Out of Oddity – New Cryptanalytic Techniques against Symmetric Primitives Optimized for Integrity Proof Systems

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Symmetric primitives optimized for a specific cost metric

- **FHE-friendly encryption:** Low-MC [Albrecht et al. 15], Flip [Méaux et al. 16], Kreyvium [Canteaut et al. 16], Rasta [Dobraunig et al. 18]...
- **MPC-friendly block ciphers:** MiMC [Albrecht et al. 16] and its variants
- **Primitives dedicated to new integrity proof systems (STARKs, SNARKs, Bulletproof)**: hash functions specified as sequences of low-degree polynomials or low-degree rational maps over a finite field.

Older examples:

Cradic [Knudsen Nyberg 92], Misty [Matsui 97].

SNARK-friendly and STARK-friendly primitives

Performance.

- the size of the polynomial relations representing the execution trace over a large finite field should be minimized.
- finite fields of odd characteristic, especially prime fields, are suitable.

Security.

- algebraic attacks based on Gröbner basis [Albrecht et al. 19]...
- all other cryptanalytic techniques.

Focus on STARK-friendly primitives

StarkWare challenges `https://starkware.co/hash-challenge/`

Keyed permutations.

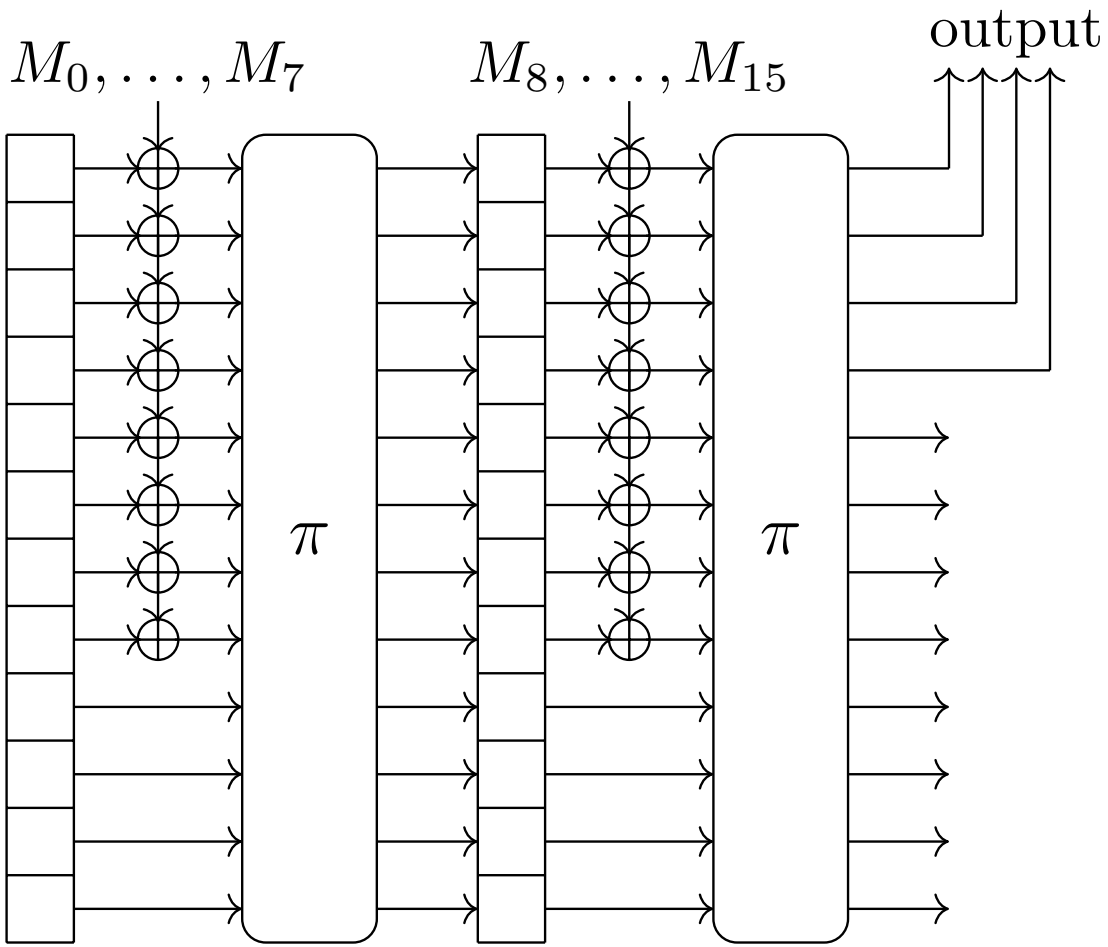
- GMiMC i.e. $\text{GMiMC}_{\text{erf}}$ over $\mathbb{F}_p$ [Albrecht et al. 19]
- HadesMiMC permutation: Starkad ($\mathbb{F}_{2^m}$) and Poseidon ($\mathbb{F}_p$) [Grassi et al. 19]

Hash functions.

sponges using one of the previous functions as inner permutation.

Sponge construction

Sponge construction with blocksize t and capacity c.



Parameters

Security level	$\log_2 q$	q (prime)	q (binary)	c	t	Variant
128 bits	64	$2^{61} + 20 \times 2^{32} + 1$	2^{63}	4	12	128-d
	128	$2^{125} + 266 \times 2^{64} + 1$	2^{125}	2	4	128-a
				2	12	128-c
	256	$2^{253} + 2^{199} + 1$	2^{255}	1	3	128-b
				1	11	128-e
256 bits	128	$2^{125} + 266 \times 2^{64} + 1$	2^{125}	4	8	256-a
				4	14	256-b

Keypoints

- generalization of attacks to fields of any characteristic.
- use of the specific algebraic structure to improve classical attacks.

Outline

- Integral attacks over fields of any characteristic
- Integral distinguishers on the full GMiMC
- Algebraically-controlled differential attacks on GMiMC

Integral attacks over $\mathbb{F}_q$

When $q = 2^m$.

For any $F : \mathbb{F}_{2^m} \to \mathbb{F}_{2^m}$, for any (affine) subspace $V \subset \mathbb{F}_2^m$ with $\deg(F) < |V| - 1$,

$$\sum_{x \in V} F(x) = 0.$$

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Because, for $V = b + \langle a_1, \ldots, a_v \rangle$,

$$D_{a_1} D_{a_2} \ldots D_{a_v} F(b) = \sum_{x \in V} F(x)$$

Not valid in odd characteric.

But for any q

For any exponent k with $0 \le k < q-1$,

$$\sum_{x \in \mathbb{F}_q} x^k = 0$$

General result.

For any $F : \mathbb{F}_q \to \mathbb{F}_q$ with $\deg(F) < q-1$,

$$\sum_{x \in \mathbb{F}_q} F(x) = 0 \ .$$

However, this only works when an input is saturated

For any $F : \mathbb{F}_q \to \mathbb{F}_q$ with $\deg(F) < q - 1$,

$$\sum_{x \in \mathbb{F}_q} F(x) = 0 \ .$$

Less general than the property over $\mathbb{F}_{2^m}$:

For any (affine) subspace $V \subset \mathbb{F}_2^m$ such that $\deg(F) < |V| - 1$,

$$\sum_{x \in V} F(x) = 0.$$

Using multiplicative subgroups

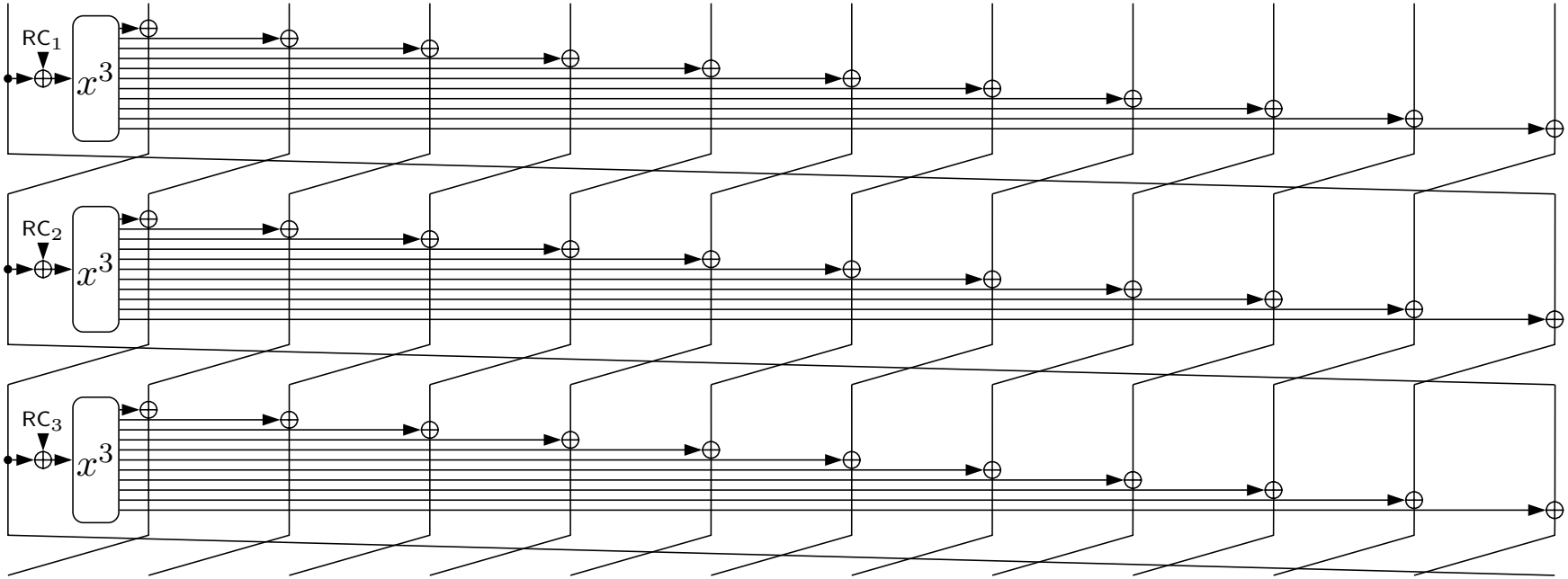
Let G be a multiplicative subgroup of $\mathbb{F}_q^\times$.

For any $F : \mathbb{F}_q \to \mathbb{F}_q$ such that $\deg(F) < |G|$,

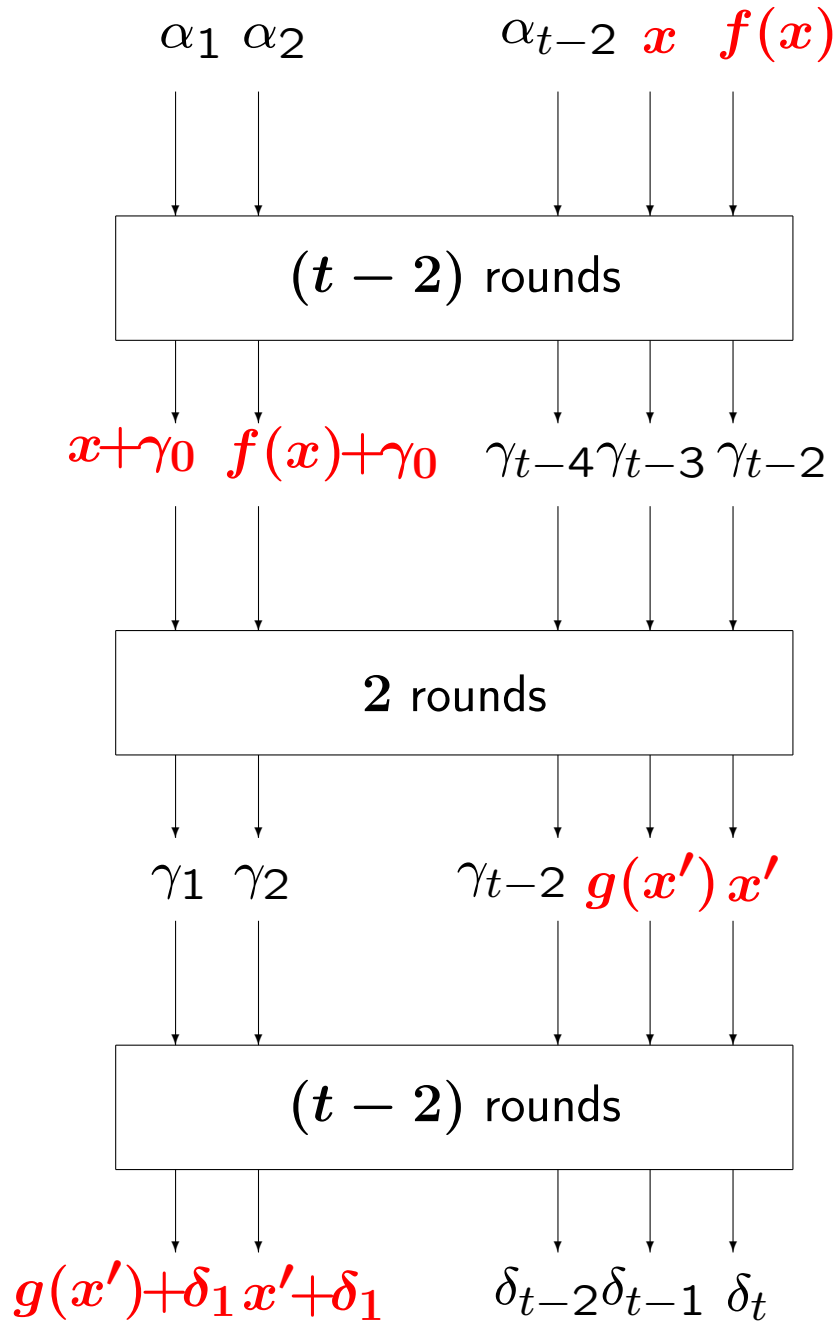
$$\sum_{x \in G} F(x) = F(0) \cdot |G| \ .$$

Integral attacks on GMiMC

GMiMC with 101 rounds



A differential property on $(2t-2)$ rounds



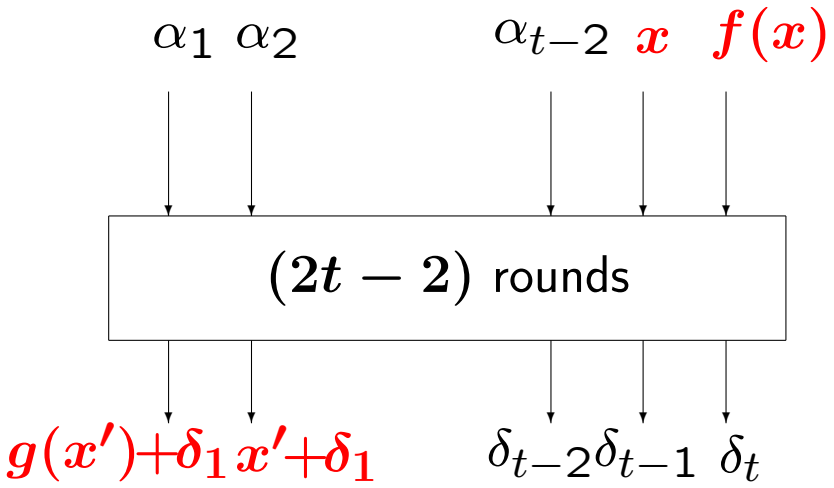
$x \in \mathbb{F}_q$

$\deg f = 3$

$x' \in \mathbb{F}_q$

$\deg g = 3$

A differential property on $(2t-2)$ rounds



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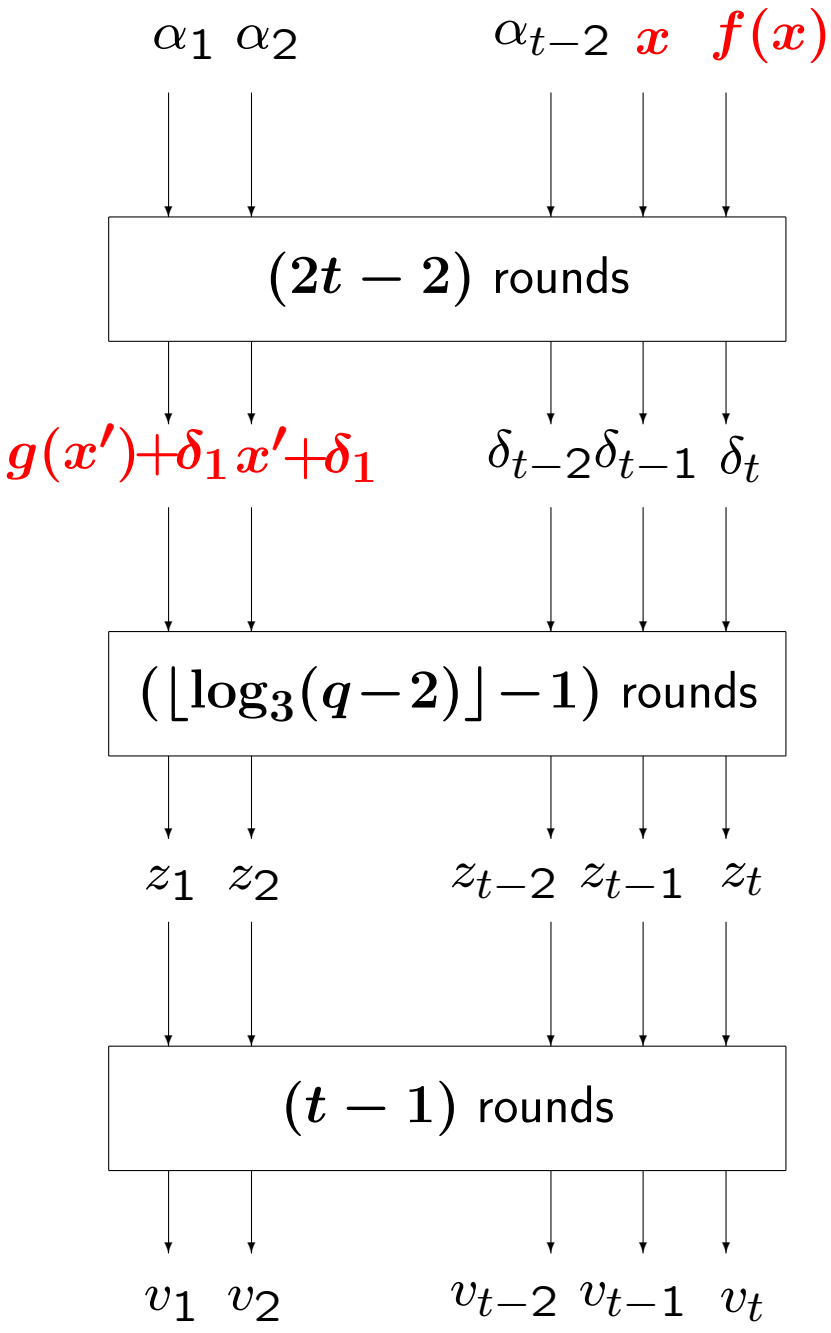
Integral distinguisher on GMiMC



$x \in \mathbb{F}_q$

$\deg f = 3$

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$\deg_x z_i = 3^{r+1} < q-1$

for $r \leq \lfloor \log_3(q-2) \rfloor - 1$

$\sum_{i=2}^{t} v_i - (t-2)v_1$

is a linear combination of the z_i

Integral distinguisher on GMiMC

With complexity q.

After $3t - 4 + \lfloor \log_3(q-2) \rfloor$ rounds,

$$\sum_{i=2}^{t} v_i - (t-2)v_1$$

is a polynomial of degree at most $(q-2)$ in x.

$\Rightarrow$ It sums to 0 when x varies in $\mathbb{F}_q$.

$\log_2 q$	t	Full nb of rounds	Nb of rounds of the distinguisher
61	12	101	70
125	4	166	86
125	12	182	110
253	3	326	–
253	11	342	–

Integral distinguisher on GMiMC using multiplicative subgroups

For $q = 2^{253} + 2^{199} + 1$.

After $3t - 4 + \lfloor \log_3(|G| - 1) \rfloor$ rounds,

$$\sum_{i=2}^{t} v_i - (t-2)v_1$$

is a polynomial of degree at most $(|G| - 1)$ in x.

$\Rightarrow$ It sums to 0 when x varies in G.

$\log_2 q$	t	Full nb of rounds	Nb of rounds of the distinguisher
61	12	101	70
125	4	166	86
125	12	182	110
253	3	326	85 with $\lvert G\rvert = 2^{128}$
253	11	342	109 with $\lvert G\rvert = 2^{128}$

Zero-sum distinguisher on GMiMC

With a multiplicative subgroup G.

After $4t - 6 + 2\lfloor \log_3(|G| - 1) \rfloor$ rounds,

$$\sum_{i=1}^{t-1} u_i - (t-2)u_t \quad \text{and} \quad \sum_{i=2}^{t} v_i - (t-2)v_1$$

sum to 0 when x varies in G.

$\log_2 q$	t	Full nb of rounds	Nb of rounds of the ZS	$\lvert G \rvert$
61	12	101	118	q
61	12	101	102	$2^{33} \cdot 167 \cdot 211 \simeq 2^{48}$
125	4	166	166	q
125	12	182	198	q
253	3	326	166	2^{128}
253	11	342	198	2^{128}

Algebraically-controlled differential attacks on GMiMC

Algebraically-controlled differential attacks

Idea:

use algebraic techniques to efficiently find hash function inputs that satisfy a differential characteristic (avoid expensive probabilistic cost)

Method:

represent the conditions of differential transitions as (efficiently solvable) algebraic equations

Application to GMiMC:

- exploit algebraic structure to penetrate deep into internal state
- attack almost entirely algebraic — differential transitions too expensive to bypass probabilistically

Results:

- basic method on $3t - 2$ rounds of permutation
- extend to more rounds and attack the hash function (e.g., practical 40-round collision on GMiMC-128-d)

Application to GMiMC

Differential characteristic:

Δ_0, Δ_0' arbitrary non-zero differences

$$(\Delta_0, \Delta_0', 0, \ldots, 0) \xrightarrow{\mathcal{R}} (\Delta_0' + \Delta_1, \Delta_1, \ldots, \Delta_1, \Delta_0) \qquad \Delta_0 \xrightarrow{S} \Delta_1$$

$$\xrightarrow{\mathcal{R}} (\Delta_1 + \Delta_1', \ldots, \Delta_1 + \Delta_1', \Delta_0 + \Delta_1', \Delta_0' + \Delta_1) \qquad \Delta_0' + \Delta_1 \xrightarrow{S} \Delta_1'$$

Application to GMiMC

Differential characteristic:

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$$(\boldsymbol{\Delta_0}, \boldsymbol{\Delta_0'}, 0, \ldots, 0) \xrightarrow{\mathcal{R}} (\boldsymbol{\Delta_0'}+\boldsymbol{\Delta_1}, \boldsymbol{\Delta_1}, \ldots, \boldsymbol{\Delta_1}, \boldsymbol{\Delta_0}) \qquad \boldsymbol{\Delta_0} \xrightarrow{S} \boldsymbol{\Delta_1}$$

$$\xrightarrow{\mathcal{R}} (\boldsymbol{\Delta_1}+\boldsymbol{\Delta_1'}, \ldots, \boldsymbol{\Delta_1}+\boldsymbol{\Delta_1'}, \boldsymbol{\Delta_0}+\boldsymbol{\Delta_1'}, \boldsymbol{\Delta_0'}+\boldsymbol{\Delta_1}) \quad \boldsymbol{\Delta_0'}+\boldsymbol{\Delta_1} \xrightarrow{S} \boldsymbol{\Delta_1'}$$

If $\boldsymbol{\Delta_1} + \boldsymbol{\Delta_1'} = 0$, we get an iterative differential characteristic

$$(\boldsymbol{\Delta_0}, \boldsymbol{\Delta_0'}, 0, \ldots, 0) \xrightarrow{\mathcal{R}^t} (\boldsymbol{\Delta_0} - \boldsymbol{\Delta_1}, \boldsymbol{\Delta_0'} + \boldsymbol{\Delta_1}, 0, \ldots, 0)$$

Application to GMiMC

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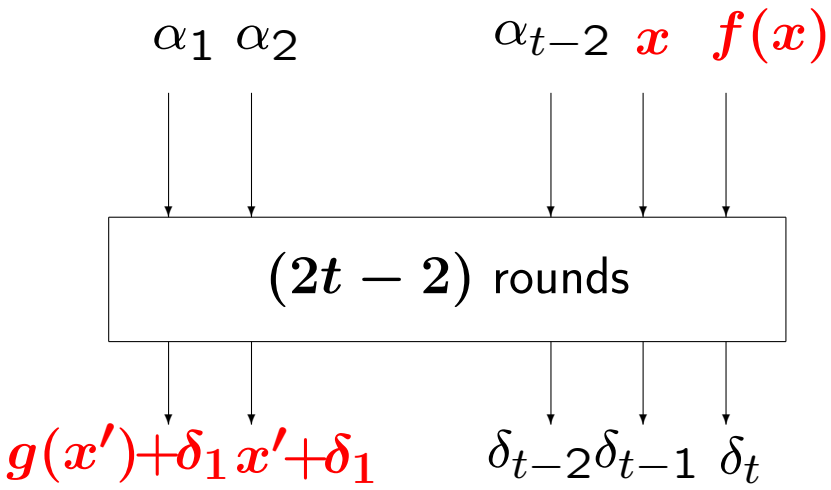
If $\Delta_1 + \Delta_1' = 0$, we get an iterative differential characteristic

$$(\Delta_0, \Delta_0', 0, \ldots, 0) \xrightarrow{\mathcal{R}^t} (\Delta_0 - \Delta_1, \Delta_0' + \Delta_1, 0, \ldots, 0)$$

$\Delta_1 + \Delta_1' = 0$ occurs with probability $\approx 1/q$

Condition $\Delta_1 + \Delta_1' = 0$ is viewed as an equation on values

A differential property on $(2t-2)$ rounds



$x \in \mathbb{F}_q$

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$\deg g = 3$

Satisfying $(3t-2)$ rounds

Satisfying $(2t-2)$ rounds with special states values:

$$X_0 = (\alpha_1, \ldots, \alpha_{t-2}, x, f(x)) \xrightarrow{\mathcal{R}^{2t-2}} X_{2t-2} = (g(x') + \delta_1, x' + \delta_1, \delta_3, \ldots, \delta_t)$$

$$Y_0 = (\alpha_1, \ldots, \alpha_{t-2}, y, f(y)) \xrightarrow{\mathcal{R}^{2t-2}} Y_{2t-2} = (g(y') + \delta_1, y' + \delta_1, \delta_3, \ldots, \delta_t)$$

Therefore

$$X_{2t-2} - Y_{2t-2} = (g(x') - g(y'), x' - y', 0, \ldots, 0)$$

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Adding t rounds with the differential characteristic

$$(\Delta_0, \Delta_0', 0, \ldots, 0) \xrightarrow{\mathcal{R}^t} (\Delta_0 - \Delta_1, \Delta_0' + \Delta_1, 0, \ldots, 0)$$

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The differential transition after $(3t-2)$ rounds is assured if $\Delta_2(x,y) + \Delta_2'(x,y) = 0$

Degree of the equation?

Satisfying $(3t-2)$ rounds

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Adding t rounds with the differential characteristic

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Degree of the equation

$$\xrightarrow{\mathcal{R}^{2t-2}} \deg g = 3 \xrightarrow{\mathcal{R}} \deg = 3^2 \xrightarrow{\mathcal{R}} \deg = 3^3 = 27$$

Satisfying $(3t-2)$ rounds

Satisfying $(2t-2)$ rounds with special states values:

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Therefore

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Degree **only** $3^3 = 27$

Set $y =$ **const** and solve for x (factor polynomial)

Conclusions

	Rounds		Type	Rounds	Cost
GMiMC (128 bits)	101	permutation	integral distinguisher	70	2^{61}
			ZS distinguisher	102	$\mathbf{2^{48}}$
			diff. distinguisher	66	practical
		hash function	collisions	40	practical
			collisions	52	2^{83}
HadesMiMC (128 bits)	8+40	permutation	ZS distinguisher	6+45	2^{61}
GMiMC (256 bits)	186	permutation	integral distinguisher	116	2^{125}
			ZS distinguisher	206	$\mathbf{2^{125}}$
HadesMiMC (256 bits)	8+83	permutation	ZS distinguisher	6+87	2^{125}
		hash function*	preimages	8+any	$\mathbf{2^{160}}$

Need for new tools for analyzing primitives over fields of odd characteristic.

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