The Multiple Number Field Sieve with Conjugation and Generalized Joux-Lercier Methods

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- Multiplicative group G generated by g: solving the discrete logarithm problem in G, is inverting the map $x \mapsto g^x$
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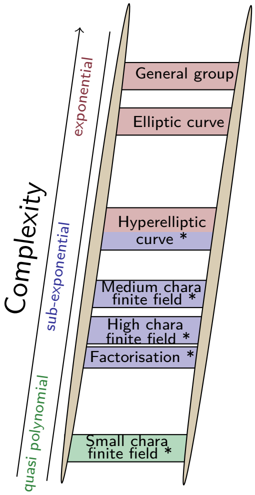

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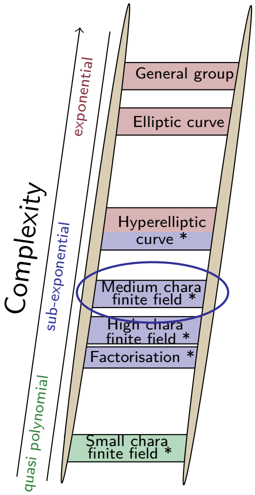

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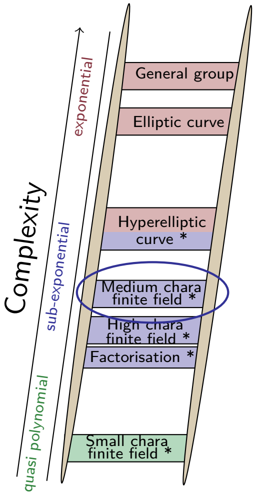

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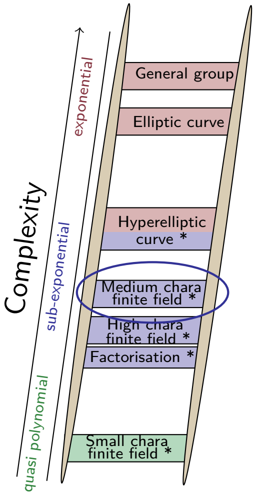

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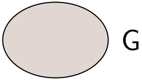

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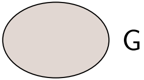

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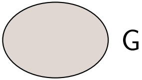

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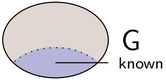

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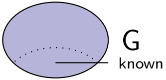

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$$\prod_{b_i \in B_2} v_1(b_i) = \prod_{b_i \in B_2} v_2(b_i)$$ thanks to linearity.

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Factor base ? $B_i :=$ prime ideals (of the ring of integers) with a norm smaller than a certain smoothness* bound.

*An ideal $\mathfrak{I}$ is B-smooth if all its factors have norms lower than B.

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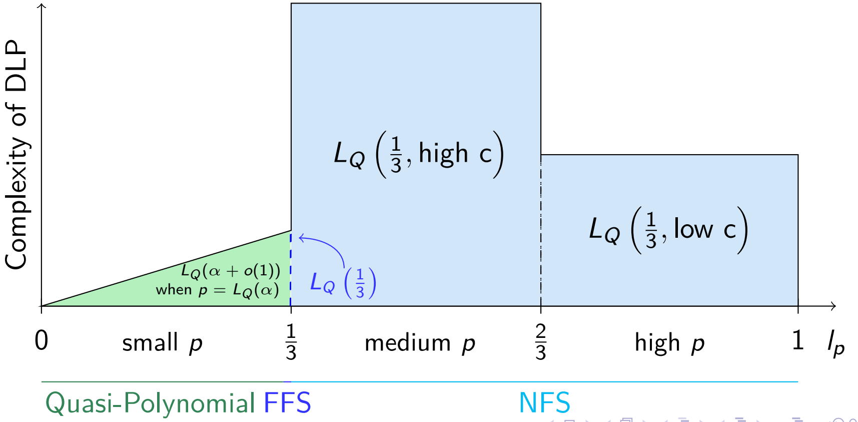

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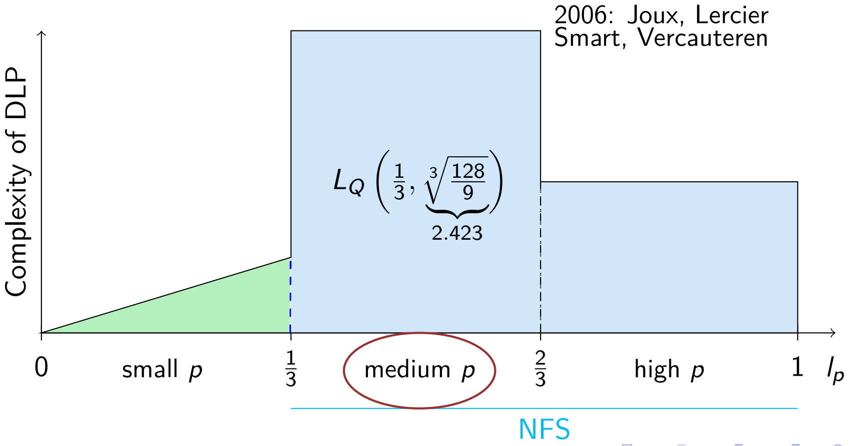

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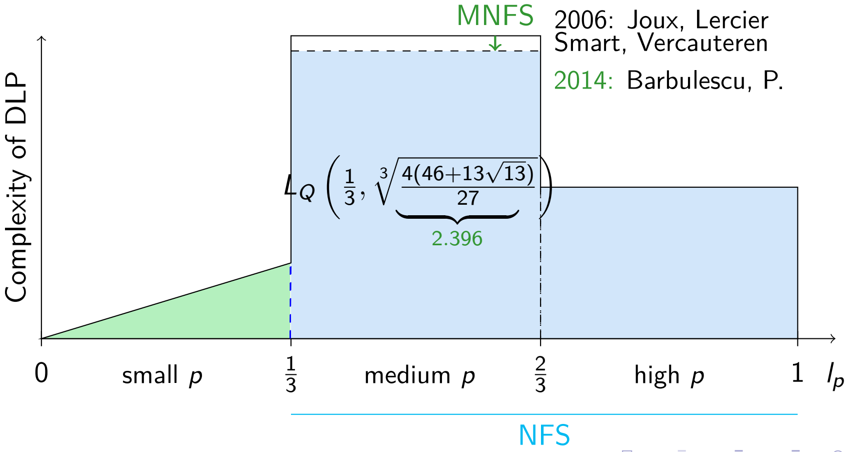

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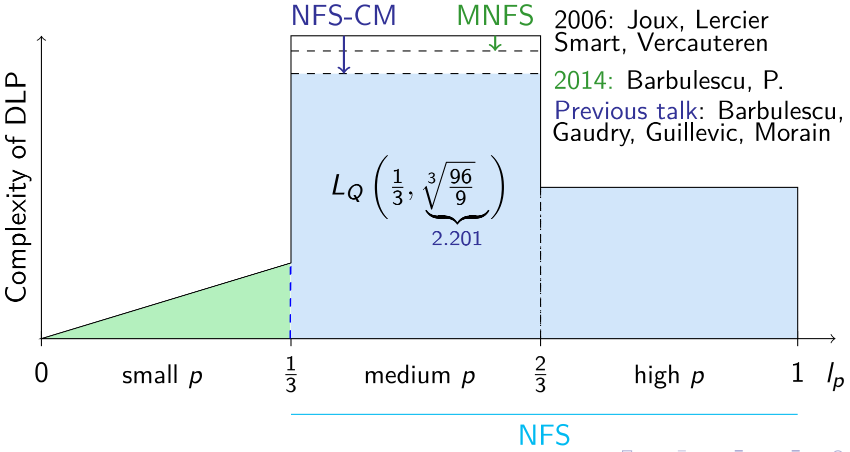

2006: Joux, Lercier
Smart, Vercauteren

2014: Barbulescu, P.

Previous talk: Barbulescu, Gaudry, Guillevic, Morain

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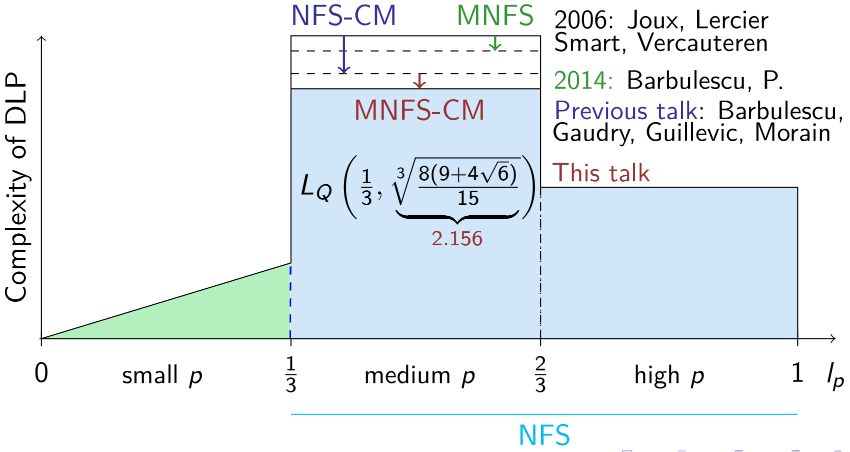
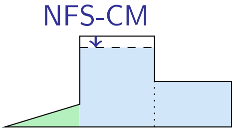


Polynomial Selection



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Find two polynomials f_1 and f_2 with an irreducible factor $\mathcal{I}$ of degree n modulo p.

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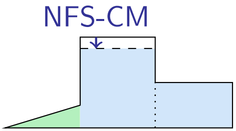

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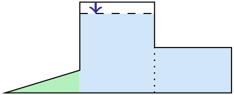

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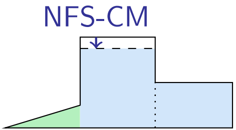

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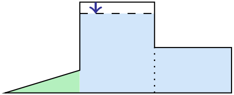
NFS-CM



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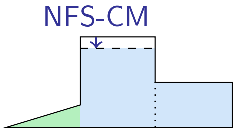

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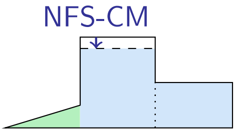

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New polynomial selection proposed by Barbulescu, Gaudry, Guillevic and Morain: the Conjugation Method.

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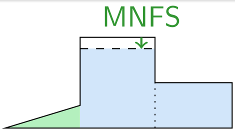
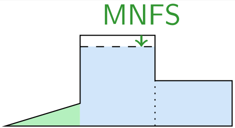


The Multiple Number Field Sieve

Main idea: from 2 to V number fields.

- Idea from integer factorization [Coppersmith 93], prime fields [Matyukhin 03], high and medium characteristic [Barbulescu, P. 14].



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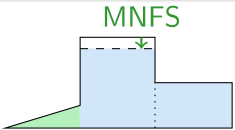

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- Choice of polynomials f_1 and f_2 with a common root m in $\mathbb{F}_{p^n}$
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Dissymetric MNFS in one slide

Dissymmetric = when a polynomial is better than the other.

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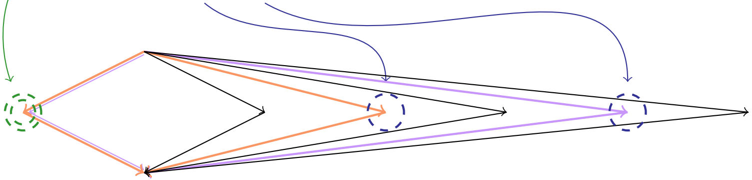

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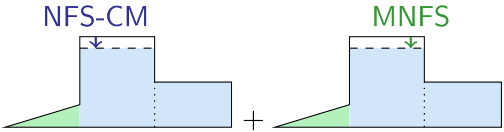

This Talk

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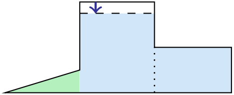
NFS-CM



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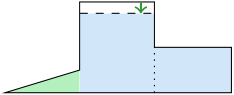
MNFS



$\Rightarrow$ Best algorithm to solve the DLP for medium characteristic finite fields $\mathbb{F}_{p^n}$.

=

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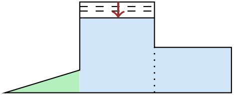

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n $\qquad\qquad\qquad\qquad$ $\sqrt{p}$

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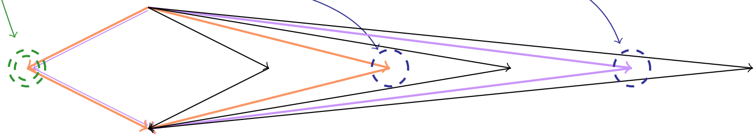
Degrees $\qquad\qquad\qquad\qquad$ Coeffs.

And then ?

Construct a Multiple NFS thanks to:

- $\mathbb{Q}[X]/(f_1(X))$ on one side
- $\mathbb{Q}[X]/(f_i(X))$ on the other side, where the $V-1$ polynomials are defined as $f_i = \alpha_i f_2 + \beta_i f_3$ with $\alpha_i, \beta_i \approx \sqrt{V}$



Asymptotic Complexity Analysis

The idea is classical:

1. Choose parameters of size:
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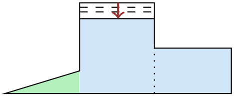
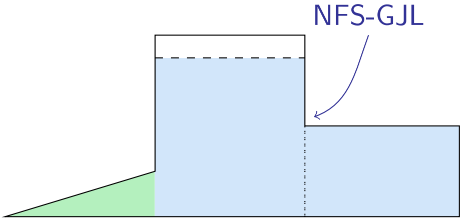
$$\Rightarrow L_Q\left(\frac{1}{3}, \sqrt[3]{\frac{8(9+4\sqrt{6})}{15}}\right)$$

MNFS-CM



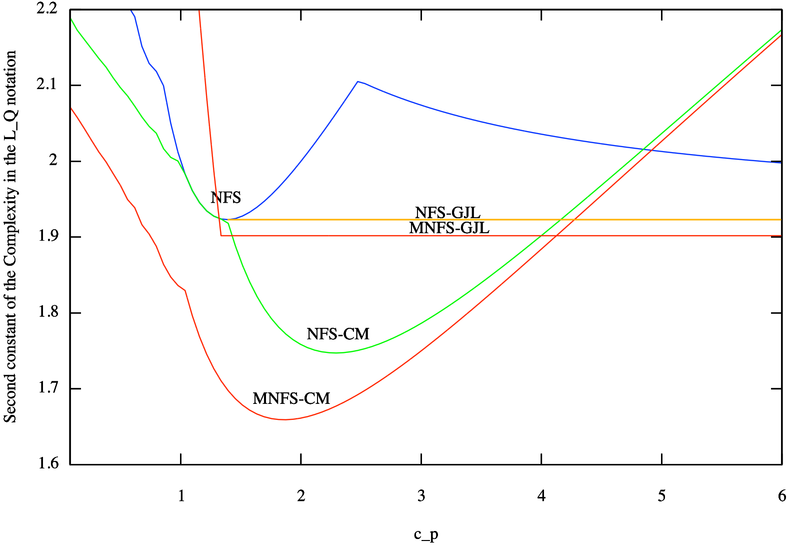
A similar approach permits to combine:

- the Generalized Joux-Lercier Method [BGGM 15]
- with MNFS.

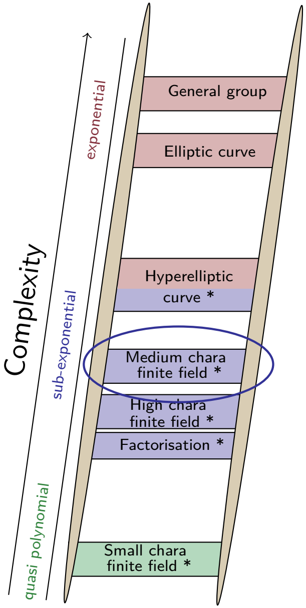


$$p = L_{p^n}(2/3, c_p)$$

Complexities at $p = L_{p^n}(2/3, c_p)$



Thank you for your attention !



Going further

Implementation ?

- For factoring (as a comparison): implemented in 1996 by Elkenbracht-Huizing, but not as efficient as the classical NFS for those parameters.
- For Discrete Logs: tested for small parameters but MNFS in a more realistic context is still to do.

Choice of Polynomials

Previously (NFS) :

- For medium p : f_1 irreducible of degree n over $\mathbb{F}_p$ and $f_2 = f_1 + p$
  Small degrees but high coeffs for f_2
- For high p : based on lattice reduction of $(f_1, Xf_1, \cdots, X^{d-n}f_1, p, Xp, \cdots, X^d p)$
  $\Rightarrow$ f_2 is a multiple of f_1 modulo p but with smaller coeffs
  f_1 with not too small coeffs (otherwise we get trivial multiples)

Some Obstructions Coming from Number Fields and its Solutions

How to go down ?

$$\begin{array}{ccc} \mathbb{Q}[\theta] & & \\ & \searrow & \\ & & \mathbb{F}_{p^n} \end{array}$$

- No unique factorization over elements $\Rightarrow$ we consider ideals in the ring of integers of $\mathbb{Q}[\theta]$.
- Ideals are not principal $\Rightarrow$ we (virtually) raise them to the power of the class number of $\mathbb{Q}[\theta]$.
- Generators are not unique $\Rightarrow$ Schirokauers maps.

Extension of NFS in the boundary case $p = L_{p^n}(1/3)$

- We want to upper-bound the resultant :
  $|\det \mathrm{Sylv}(h, f)| \leqslant \Theta \|f\|^{\deg h} \|h\|^{\deg f}$ with Θ = number of permutations with non zero contributions in the sum.
- Θ ? Let deg $(h) = n$ and deg $(f) = t$.
  Before : $\Theta \leqslant n^t t^n$. Kalkbrener gives : $\Theta \leqslant \binom{n+t}{n} \cdot \binom{n+t-1}{t}$.
  Because of the following inequalities:

$$\begin{aligned}
\binom{n+t}{n} \cdot \binom{n+t-1}{t} &= \frac{n}{n+t} \left(\frac{(n+t)!}{n!t!} \right)^2 \\
&\leq \frac{n}{n+t} \left(\frac{(n+1)\cdots(n+t)}{t!} \right)^2 \\
&\leq \frac{n}{n+t} \left(\prod_{i=1}^{t} \frac{(n+i)}{i} \right)^2 \\
&\leq \frac{n}{n+t} \prod_{i=1}^{t} \left(\frac{n}{i} + 1 \right)^2
\end{aligned}$$

we obtain that $\Theta \leqslant (n+1)^{2t}$.