b}}_{k+1}$ from $\tilde{\mathbf{b}}_k$

$$\boxed{\tilde{\mathbf{b}}_k \text{ is the reduction of } \mathbf{b}_k \text{ w.r.t. all the previous vectors}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{array}{rcl} \tilde{\mathbf{b}}_k & \perp & \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{k-1} \\ \Rightarrow\ r(\tilde{\mathbf{b}}_k) & \perp & r(\mathbf{b}_1), r(\mathbf{b}_2), \ldots, r(\mathbf{b}_{k-1}) \\ \Rightarrow\ r(\tilde{\mathbf{b}}_k) & \perp & \cancel{\mathbf{b}_1}, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_k \end{array}$$

$$\begin{array}{rcl} \mathbf{b}_k - \tilde{\mathbf{b}}_k & \in & Span(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{k-1}) \\ \Rightarrow\ r(\mathbf{b}_k) - r(\tilde{\mathbf{b}}_k) & \in & Span(r(\mathbf{b}_1), r(\mathbf{b}_2), \ldots, r(\mathbf{b}_{k-1})) \\ \Rightarrow\ \mathbf{b}_{k+1} - r(\tilde{\mathbf{b}}_k) & \in & Span(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_k) \end{array}$$

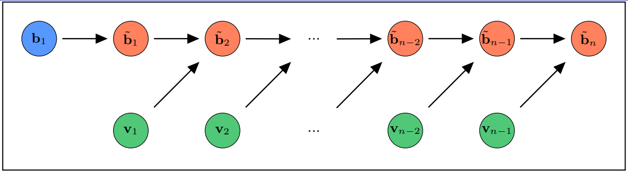
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{r(\tilde{\mathbf{b}}_k) \text{ is ``almost'' the reduction of } \mathbf{b}_{k+1} \text{ w.r.t. all the previous vectors}}$$

How to turn $r(\tilde{\mathbf{b}}_k)$ into a complete reduction of $\mathbf{b}_{k+1}$?

- Reduce $r(\tilde{\mathbf{b}}_k)$ w.r.t. to $\mathbf{b}_1$? Breaks orthogonality ✗
- Reduce $r(\tilde{\mathbf{b}}_k)$ w.r.t. to $\boxed{\mathbf{v}_k \triangleq \mathbf{b}_1 - \mathbf{Proj}(\mathbf{b}_1, Span(\mathbf{b}_2, \ldots, \mathbf{b}_k))}$? ✓

GSO for Isometric Bases



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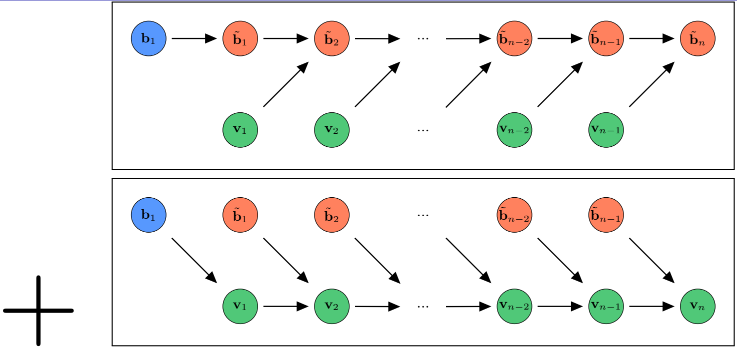

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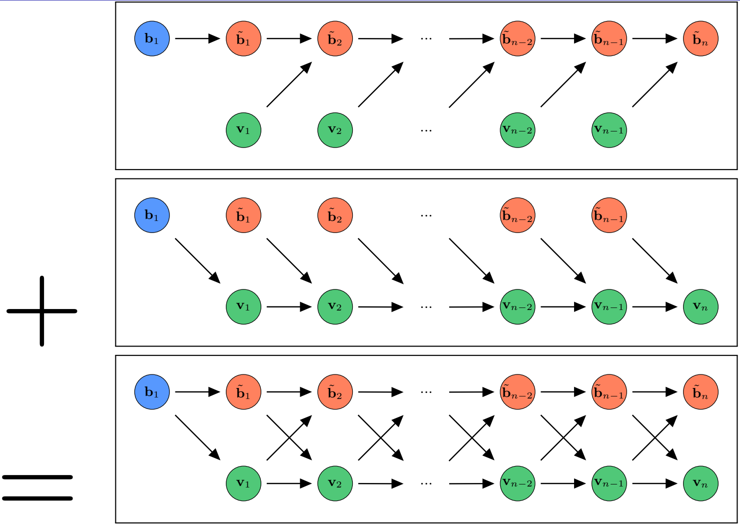
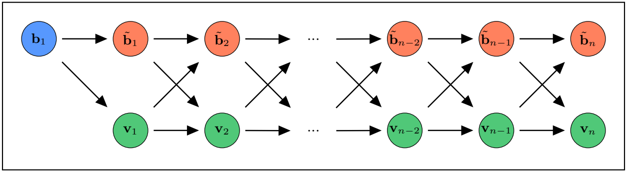


Figure : Computing the Gram-Schmidt Orthogonalization with a double recursion

One $\longrightarrow$ arrow takes time $O(n)$. Total time complexity $= O(n^2)$

GSO for Isometric Bases



Algorithm `Isometric_GSO`$(\mathbf{B})$

Require: Basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$

Ensure: GSO basis $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n\}$

$\tilde{\mathbf{b}}_1 \leftarrow \mathbf{b}_1$

$\mathbf{v}_1 \leftarrow \mathbf{b}_1$

for $k = 1, \ldots, n-1$ **do**

$\tilde{\mathbf{b}}_{k+1} \leftarrow r(\tilde{\mathbf{b}}_k) - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k)\rangle}{\|\mathbf{v}_k\|^2}\mathbf{v}_k$

$\mathbf{v}_{k+1} \leftarrow \mathbf{v}_k - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k)\rangle}{\|\tilde{\mathbf{b}}_k\|^2} r(\tilde{\mathbf{b}}_k)$

end for

return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n\}$

Algorithm `Classical_GSO`$(\mathbf{B})$

Require: Basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$

Ensure: GSO basis $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n\}$

$\tilde{\mathbf{b}}_1 \leftarrow \mathbf{b}_1$

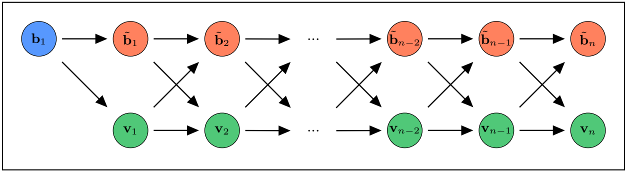
for $k = 1, \ldots, n-1$ **do**

$\tilde{\mathbf{b}}_{k+1} \leftarrow r(\mathbf{b}_k) - \sum_{j<k} \frac{\langle \tilde{\mathbf{b}}_j, r(\mathbf{b}_k)\rangle}{\|\tilde{\mathbf{b}}_j\|^2}\tilde{\mathbf{b}}_j$

end for

return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n\}$

GSO for Isometric Bases



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Ensure: GSO basis $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n\}$

$\tilde{\mathbf{b}}_1 \leftarrow \mathbf{b}_1$
$\mathbf{v}_1 \leftarrow \mathbf{b}_1$
for $k = 1, \ldots, n-1$ **do**
$\quad \tilde{\mathbf{b}}_{k+1} \leftarrow r(\tilde{\mathbf{b}}_k) - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k)\rangle}{\|\mathbf{v}_k\|^2}\mathbf{v}_k$
$\quad \mathbf{v}_{k+1} \leftarrow \mathbf{v}_k - \frac{\langle \mathbf{v}_k, r(\tilde{\mathbf{b}}_k)\rangle}{\|\tilde{\mathbf{b}}_k\|^2} r(\tilde{\mathbf{b}}_k)$
end for
return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n\}$

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Require: Basis $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$
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$\quad \tilde{\mathbf{b}}_{k+1} \leftarrow r(\mathbf{b}_k) - \sum_{j<k} \frac{\langle \tilde{\mathbf{b}}_j, r(\mathbf{b}_k)\rangle}{\|\tilde{\mathbf{b}}_j\|^2}\tilde{\mathbf{b}}_j$
end for
return $\tilde{\mathbf{B}} = \{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n\}$

Further optimizations: We can avoid 2 out 3 scalar products $\Rightarrow$ up to 67% faster

Gram-Schmidt Decomposition (GSD)

Definition (Gram-Schmidt Decomposition (GSD))

Write $\mathbf{B} = \mu \times \tilde{\mathbf{B}}$, where $\tilde{\mathbf{B}}$ is the GSO of $\mathbf{B}$

Applications:

- In cryptology: lattice reduction (LLL, BKZ...), Gaussian Sampling
- Outside cryptology: solving least square problems, linear systems, computing eigenvalues...

Speeding up GSD for Isometric Bases

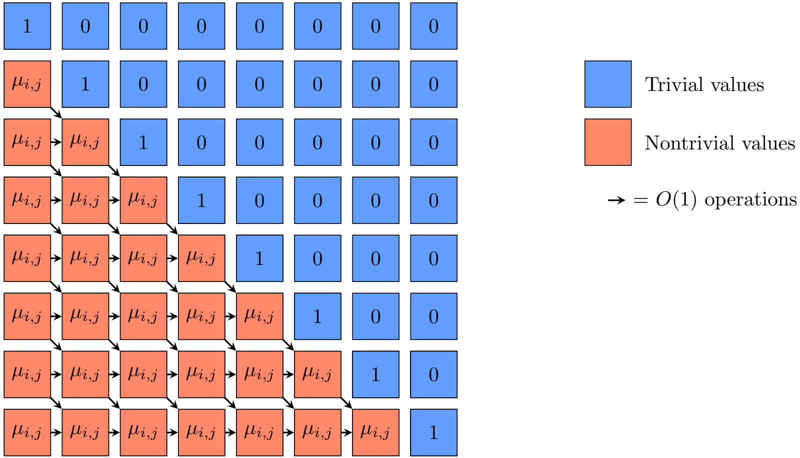

Figure : Fast computation of the matrix μ for an isometric basis.
Overall time and space complexity: $O(n^2)$

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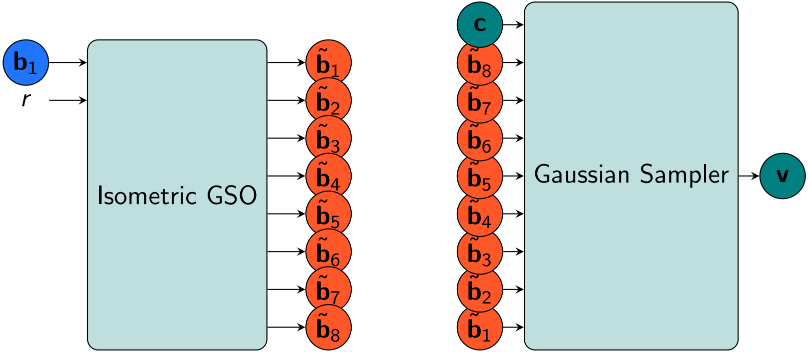
GSO and Gaussian Sampler

Now both GSO and the Gaussian Sampler run in time $O(n^2)$.

Can we directly plug together the Isometric GSO
and the Gaussian Sampler to both run on-the-fly?



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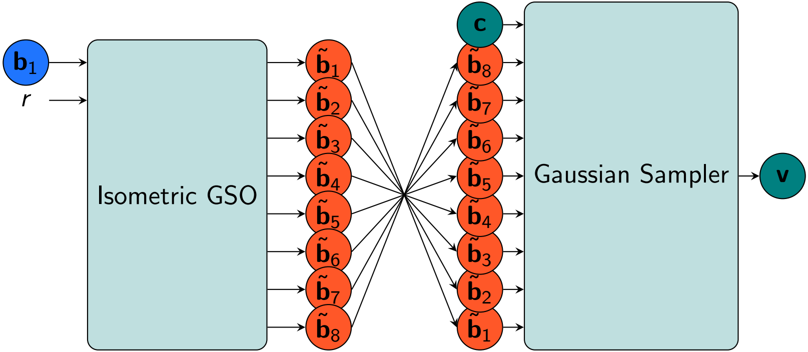


Not really.

Linear-Space Gaussian Sampler

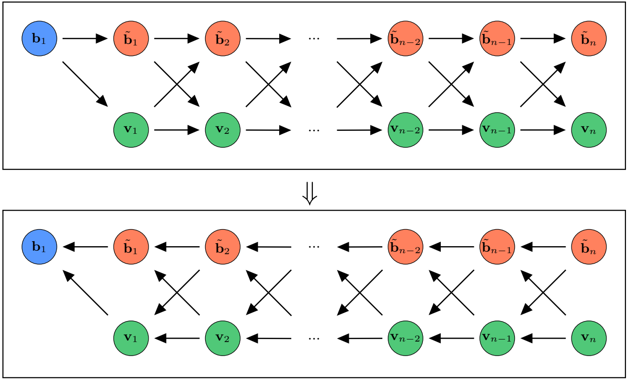
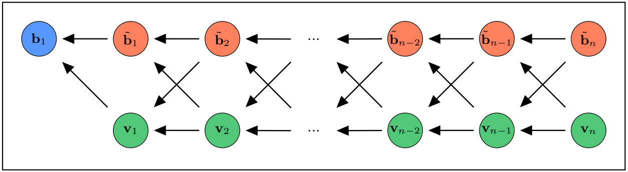


Figure : Reverting Isometric GSO in order to do linear-space Gaussian Sampling

One $\longrightarrow$ arrow takes time $O(n)$ and space $O(1)$ (besides the vectors $\tilde{\mathbf{b}}_k, \mathbf{v}_k$).

A Linear Space Gaussian Sampler

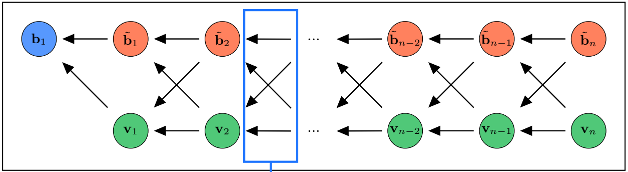


Algorithm
`Classic_Sampler`$(\mathbf{B}, \tilde{\mathbf{B}}, \sigma, \mathbf{c})$

Require: $\mathbf{B}$, $\tilde{\mathbf{B}}$, σ, $\mathbf{c}$
Ensure: $\mathbf{z}$ sampled in $\mathcal{D}_{\Lambda(\mathbf{B}),\sigma,\mathbf{c}}$

$\mathbf{c}_n \leftarrow \mathbf{c}$
for $k \leftarrow n, ..., 1$ **do**
  $d_k \leftarrow \langle \mathbf{c}_k, \tilde{\mathbf{b}}_k \rangle / \|\tilde{\mathbf{b}}_k\|^2$
  $\sigma_k \leftarrow \sigma / \|\tilde{\mathbf{b}}_k\|$
  $z_k \leftarrow D_{\mathbb{Z},\sigma_k,d_k}$
  $\mathbf{c}_{k-1} \leftarrow \mathbf{c}_k - z_k \mathbf{b}_k$
end for
return $\mathbf{c} - \mathbf{c}_0$

A Linear Space Gaussian Sampler



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Require: $\mathbf{B}$, $\tilde{\mathbf{B}}$, σ, $\mathbf{c}$
Ensure: $\mathbf{z}$ sampled in $\mathcal{D}_{\Lambda(\mathbf{B}),\sigma,\mathbf{c}}$

- $\mathbf{c}_n \leftarrow \mathbf{c}$
- **for** $k \leftarrow n, \ldots, 1$ **do**
  - $d_k \leftarrow \langle \mathbf{c}_k, \tilde{\mathbf{b}}_k \rangle / \|\tilde{\mathbf{b}}_k\|^2$
  - $\sigma_k \leftarrow \sigma / \|\tilde{\mathbf{b}}_k\|$
  - $z_k \leftarrow D_{\mathbb{Z},\sigma_k,d_k}$
  - $\mathbf{c}_{k-1} \leftarrow \mathbf{c}_k - z_k \mathbf{b}_k$
- **end for**
- **return** $\mathbf{c} - \mathbf{c}_0$

Algorithm `Compact_Sampler`$(\mathbf{B}, \tilde{\mathbf{b}}_n, \mathbf{v}_n, \sigma, \mathbf{c}, (H_k, I_k)_k)$

Require: $\mathbf{B}$, $\tilde{\mathbf{b}}_n, \mathbf{v}_n$, σ, $\mathbf{c}$
Ensure: $\mathbf{z}$ sampled in $\mathcal{D}_{\Lambda(\mathbf{B}),\sigma,\mathbf{c}}$

- $\mathbf{c}_n \leftarrow \mathbf{c}$
- **for** $k \leftarrow n, \ldots, 1$ **do**
  - $d_k \leftarrow \langle \mathbf{c}_k, \tilde{\mathbf{b}}_k \rangle / \|\tilde{\mathbf{b}}_k\|^2$
  - $\sigma_k \leftarrow \sigma / \|\tilde{\mathbf{b}}_k\|$
  - $z_k \leftarrow D_{\mathbb{Z},\sigma_k,d_k}$
  - $\mathbf{c}_{k-1} \leftarrow \mathbf{c}_k - z_k \mathbf{b}_k$
  - $\tilde{\mathbf{b}}_{k-1} = r^{-1}(H_k \tilde{\mathbf{b}}_k + I_k \mathbf{v}_k)$
  - $\mathbf{v}_{k-1} = I_k \tilde{\mathbf{b}}_k + H_i \mathbf{v}_k$
- **end for**
- **return** $\mathbf{c} - \mathbf{c}_0$

Timings and Space Requirements

Table : Timings and space requirements of the classic and compact Gaussian Samplers (Classic GS and Compact GS)[1].

Statistical distance from ideal		2^{-128}
Precision needed	Classic GS	163 bits
	Compact GS	193 bits
Running time	Classic GS	170 ms
	Compact GS	521 ms
Space requirement	Classic GS	163 Mb
	Compact GS	0.47 Mb

- Running time: $\times$**3.06**
- Space requirement: $/$**340**

[1]The implementation was done in C++ using GMP. Timings were performed on an Intel Core i5-3210M laptop with a 2.5GHz CPU and 6GB RAM.

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A few open questions:

- Precision analysis of Isometric GSO?
- Better precision analysis of Gaussian Sampler?
- Combine with ideas of [DN12]?
- How does this link to Arnoldi iteration and Lanczos Algorithm?
- Use same principles to improve other algorithms?

Conclusion

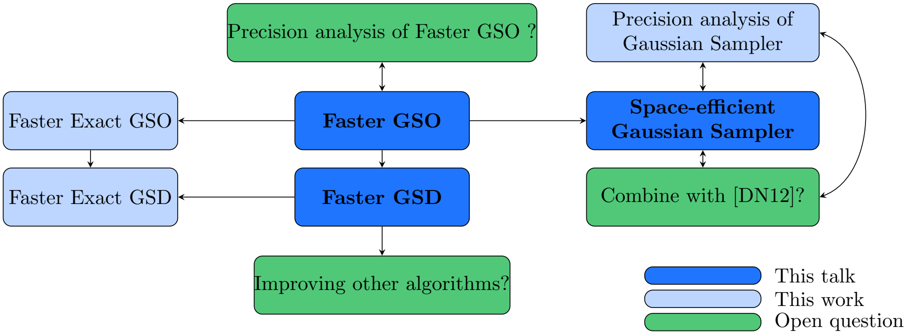

Figure : Present and future work

Conclusion

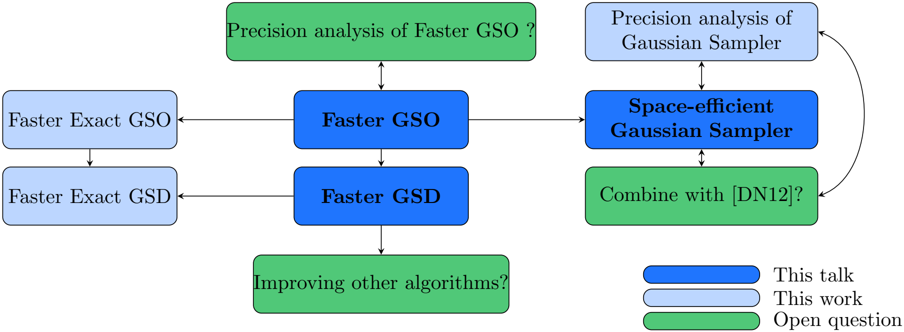

Figure : Present and future work

Thank you!

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