A Classification of Computational Assumptions in the Algebraic Group Model

Balthazar Bauer, Georg Fuchsbauer, Julian Loss

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1. The Algebraic Group Model (FKL 2018)

2. Classification

3. Separation

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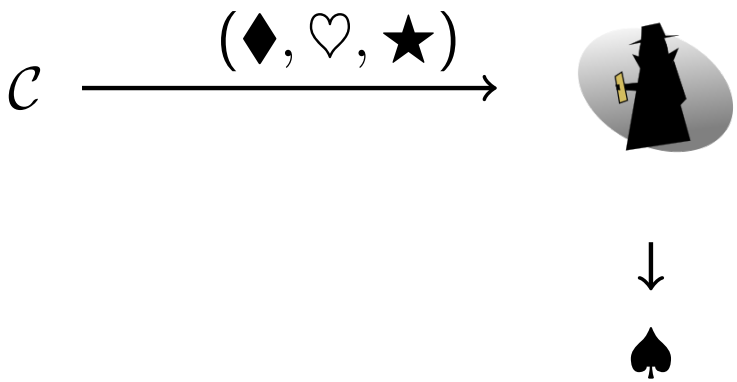

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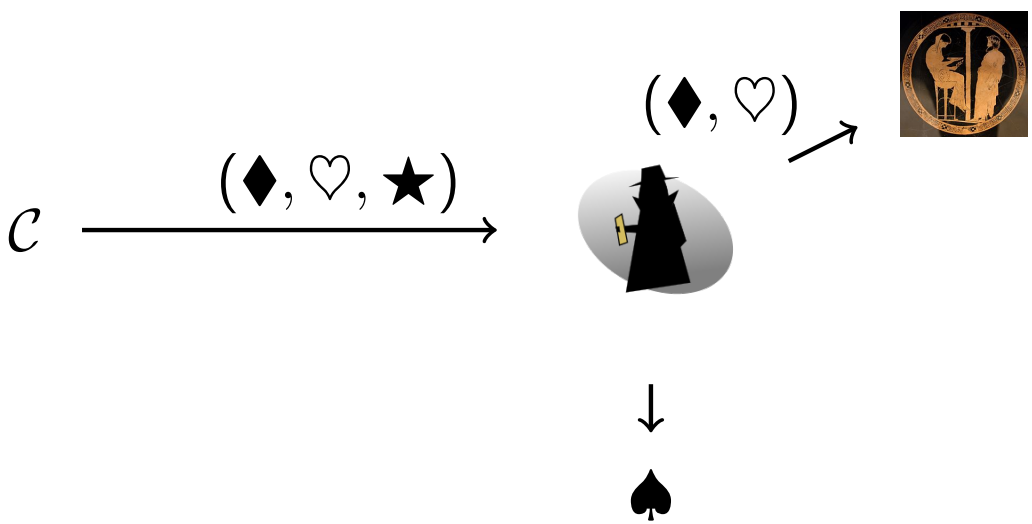

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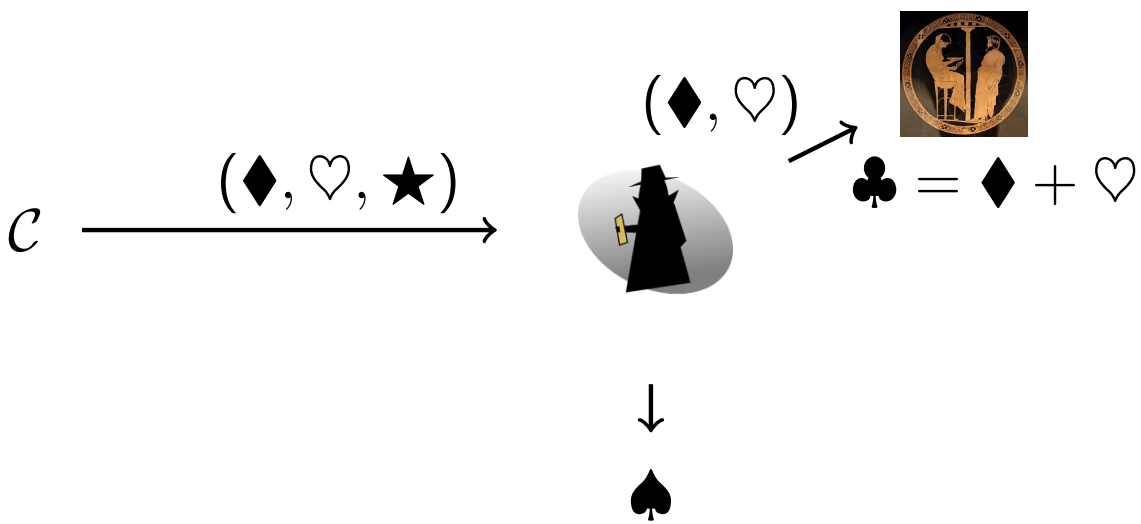

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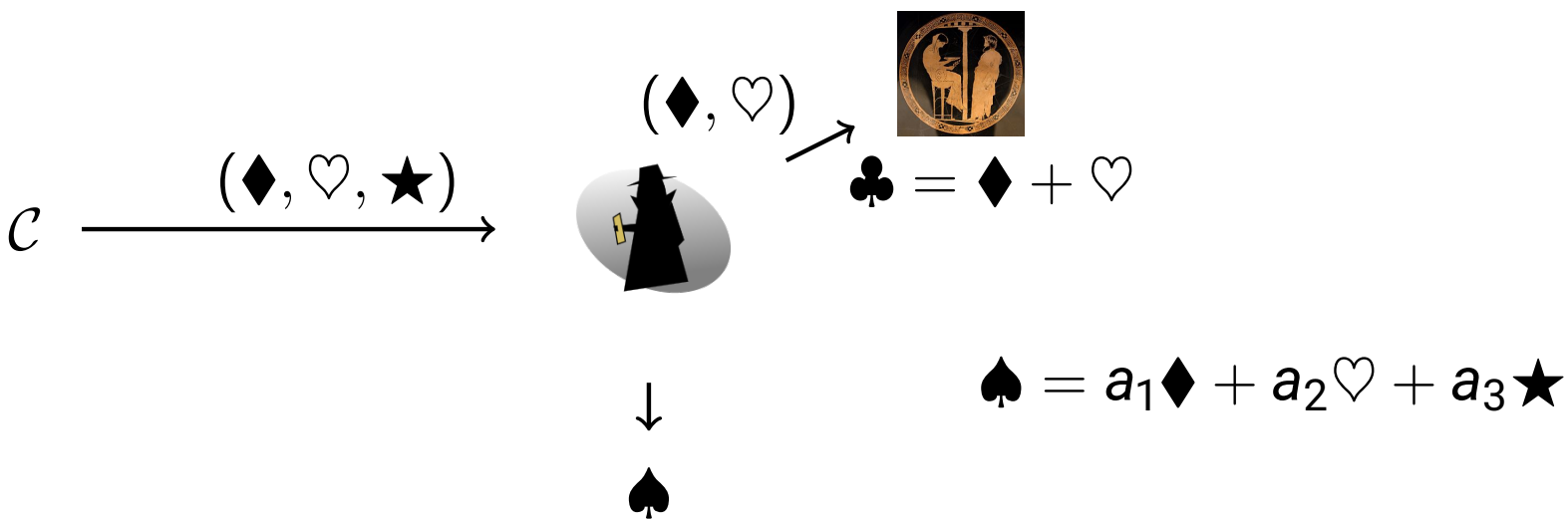

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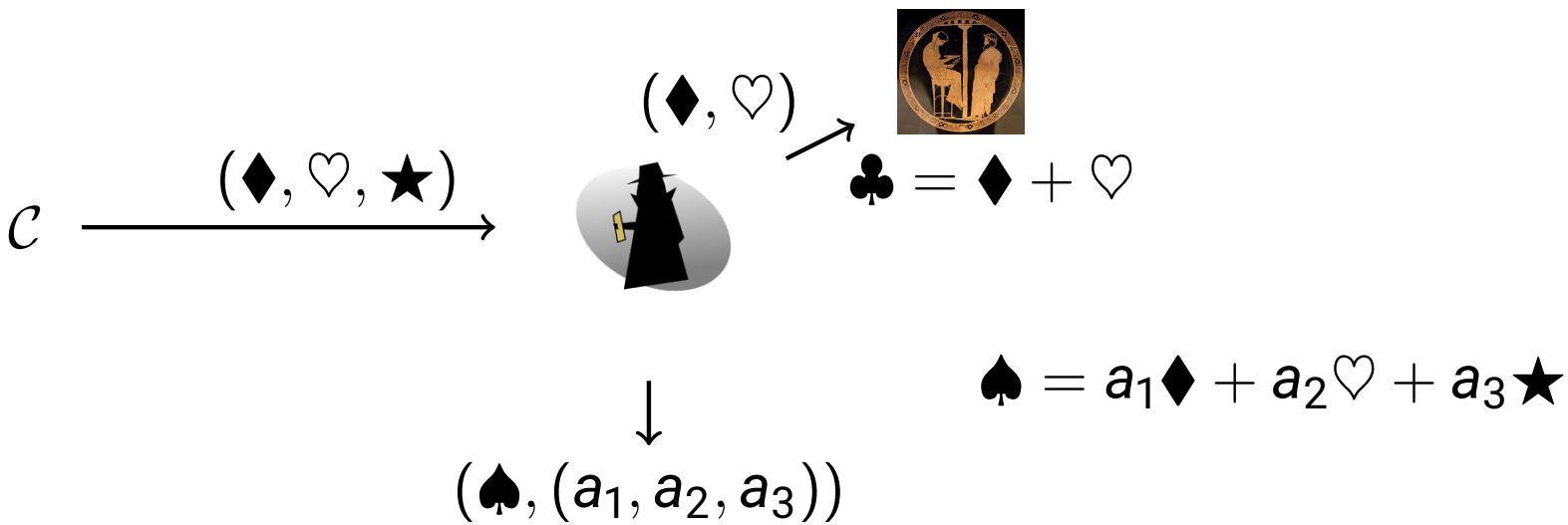

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- Conclusion: AGM enables new security reductions

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DHI

one-more DLog

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SRDH

Gap-DH

q''-DHE

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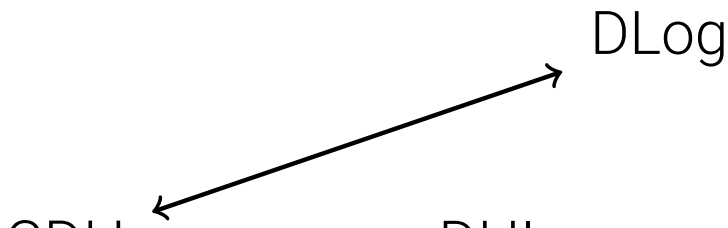

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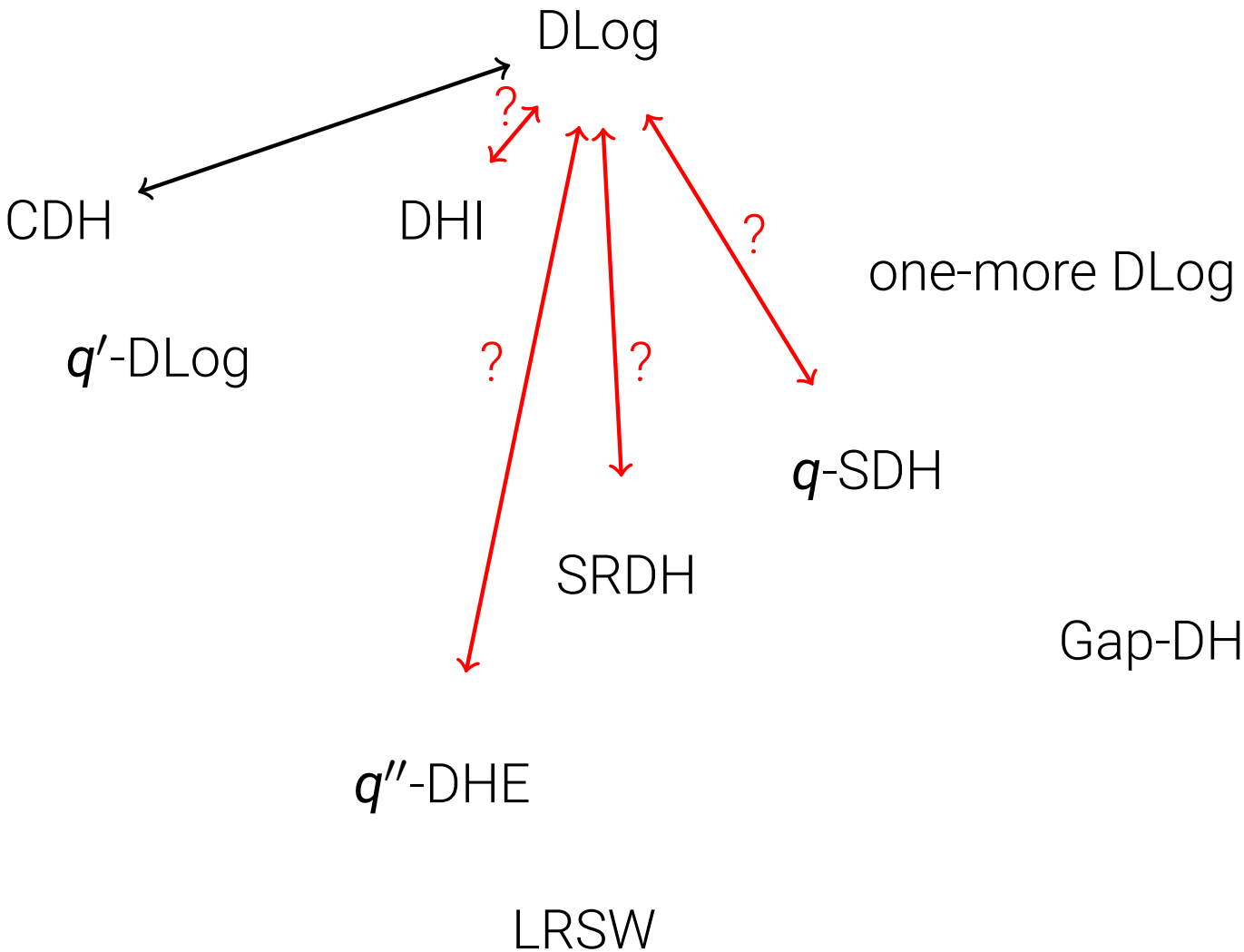
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- Hard in the GGM if $P \notin \mathrm{Span}(\vec{R})$ (non-triviality condition)

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q-DLog

- General idea: Generalize DLog assumption

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- **Ruber**: Polynomials $\rightarrow$ Rational fractions (q-SDH)

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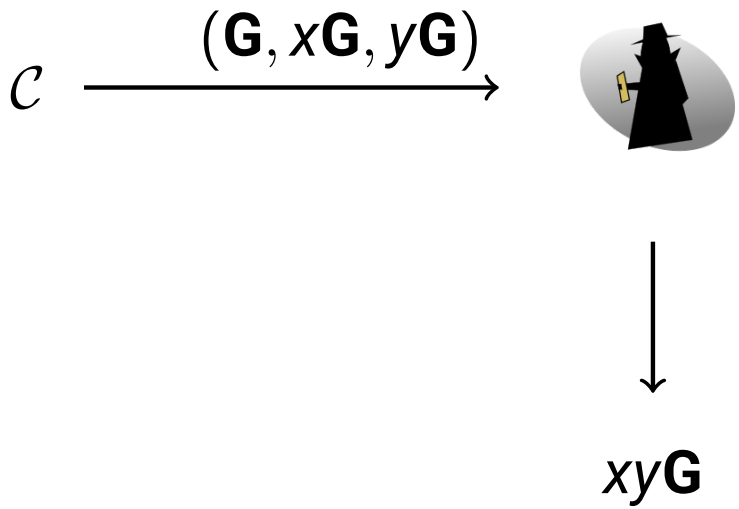
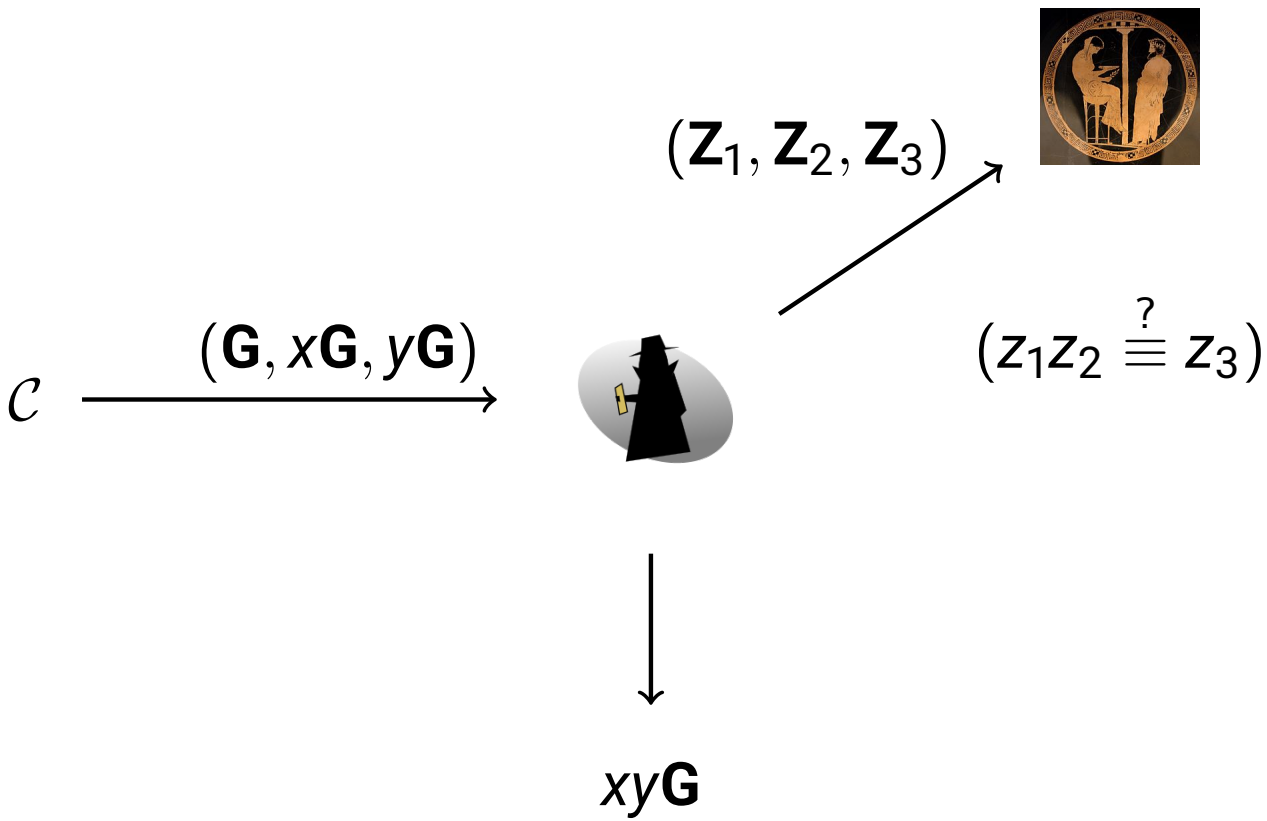
Gap-DH



Gap-DH


$(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$
$\mathcal{C}$
$(\mathbf{G}, x\mathbf{G}, y\mathbf{G})$
$(z_1 z_2 \stackrel{?}{\equiv} z_3)$
$xy\mathbf{G}$

Generalization

- **Uber**:
  - Group $\rightarrow$ Bilinear Group (types $1, 2, 3$)
  - Univariate $\rightarrow$ Multivariate (CDH)
    (embed the challenge in every coordinate: $x_i := y_i z + v_i$)
  - Fixed targets $\rightarrow$ Flexible targets
    $\mathcal{A}$ can choose $P \notin \mathrm{Span}(R)$
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$$\rightarrow \begin{pmatrix} m^* \\ a^*\mathbf{G} \\ a^*(x + m^*xy)\mathbf{G} \end{pmatrix}$$

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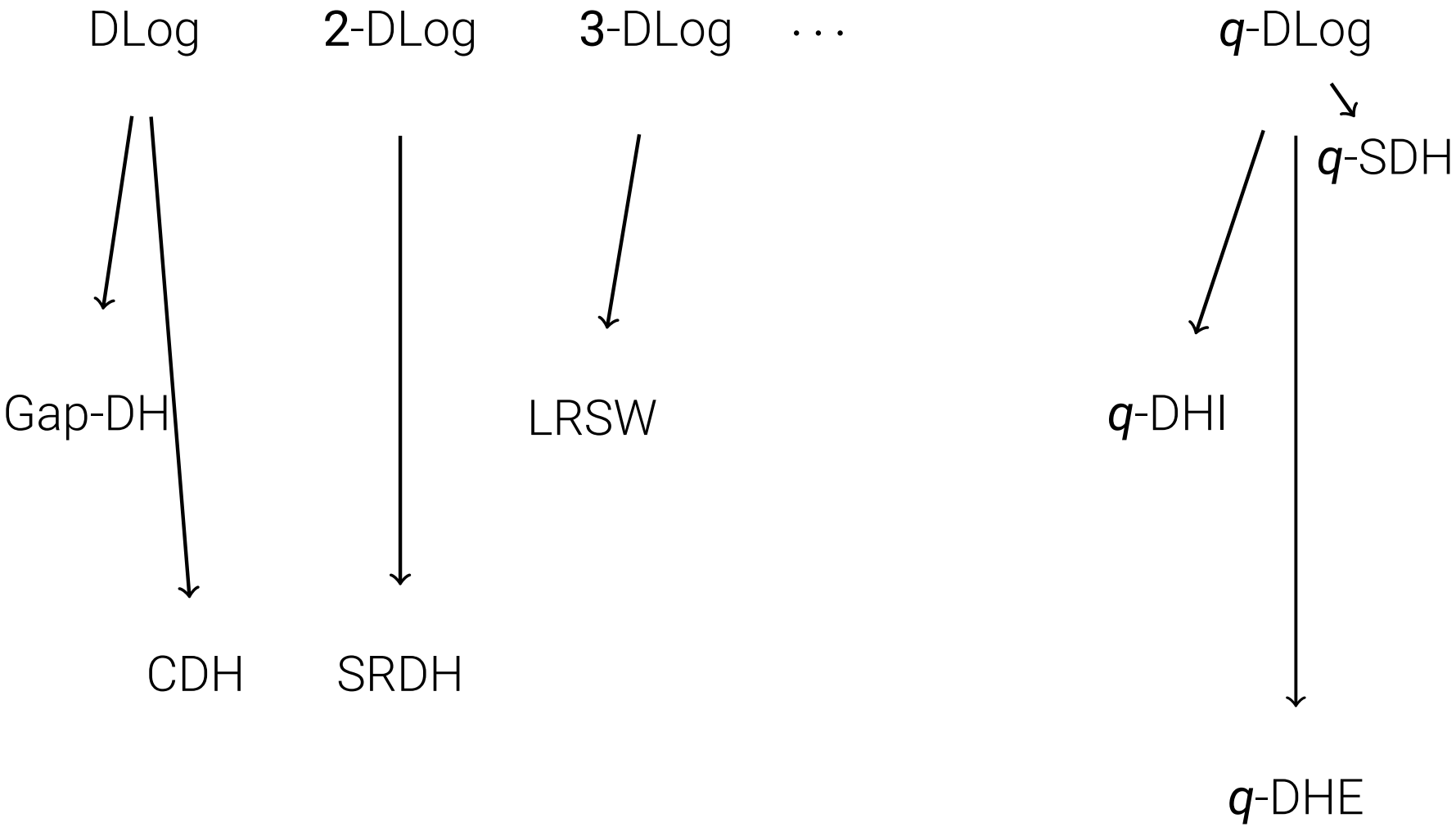
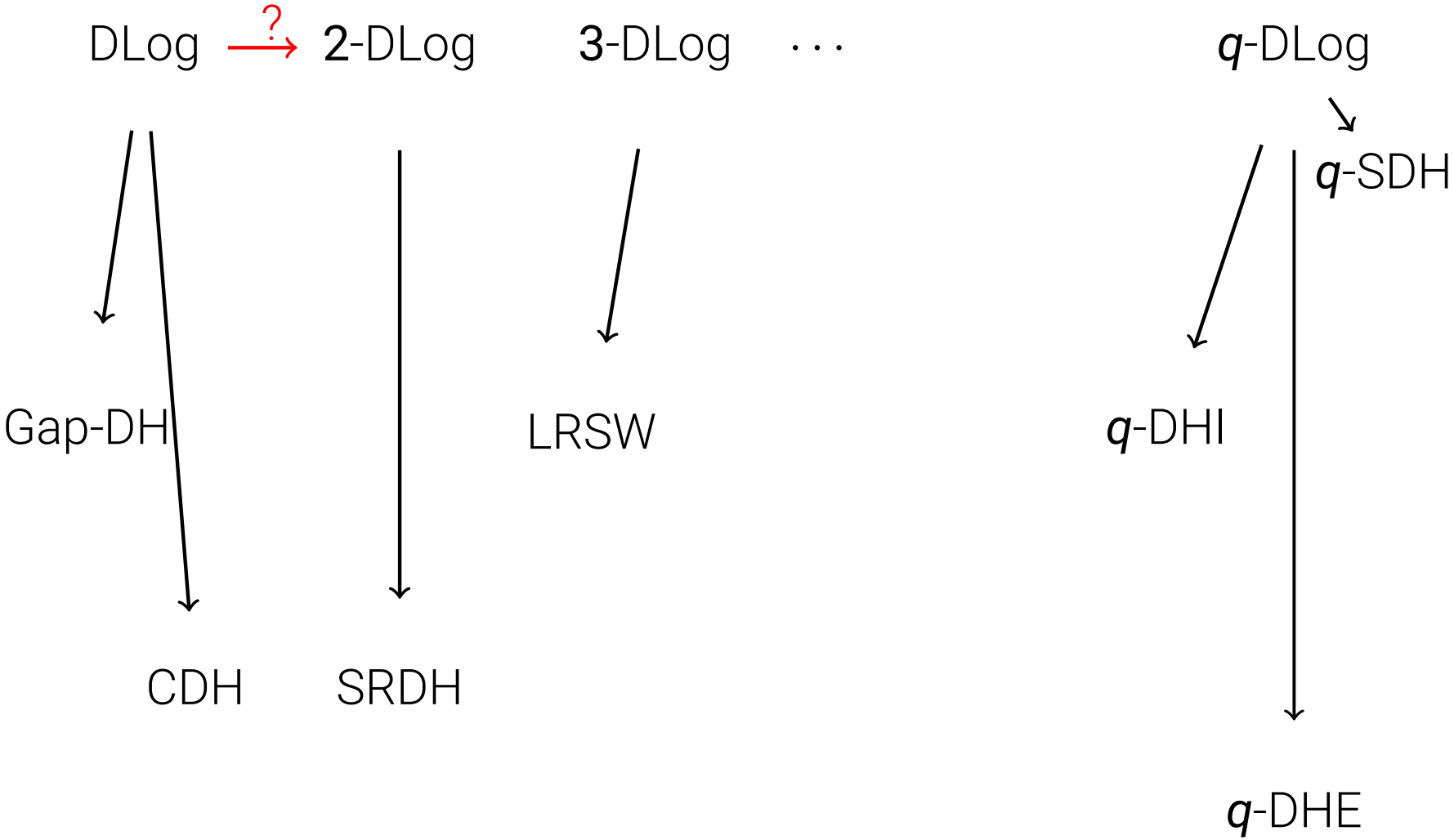
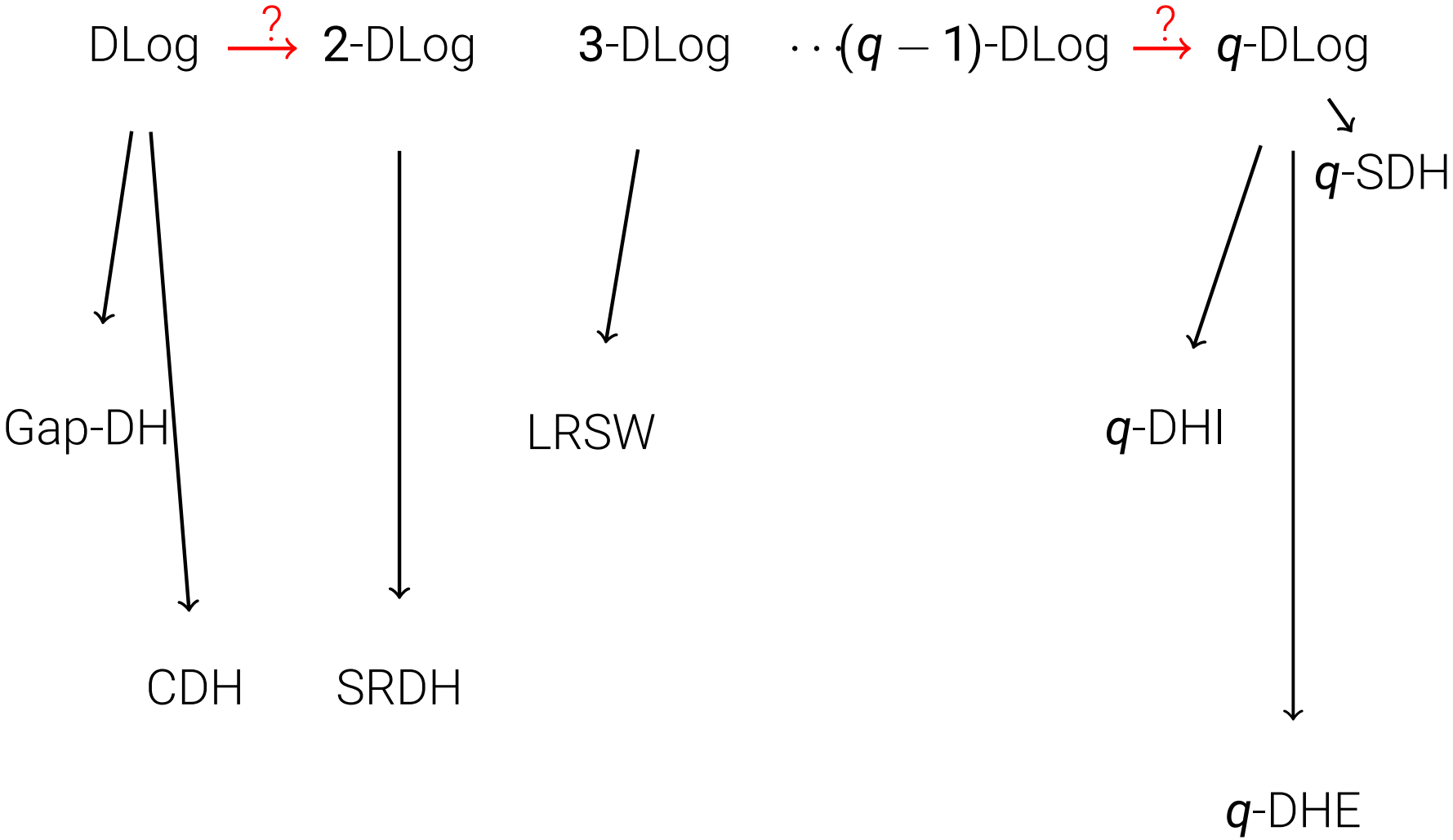
$\mathcal{A}$ can choose $\mathbf{G}'$ and return $(\mathbf{G}', P(\vec{x})\mathbf{G}')$.



Can we do better?

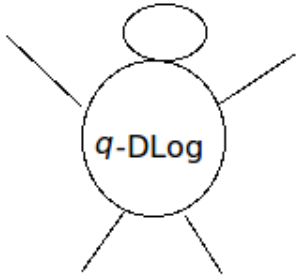


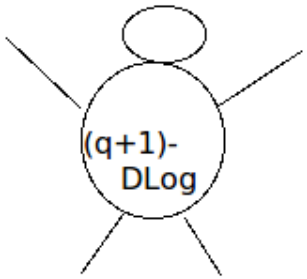
Can we do better?

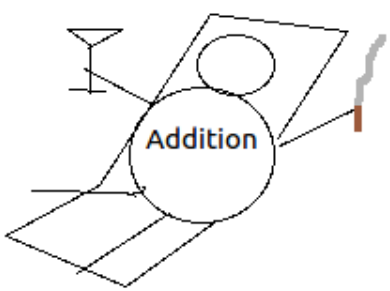


Can we do better?

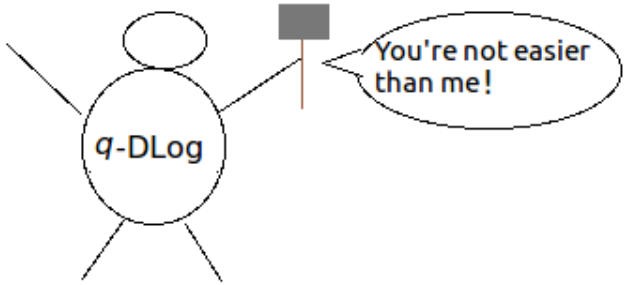
1. The Algebraic Group Model (FKL 2018)
2. Classification
3. Separation

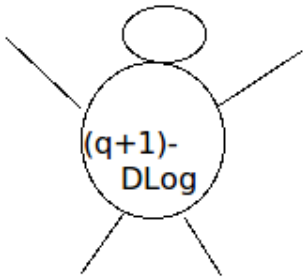


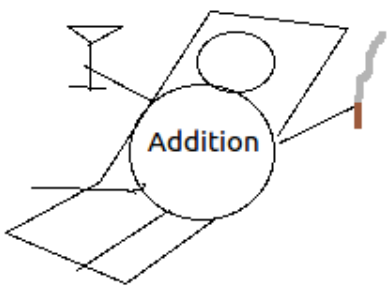




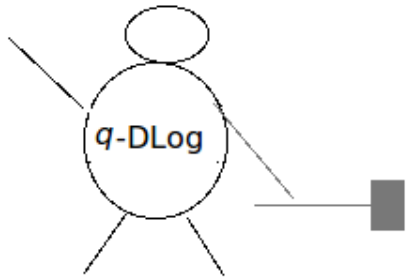
hardness

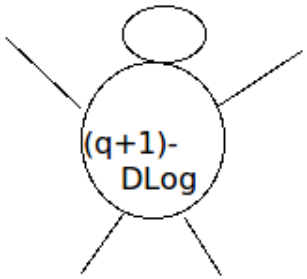


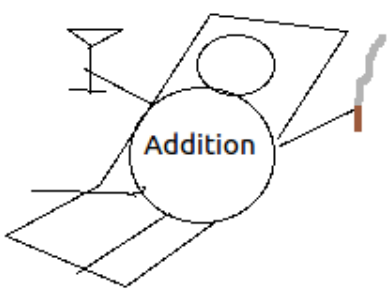




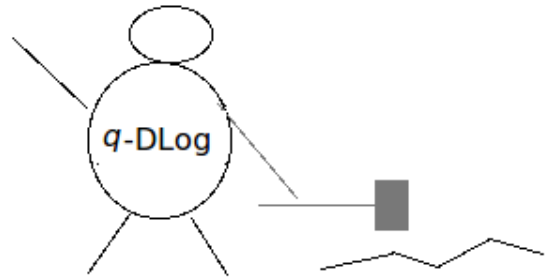
hardness

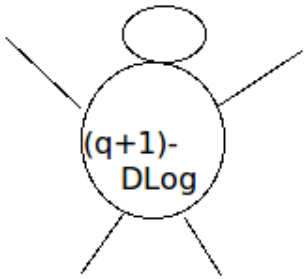


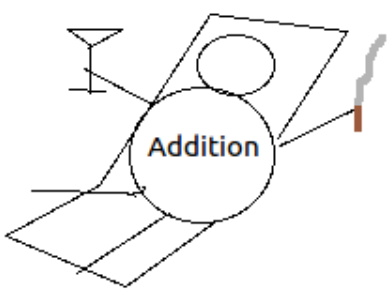




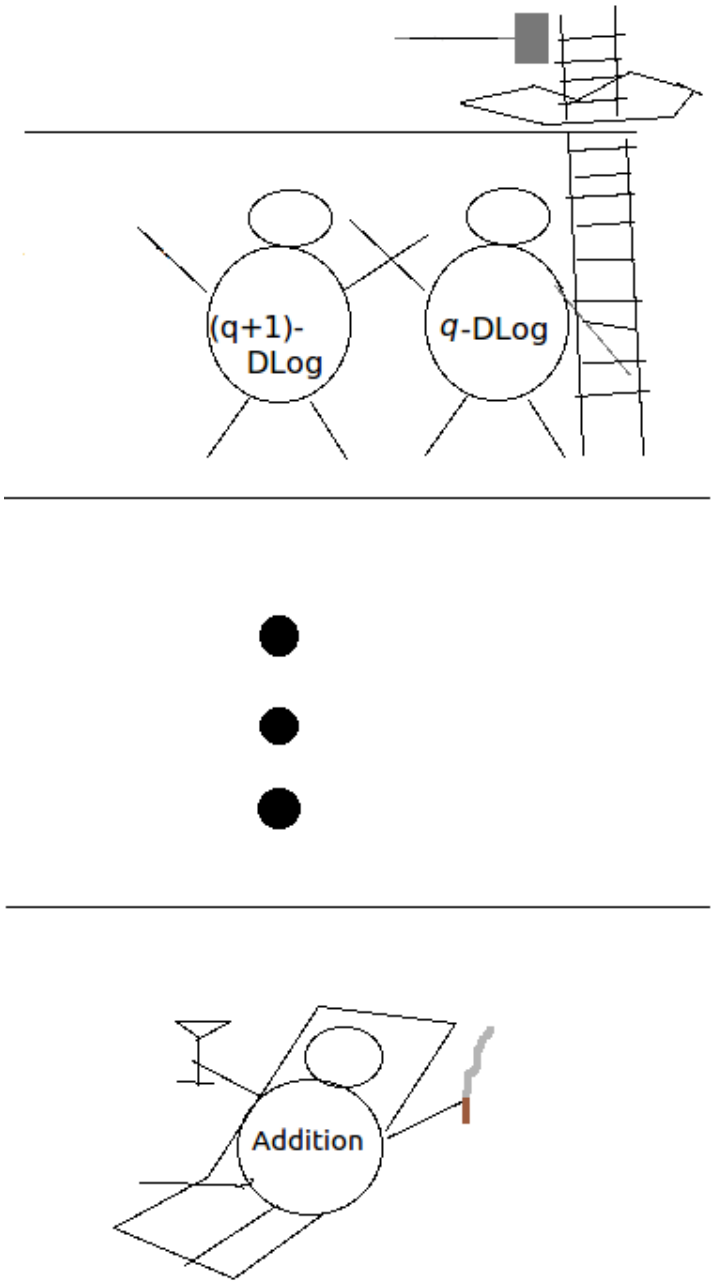
hardness







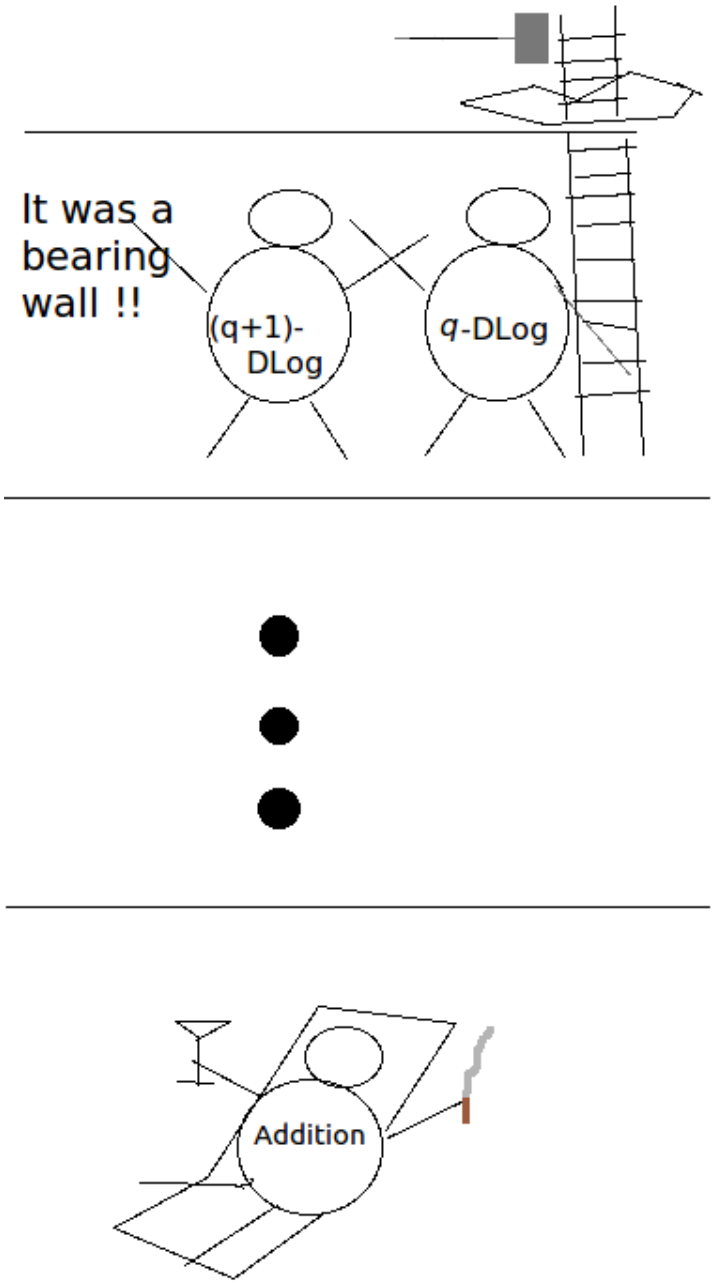
hardness



hardness

Thm:

If $(q+1)$-DLog is q-DLog-hard.



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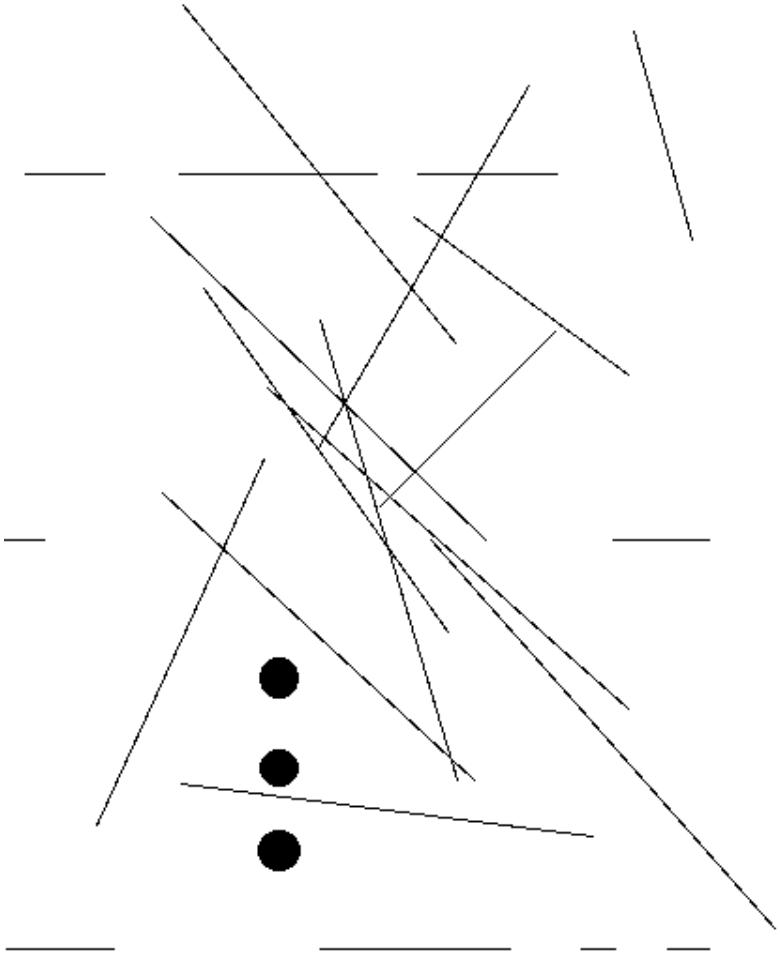
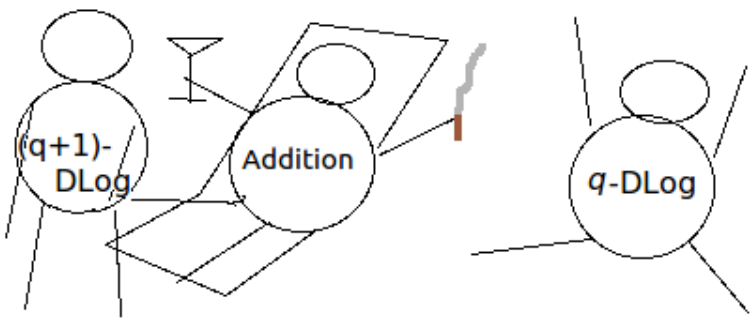


hardness



Thm:

If $(q+1)$-DLog
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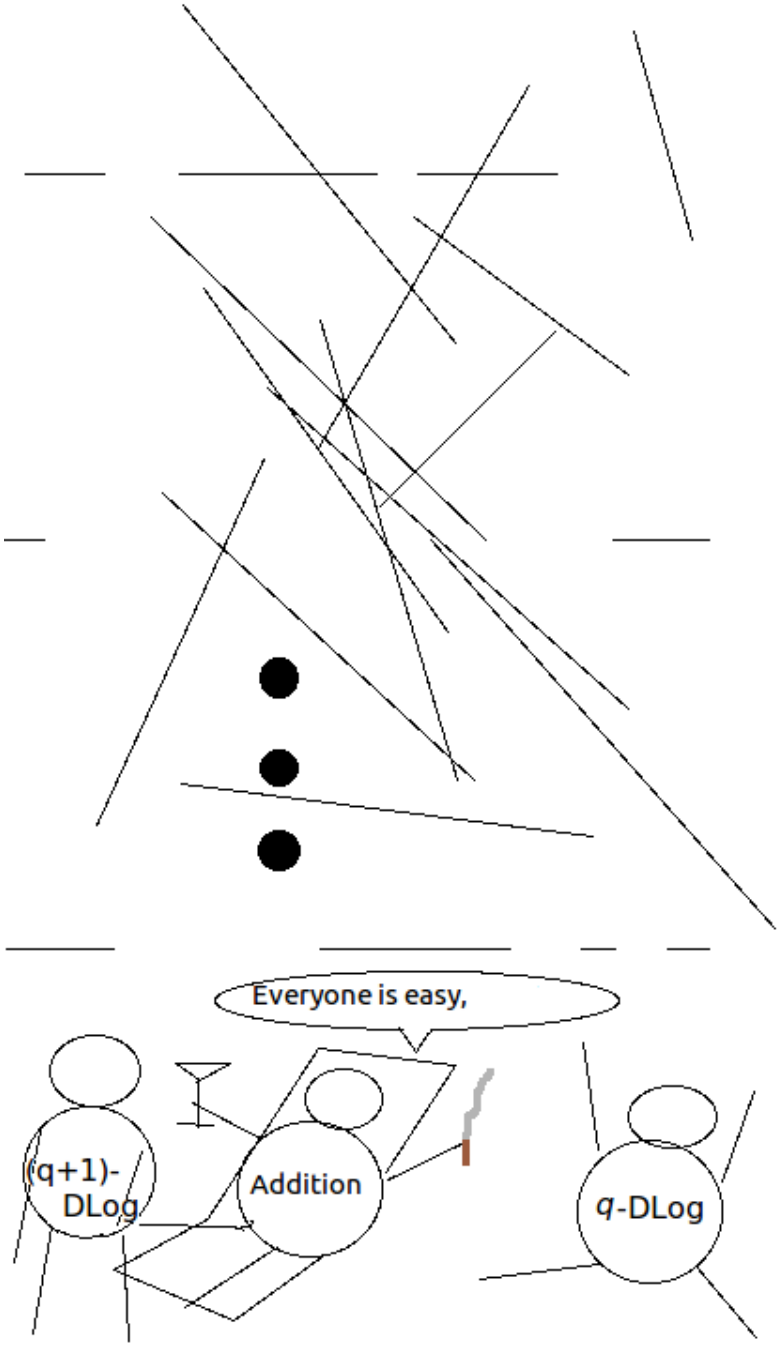
↓

q-DLog $\in$ FBPP.



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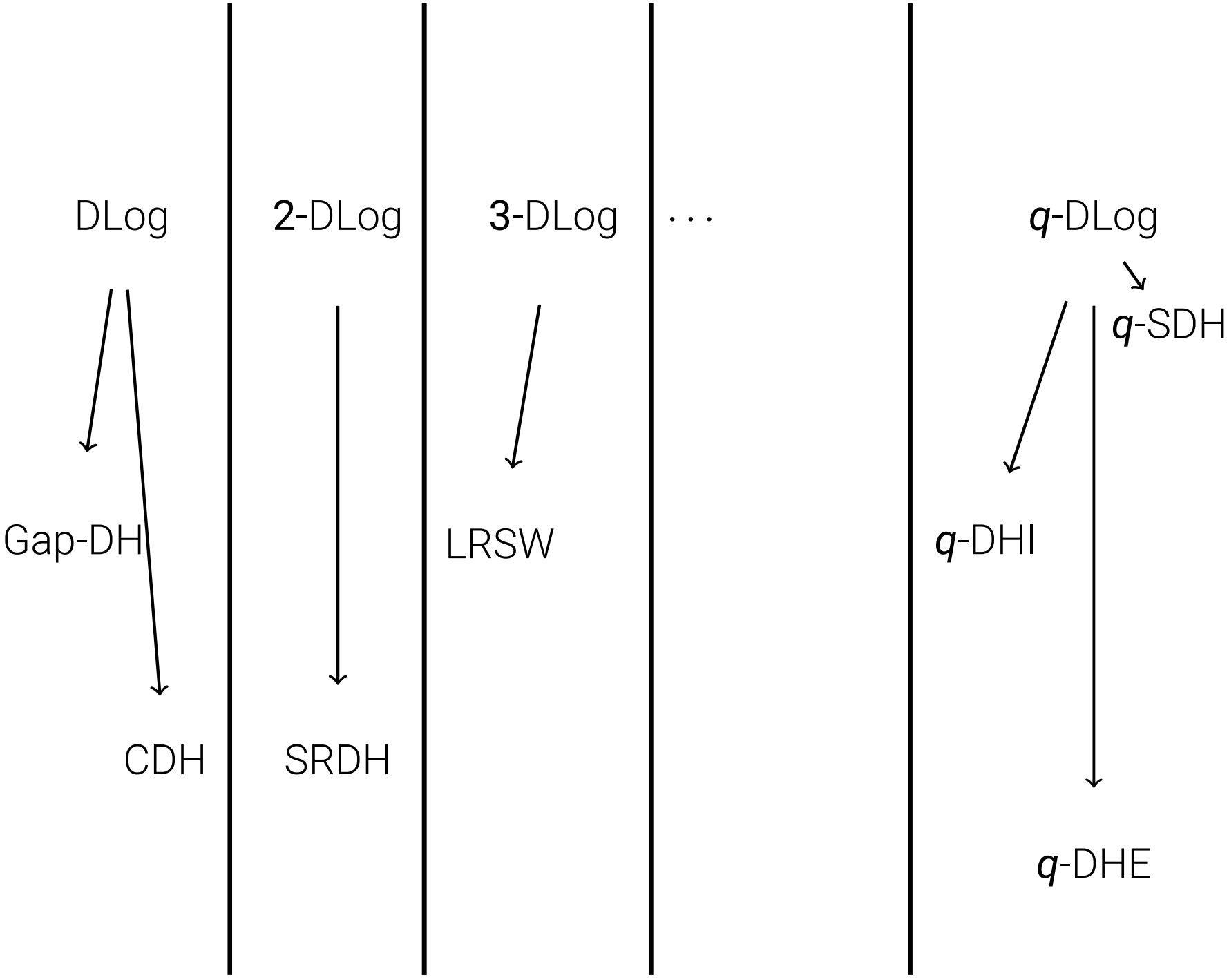
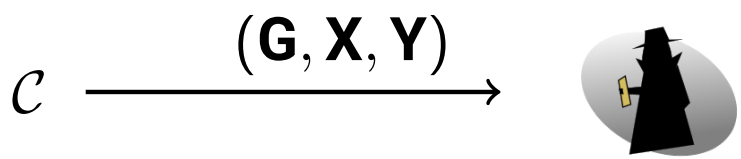

DLog
2-DLog
3-DLog
· · ·
q-DLog
q-SDH
Gap-DH
LRSW
q-DHI
CDH
SRDH
q-DHE

One-More Discrete Log



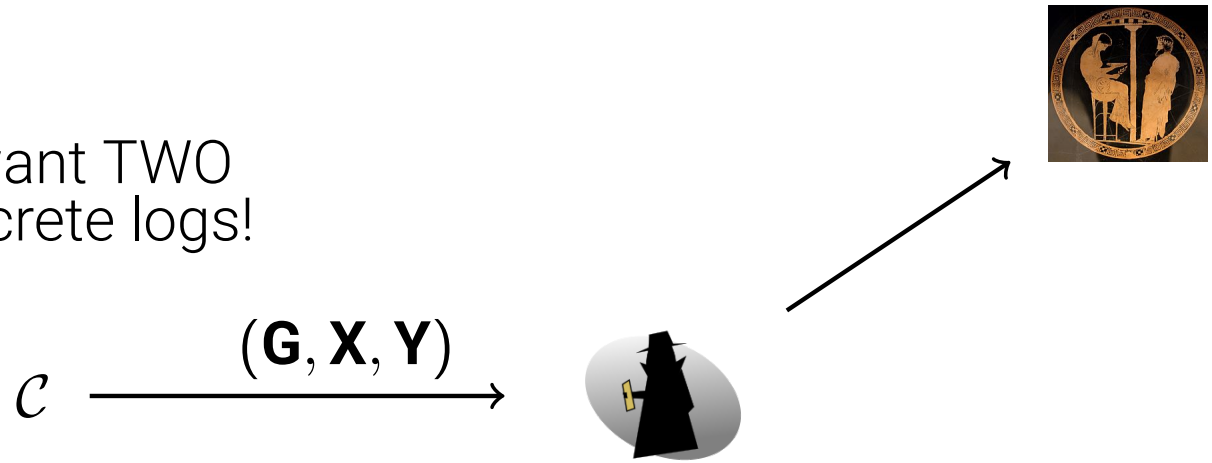
One-More Discrete Log

I want TWO discrete logs!

$\mathcal{C} \xrightarrow{(\mathbf{G}, \mathbf{X}, \mathbf{Y})}$ 

One-More Discrete Log

I want TWO discrete logs!



One-More Discrete Log

I want TWO discrete logs!

$\mathcal{C}$ $\xrightarrow{(\mathbf{G}, \mathbf{X}, \mathbf{Y})}$

I will compute only ONE discrete log for you...

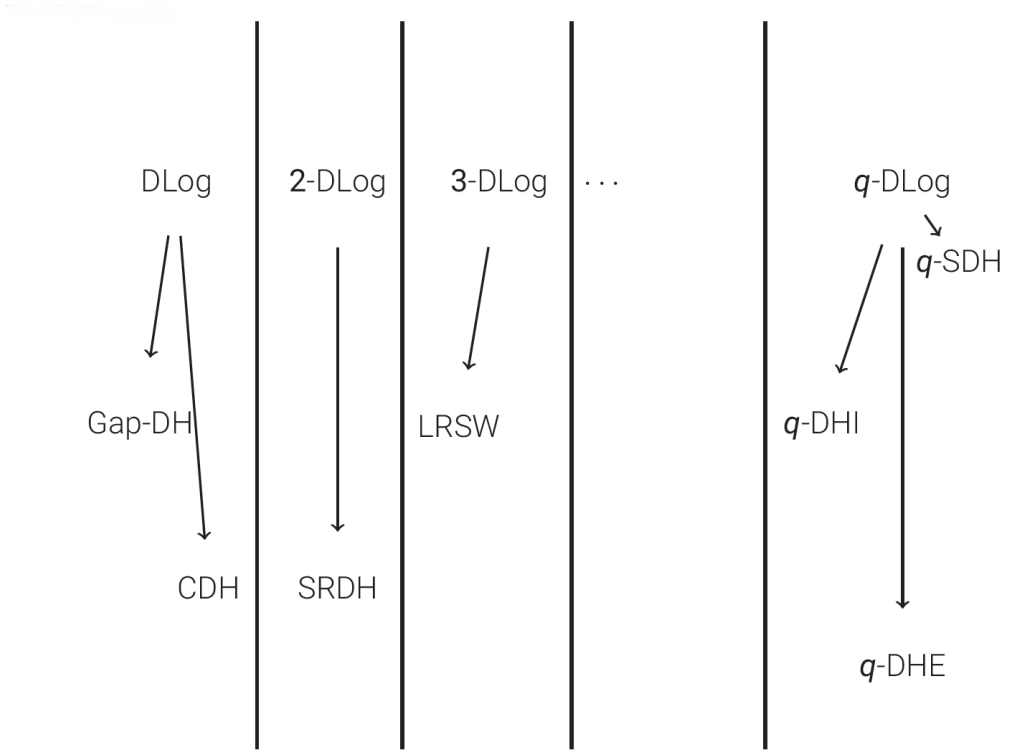
One-More Discrete Log

I want TWO discrete logs!

$\mathcal{C}$ $\xrightarrow{(\mathbf{G}, \mathbf{X}, \mathbf{Y})}$

Life is hard...

I will compute only ONE discrete log for you...



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OMDL

Thm: *q*-DLog does not imply One-More DLog in the AGM.

Thank you for your attention.