Linear Secret Sharing Schemes from Error Correcting Codes and Universal Hash Functions

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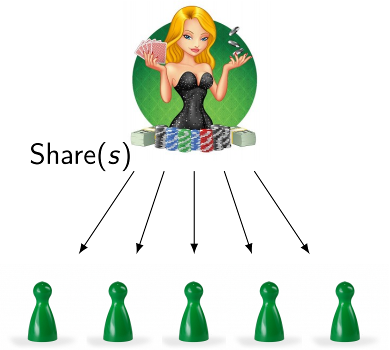

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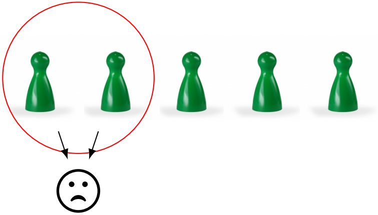

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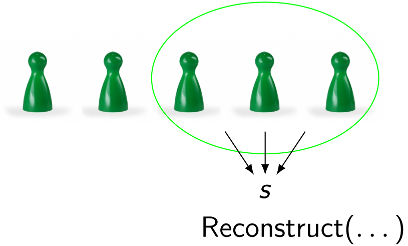

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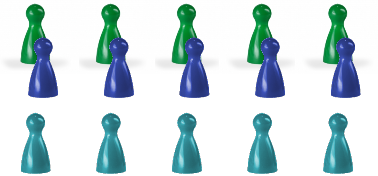
- **Linearity (LSSS):** $(c_1)_i$ and $(d_1)_i$ shares of s_1 and $s_2 \Rightarrow (\alpha c_i + \beta d_i)_i$ shares of $\alpha s_1 + \beta s_2$.



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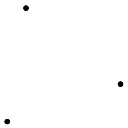

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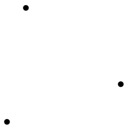

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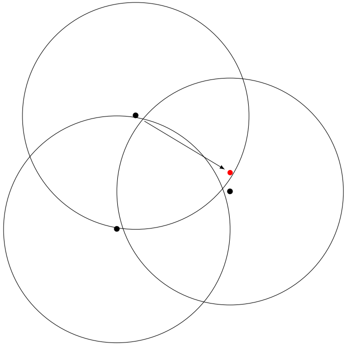

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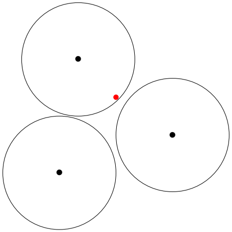

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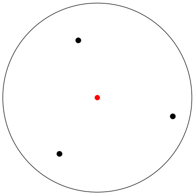

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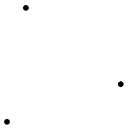

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- **Algorithmic Goals:** Efficient encoding, decoding from errors/erasures, list-decoding.

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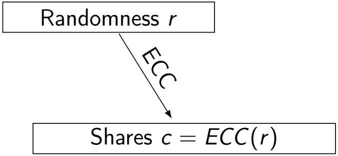
An information-theoretic look on secret sharing



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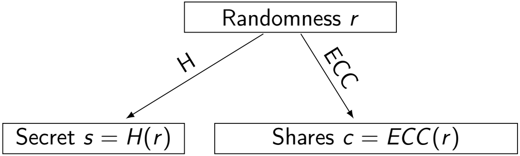

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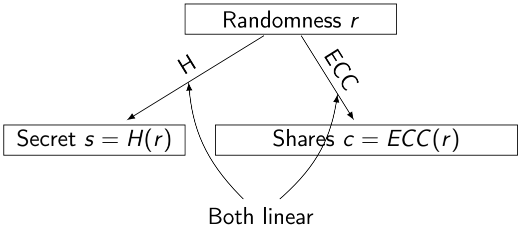

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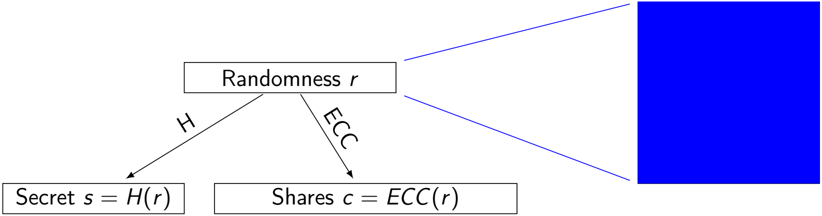

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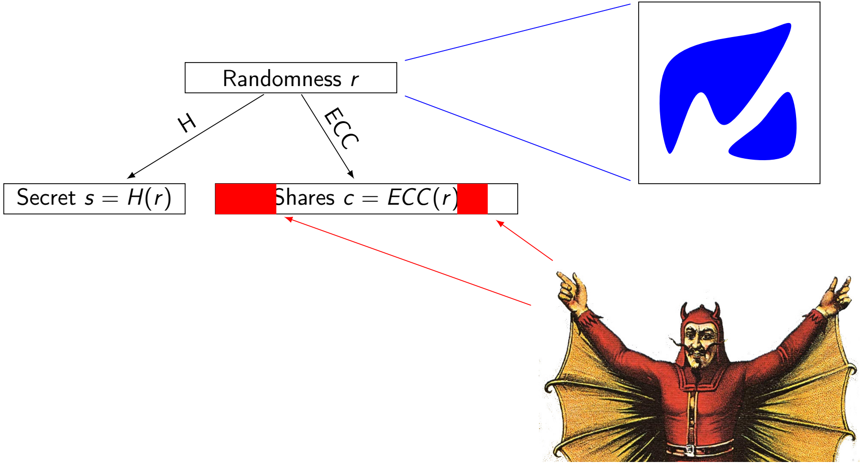

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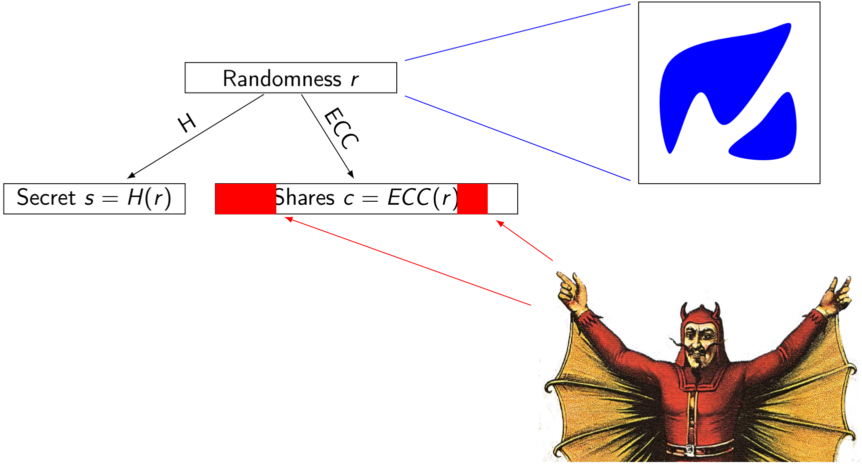

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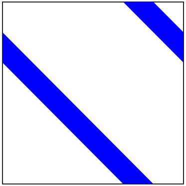
What really happens:



Randomness r

H ECC

Secret $s = H(r)$

Shares $c = ECC(r)$

Both linear

What do we require from H?

- H should be linear (we want linear SSS)
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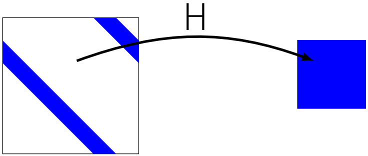

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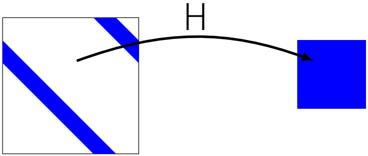

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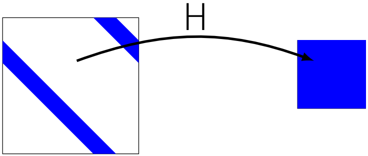

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Universal Hash Functions

- Function Ensembles with statistical collision resistance
- $\mathcal{H} : \{H : X \to Y\}$ is universal iff $\forall x_1 \neq x_2$: $\Pr_{H \leftarrow_{\$} \mathcal{H}}[H(x_1) = H(x_2)] \leq 1/|Y|$

Subspace Surjectivity

Almost all functions of a family will be surjective on a fix subspace of some minimum dimension.

- $\mathcal{H}$ a family of **linear** universal hash functions $\mathbb{F}_q^k \to \mathbb{F}_q^l$
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The Scheme

LSSS

Fix some linear hash function $H \in \mathcal{H}$, ECC linear code

Share(s):
- $r \leftarrow_{\$} H^{-1}(s)$
- $\mathbf{c} \leftarrow ECC(r)$
- Output share vector $\mathbf{c}$

Reconstruct($\tilde{c}$):
- $r \leftarrow ECC$.Decode($\tilde{c}$)
- If $r = \perp$
  - Output $\perp$
- $s \leftarrow H(r)$
- Output secret s

Main Theorem

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$$R \geq \rho + \eta + \tau + h(\tau)/(m \cdot \log(q)).$$ [a]

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- **This work:** $t < (1 - \epsilon)n/2$ for any constant ϵ, shares of size $O(1 + \lambda/n)$

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- Robust reconstruction easy for $t \leq n/3$ and impossible for $t \geq n/2$. **For $n/3 \leq t < n/2$ only statistical robustness.**
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- Best construction so far ([CFOR12]): $t \leq n/2 - 1$, shares of size $O(\lambda + n \cdot \log(n))$
- **This work:** $t < (1 - \epsilon)n/2$ for any constant ϵ, shares of size $O(1 + \lambda/n)$

Robust Secret Sharing

Idea

- **List Decodable Code $\Rightarrow$ List Decodable SSS**
- List Decodable SSS + AMD Codes $\Rightarrow$ Robust Secret Sharing

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Thank You!