Efficient Simulation of Random States and Random Unitaries

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Joint Center for Quantum Information and Computer Science

QuSoft
Research Center for Quantum Software

UCONN

Results — overview

- We study the **simulation of random quantum objects**, i.e. random states and random unitary operations
- We develop a **theory of** their **stateful simulation**, a quantum analogue of Lazy sampling
- For random states, we develop an efficient protocol for stateful simulation
- For random unitaries, we devise a simulation method that runs in polynomial space
- As an **application**, we design a **quantum money** scheme that is unconditionally unforgeable and untraceable.

Introduction

Randomness…

…is extremely useful. Applications:

- All of cryptography
- Monte Carlo simulation
- Randomized algorithms
- …

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"Lazy sampling"	$q \cdot n$	Yes	None	None

# of queries

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Haar measure

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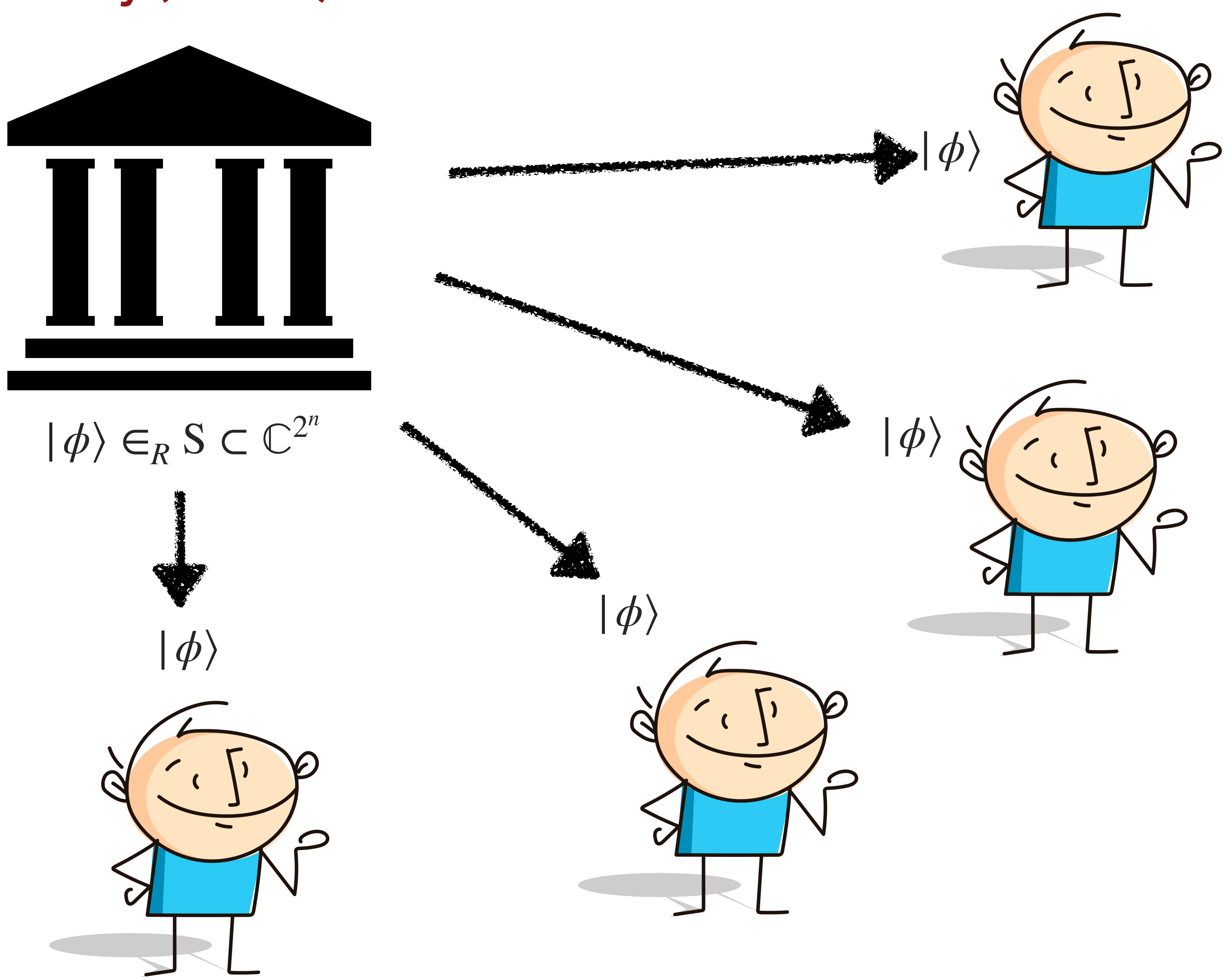

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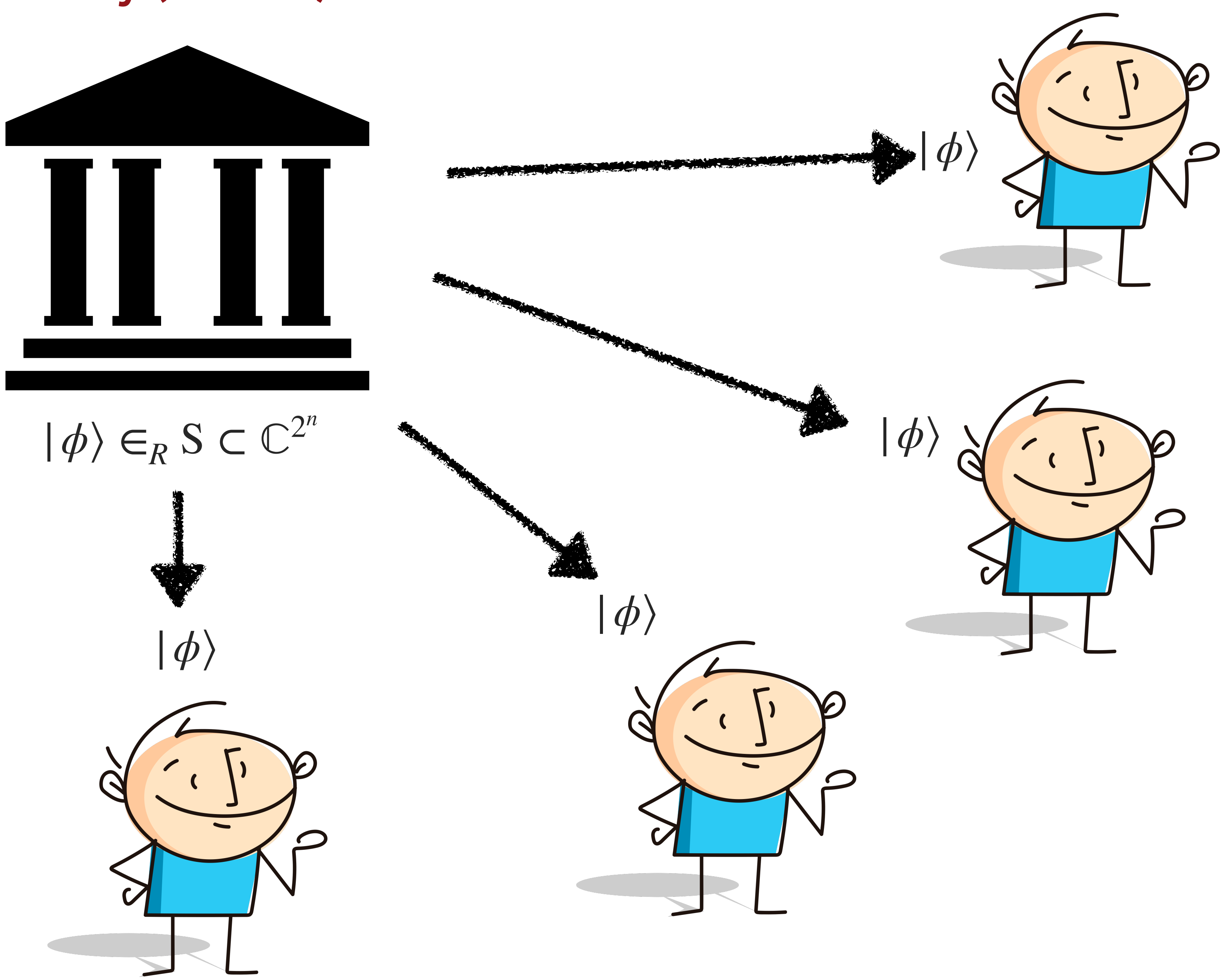

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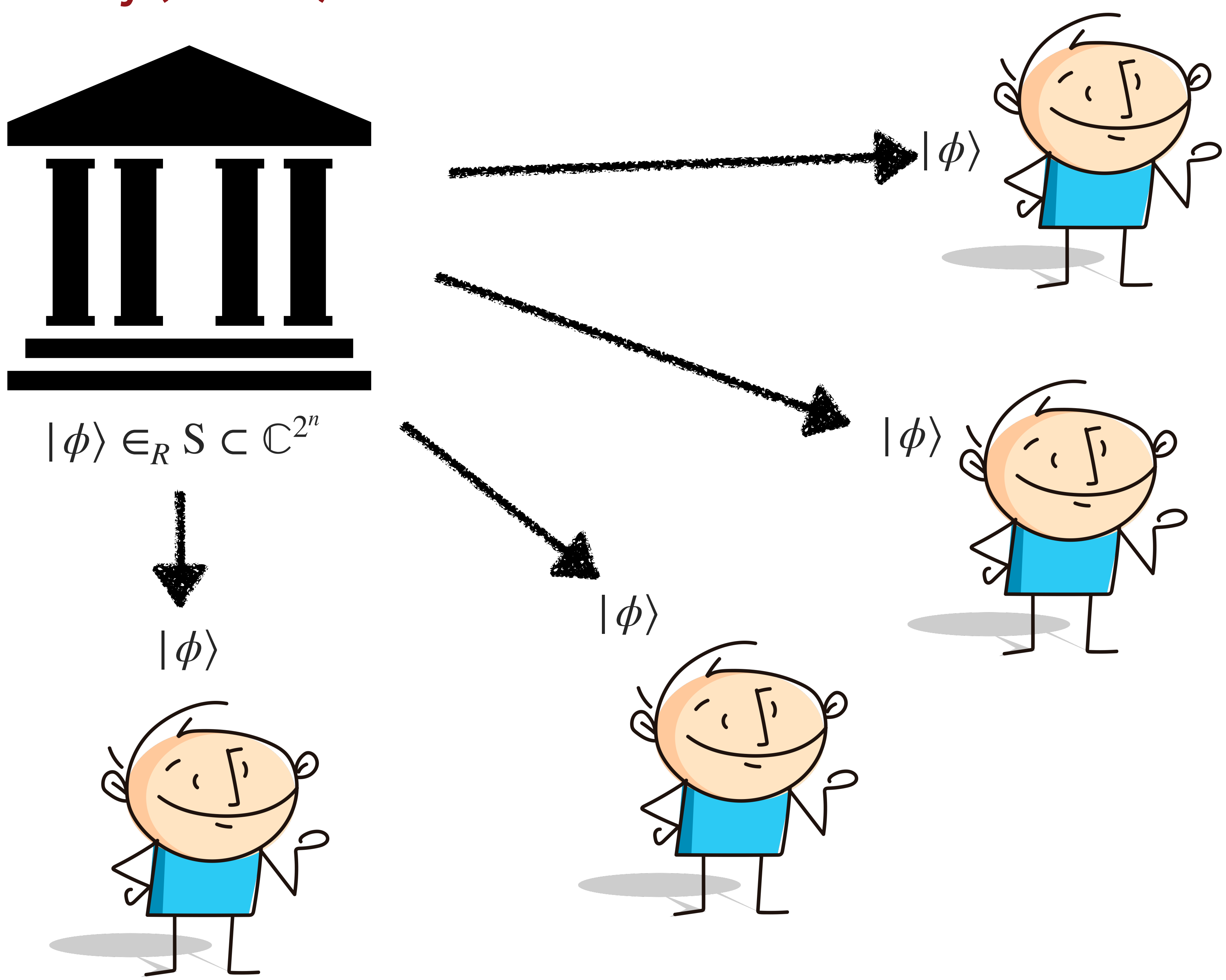

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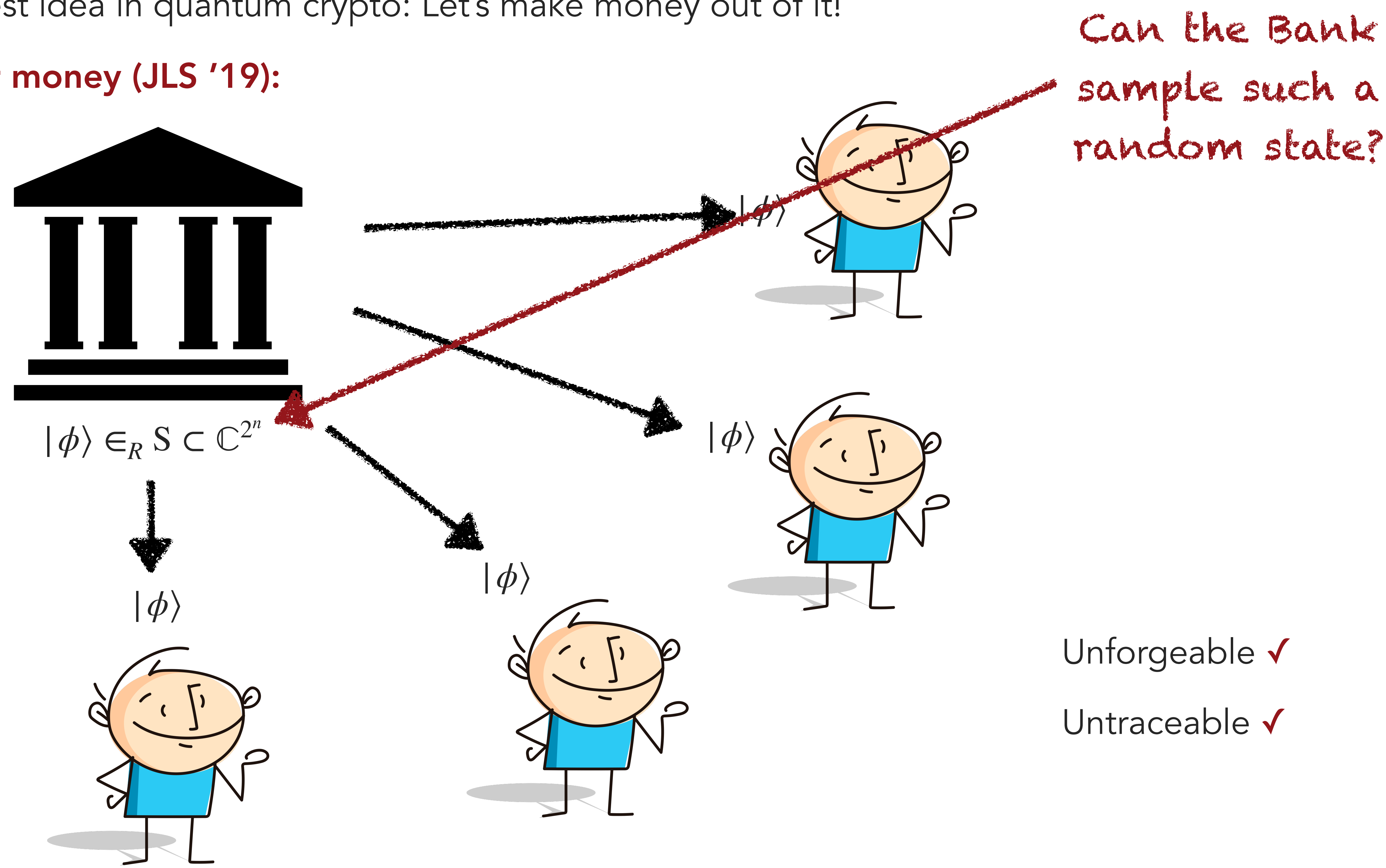

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Simulation of random quantum objects

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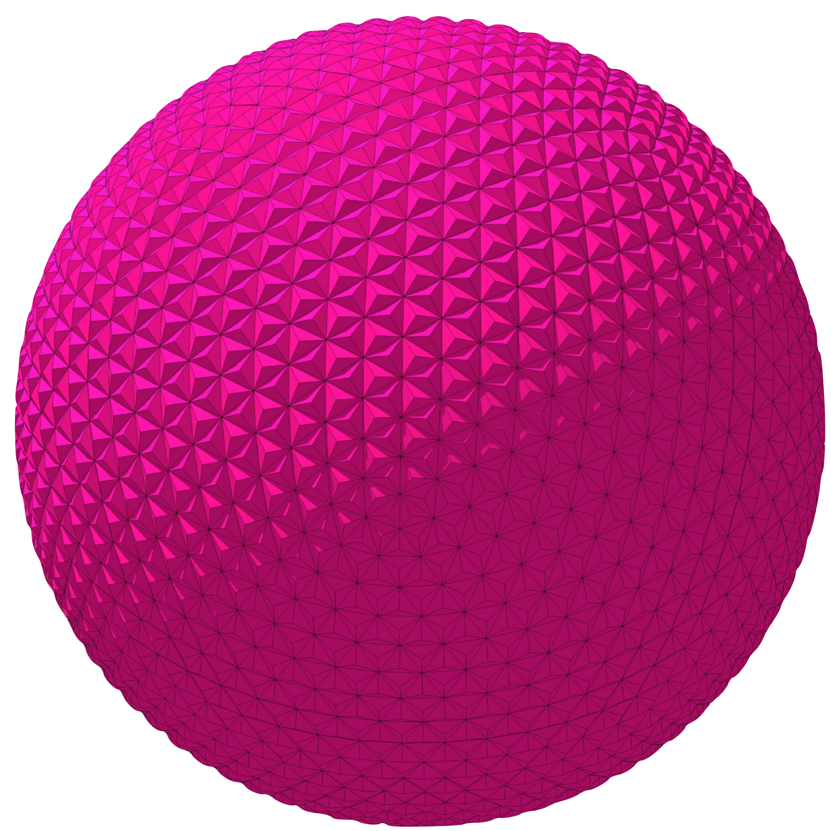

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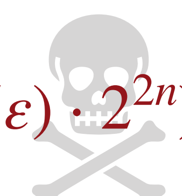
ε-Net	$O(\log(1/\varepsilon)\cdot 2^{2n})$ 	inefficient, stateless	None	$O(1/\varepsilon)$
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- **Use stateful simulation, unconditionally secure untraceable quantum money (AMR)**

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Similar argument for unitaries.

Techniques

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Fact: n copies of a Haar random state look like a single Haar random state on the symmetric subspace $\mathrm{Sym}_{d,n}$ of $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$ looks like half a maximally entangled state on $\mathrm{Sym}_{d,n} \otimes \mathrm{Sym}_{d,n}$

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  - Exact adaptive-to-nonadaptive reduction using "postselection"
  - Uniqueness property of the Stinespring dilation

Summary, open questions

Summary:

- We develop a theory of stateful simulation of random quantum primitives.
- Random quantum states can be approximately simulated efficiently using a stateful algorithm
- Random unitaries can be simulated exactly in a space-efficient using a stateful algorithm.
- The random state simulator can be used to construct unconditionally secure untraceable quantum money.

Open questions:

- Can we simulate random unitaries efficiently?
- (From JLS '19) Construct pseudorandom unitaries!