Slide Reduction, Revisited—Filling the Gaps in SVP Approximation

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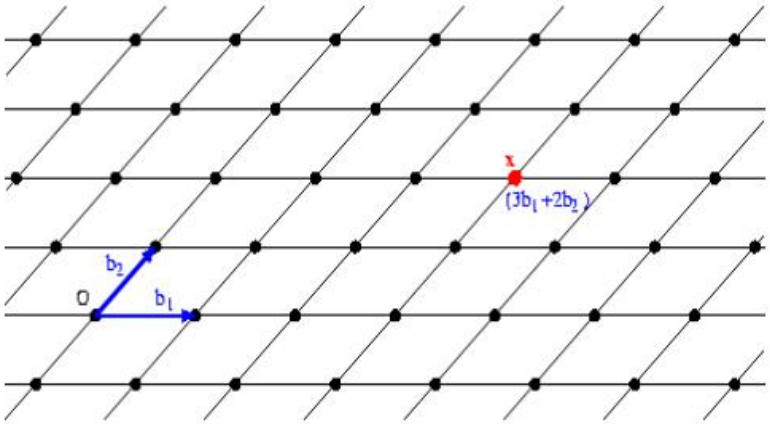
Outline

Lattice and basis

- An n-rank lattice L is a set of all integer linear combinations of n linearly independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$:

$$L = \{z_1\mathbf{b}_1 + \cdots + z_n\mathbf{b}_n, z_i \in \mathbb{Z}\}.$$

- $B := (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is called a basis of L.



A Lattice of rank 2

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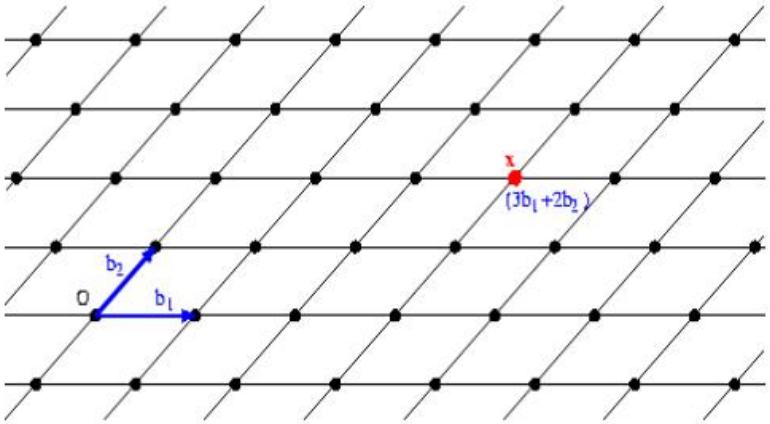

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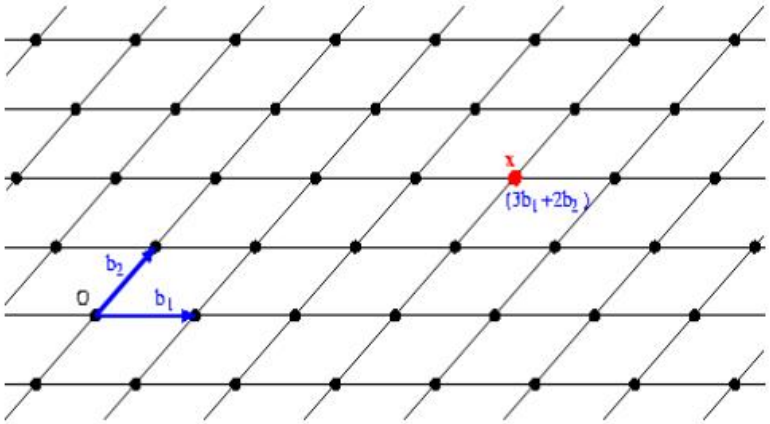

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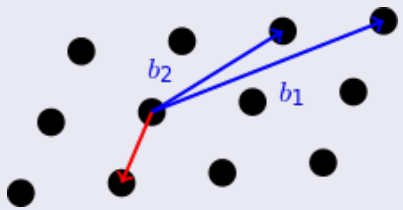
The most important lattice problem is the shortest vector problem (SVP)

Given a basis of a lattice L, SVP is to find a shortest nonzero vector $\mathbf{v}$ in L, i.e., $\|\mathbf{v}\| = \min_{\mathbf{x} \in L_{\neq \mathbf{0}}} \|\mathbf{x}\| \triangleq \lambda_1(L)$.



Two natural relaxations

- f-approximate SVP (f-SVP): Given a lattice L, find a non-zero vector $\mathbf{v} \in L$ s.t. $\|\mathbf{v}\| \leq f \cdot \lambda_1(L)$.
- f-Hermite SVP (f-HSVP): Given an n-rank lattice L, find a non-zero vector $\mathbf{v} \in L$ s.t. $\|\mathbf{v}\| \leq f \cdot \mathrm{vol}(L)^{1/n}$, where $\mathrm{vol}(L)$ is the determinant of L.

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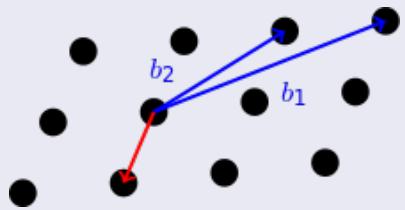

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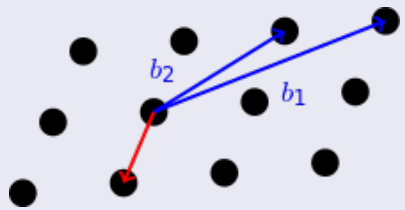

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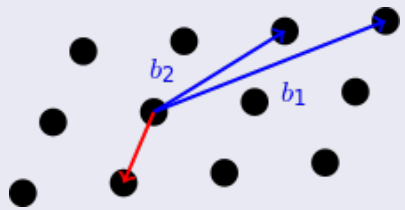

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Hardness of SVP

There is some constant $c > 0$ s.t. $n^{c/\log\log n}$-SVP on n-rank lattices is NP-hard under reasonable complexity theoretic assumptions.[a] [b] [c] [d]

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[a] M. Ajtai. The shortest vector problem in L_2 is NP-hard for randomized reductions. STOC 1998.

[b] D. Micciancio. The shortest vector in a lattice is hard to approximate to within some constant. SIAM J. Comput 2000 and FOCS 1998.

[c] S. Khot. Hardness of approximating the shortest vector problem in lattices. JACM 2005 and FOCS 2004.

[d] I. Haviv and O. Regev. Tensor-based hardness of the shortest vector problem to within almost polyno-mial factors.Theory of Computing 2012 and STOC 2007.

Cryptography VS Cryptanalysis

Lattice cryptography

From Ajtai 1996's beginning, many cryptographic primitives have been constructed whose security is based on the (worst-case) hardness of n^c-SVP for some constant c. [a]

- NIST PQC Round 3 submission

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Lattice cryptanalysis

How to do lattice cryptanalysis?

- How to estimate the concrete security of lattice cryptographic schemes?
  - Solve γ-(H)SVP
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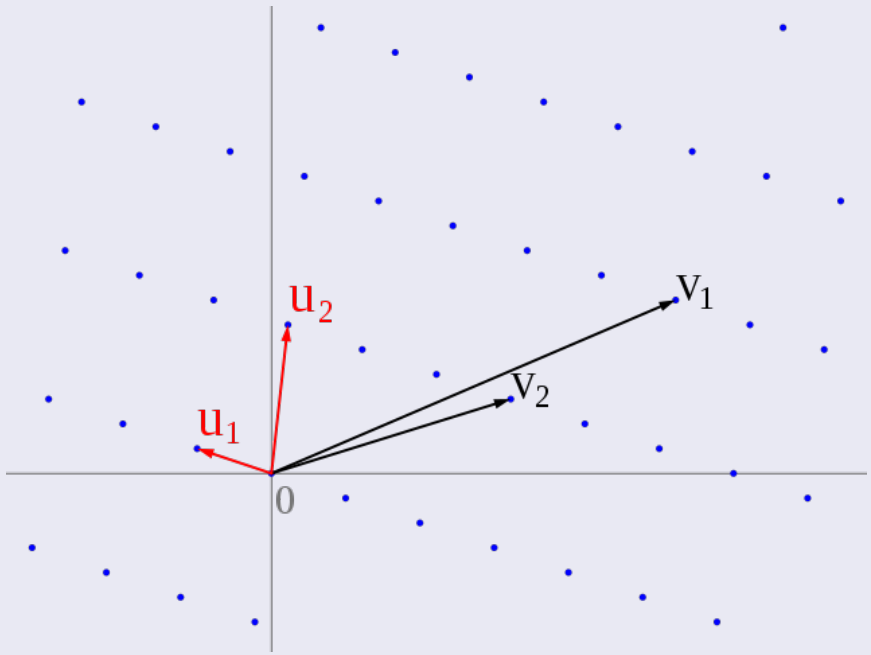
Lattice reduction

Given a lattice, find a good basis consisting of reasonably short and almost orthogonal vectors.



Importance

Lattice reduction is the classical approach for solving f-(H)SVP:

- It has proved invaluable in many fields of computer science and mathematics.
- Notably in cryptology:
  - It is a popular tool to both public-key cryptography and cryptanalysis;
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- Hermite's constant: $\gamma_n = \max \lambda_1(L)^2$ over all n-rank lattices L of unit determinant.
- $\gamma_n = \Theta(n)$ measures the output quality of lattice reduction algorithms.

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Prior work 1

Previous best algorithms for $n^{c\geq 1}$-SVP in theory

GN-slide-reduction is the previously best polynomial time lattice reduction algorithm for solving $n^{c\geq 1}$-SVP in theory: [a]

- For $n = pk \geq 2k$, with polynomially many calls to exact SVP-oracle in rank k, it outputs a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of the input lattice L s.t.

$$\begin{aligned} \|\mathbf{b}_1\| &\leq 2\gamma_k^{\frac{n-1}{2(k-1)}} \cdot \mathrm{vol}(L)^{1/n}, \\ \|\mathbf{b}_1\| &\leq 2\gamma_k^{\frac{n-k}{k-1}} \cdot \lambda_1(L). \end{aligned}$$

[a] N. Gama and P. Q. Nguyen. Finding short lattice vectors within Mordell's inequality. STOC 2008.

Prior work 2

Previous best algorithms for $n^{c\geq\frac{1}{2}}$-HSVP in theory

DBKZ is the previously best polynomial time lattice reduction algorithm for solving $n^{c\geq\frac{1}{2}}$-HSVP in theory: [a]

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[a] D. Micciancio and M. Walter. Practical, predictable lattice basis reduction. EUROCRYPT 2016.

Slide-reduction VS DBKZ

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Two natural questions

- Can we extend GN-sldie-reduction algorithm into the case that k might not divide n?
  $\Rightarrow$ Exponential speedup for solving n^c-SVP over $1 < c \notin \mathbb{Z}$.
- The best (proven) approximation factor for appximating SVP and HSVP is now achieved by a single algorithm:
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Approximating SVP with sublinear factors

- The security of many lattice-based cryptographic constructions is based on the worst-case hardness of n^c-SVP with constant $c \in [\frac{1}{2}, 1]$.
- Awkward: There is no known non-trivial algorithm for approximating SVP with sublinear factors.
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Our paper solves the three questions:

Q1 Is there an non-trivial (lattice reduction) algorithm for approximating SVP with sublinear factors?

Q2 Can we extend GN-slide-reduction algorithm into the case that k does not divide n exactly, so that it can faster solve n^c-SVP over any fractional constant $c \geq 1$?

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Our first result

Theorem (Approximating SVP with sublinear factor)

Let $2k > n \geq k \geq 2$ be integers and $\delta \geq 1$. There is an algorithm that with polynomially many calls to δ-SVP-oracle in rank k, it outputs an nonzero vector $\mathbf{b}$ of the input lattice L s.t.

$$\|\mathbf{b}\| \leq O(\delta(\delta^2\gamma_k)^{\frac{n}{2k}}) \cdot \lambda_1(L).$$

★ This is the first non-trivial algorithm for approximating SVP with sublinear factors $n^{\frac{1}{2}} \leq f \leq n^{1-\varepsilon}$.

Corollary

For any constant $c \in (1/2, 1)$ and any factor $\delta \geq 1$, there is an efficient reduction from $O(\delta^{2c+1}n^c)$-SVP in rank n to δ-SVP in rank $\lceil \frac{n}{2c} \rceil$.

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Theorem (Approximating SVP with (at least) polynomial factor)

Let $n \geq 2k \geq 4$ be integers and $\delta \geq 1$. There is an algorithm that with polynomially many calls to δ-SVP-oracle in rank k, it outputs a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ of the input lattice L s.t.

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Impact

- Our two algorithms provide currently the best polynomial-time lattice reduction algorithm:
  $\Rightarrow$ Achieve the best time/quality trade-off in theory.
- With well-chosen SVP-oracles in lower rank, our work implies the exponentially faster provable/heuristic algorithm for approximating SVP with factor $n^{1/2} \leq f \leq n^{O(1)}$:
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Impact: the faster provable/heuristic algorithm

Table: Provable algorithms for approximating SVP.[1]

Approx-factor	Previous best	This work
n^c for $c \in [\frac{1}{2}, 1)$	$2^{0.802n}$ [WLW15]	$2^{\frac{0.802n}{2c}}$
n^c for $c \geq 1$	$2^{\frac{n}{\lfloor c+1 \rfloor}}$ [GN08]+[ADRS15]	$2^{\frac{0.802n}{c+1}}$

Table: Heuristic algorithms for approximating SVP.[2]

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[1] W. Wei, M. Liu, and X. Wang. Finding shortest latticevectors in the presence of gaps. CT-RSA 2015.

[2] A. Becker, L. Ducas, N. Gama, and T. Laarhoven. New directions in nearest neighbor searching with applications to lattice sieving. SODA 2016.

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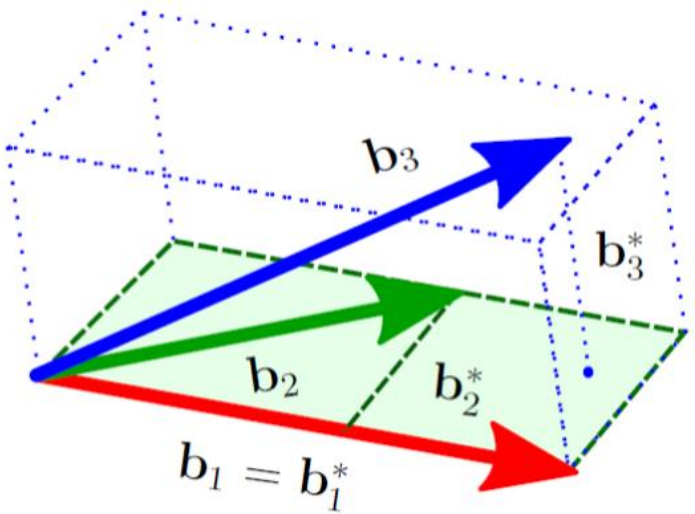
Preliminaries

GSO

Given a basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, define the orthogonal projection:

$$\pi_i : \mathrm{span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) \mapsto \mathrm{span}(\mathbf{b}_1, \ldots, \mathbf{b}_{i-1})^{\perp}.$$

- Each vector $\mathbf{b}_i^* = \pi_i(\mathbf{b}_i)$ is the Gram-Schmidt vector of B.
- The projected block $B_{[i,j]} = (\pi_i(\mathbf{b}_i), \pi_i(\mathbf{b}_{i+1}), \ldots, \pi_i(\mathbf{b}_j))$.



Preliminaries

Several reduction notions

Let B be a basis of a lattice L.

- B is *SVP-reduced* if its first basis vector is a shortest nonzero vector of L.
- B is *DSVP-reduced* if its dual basis is SVP-reduced.
- B is *DBKZ-reduced* if it is produced by the DBKZ algorithm with blocksize k.[a]
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[a] D. Micciancio and M. Walter. Practical, predictable lattice basis reduction. EUROCRYPT 2016.

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Approximating SVP with sublinear factor

Given a lattice L of rank $n \in (k, 2k)$ and a SVP-oracle in rank k.

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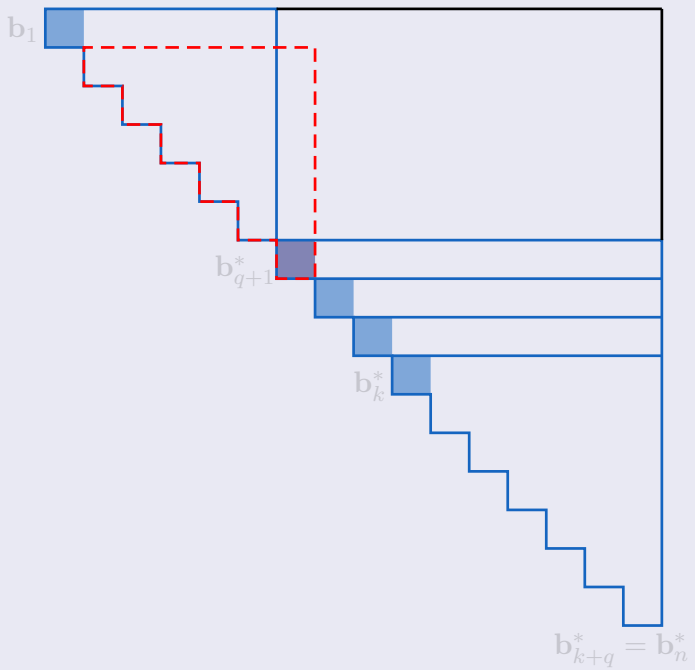

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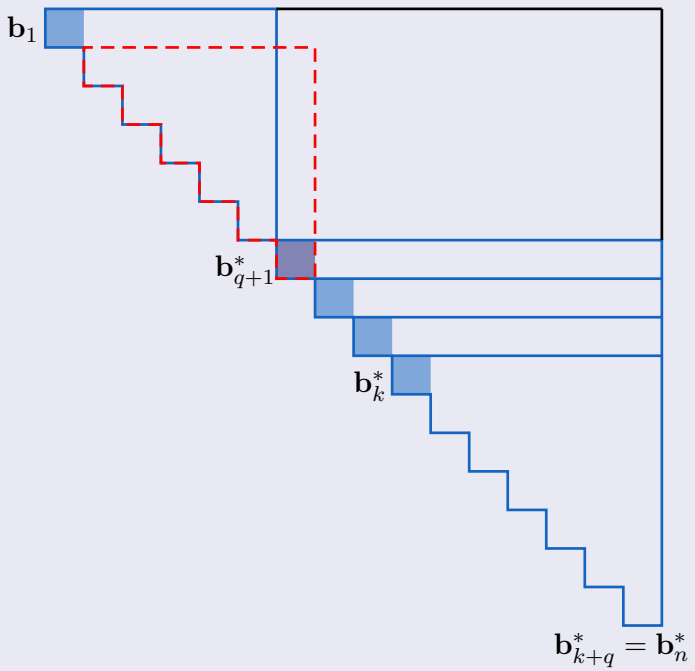

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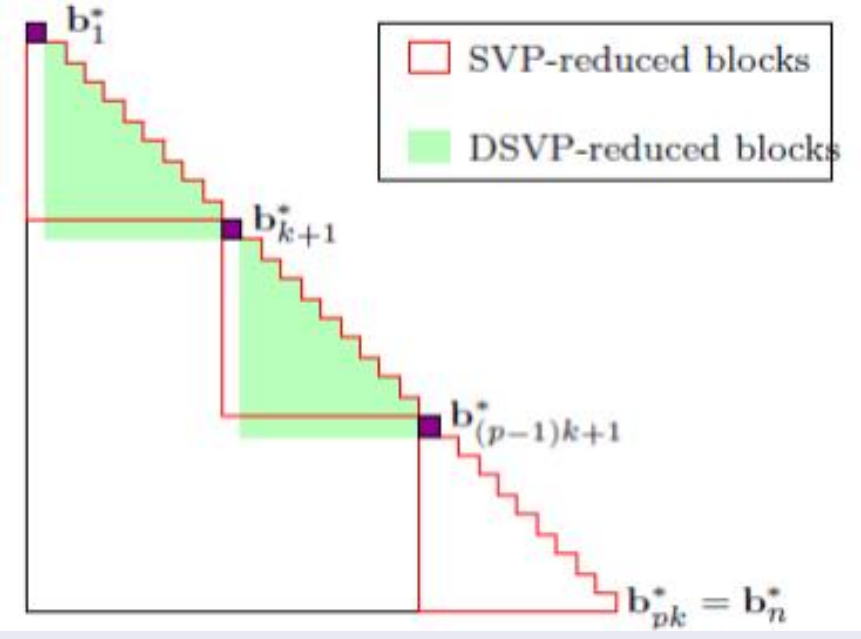
Approximating SVP with (at least) polynomial factor

Recall ideas of GN-slide-reduction

Given a basis B of rank $n = pk \geq 2k$, GN partitions the basis into p blocks of equal rank k:



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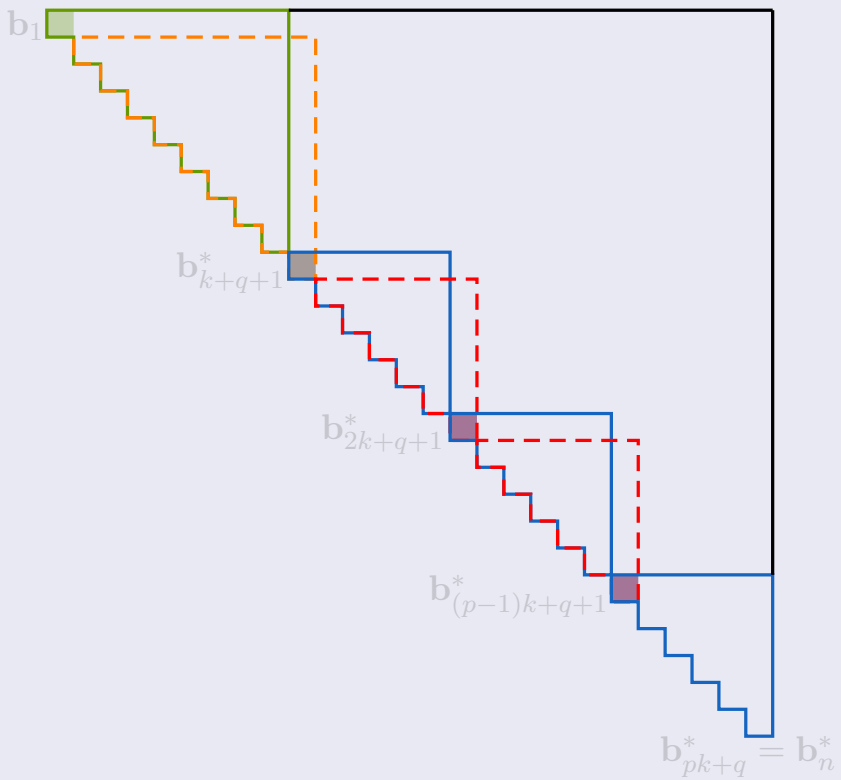

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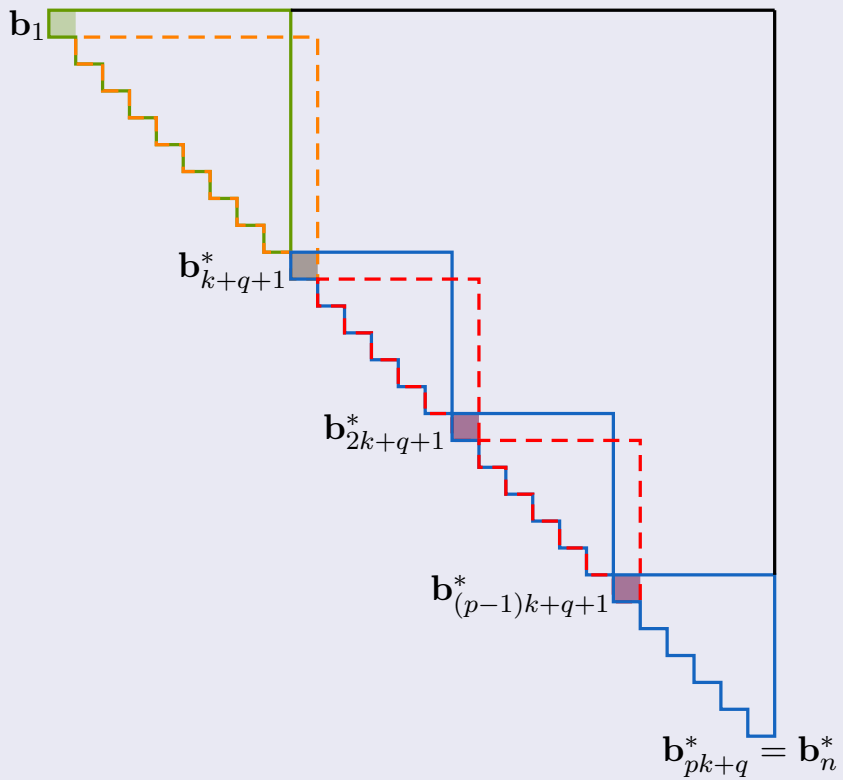

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