Impossibility Results for Lattice-Based Functional Encryption Schemes

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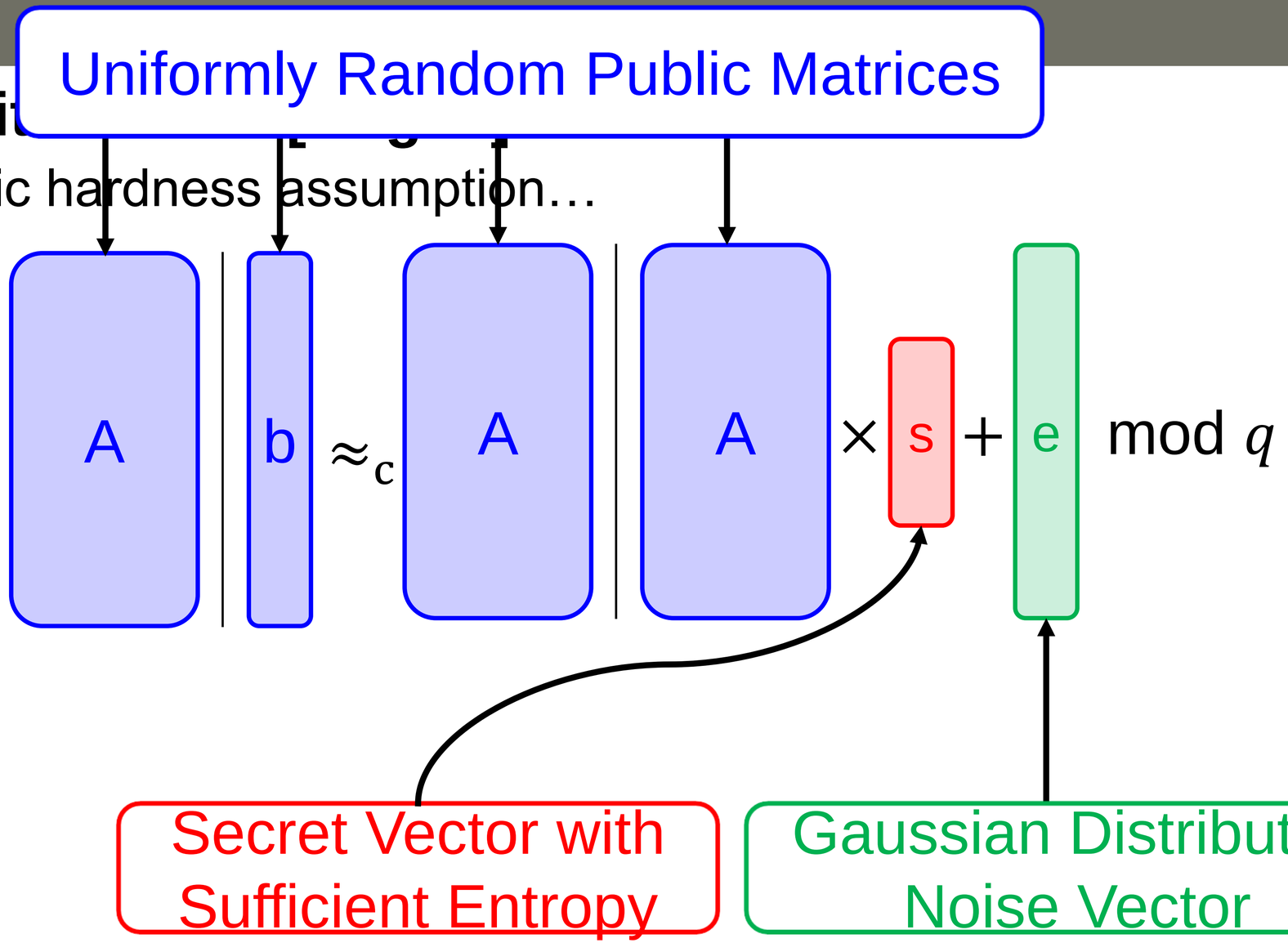
(Work done while the author was at KIT – Karlsruhe Institute of Technology, Karlsruhe, Germany.)

Learning wit

A cryptographic hardness assumption…



Learning with Errors [Reg05]

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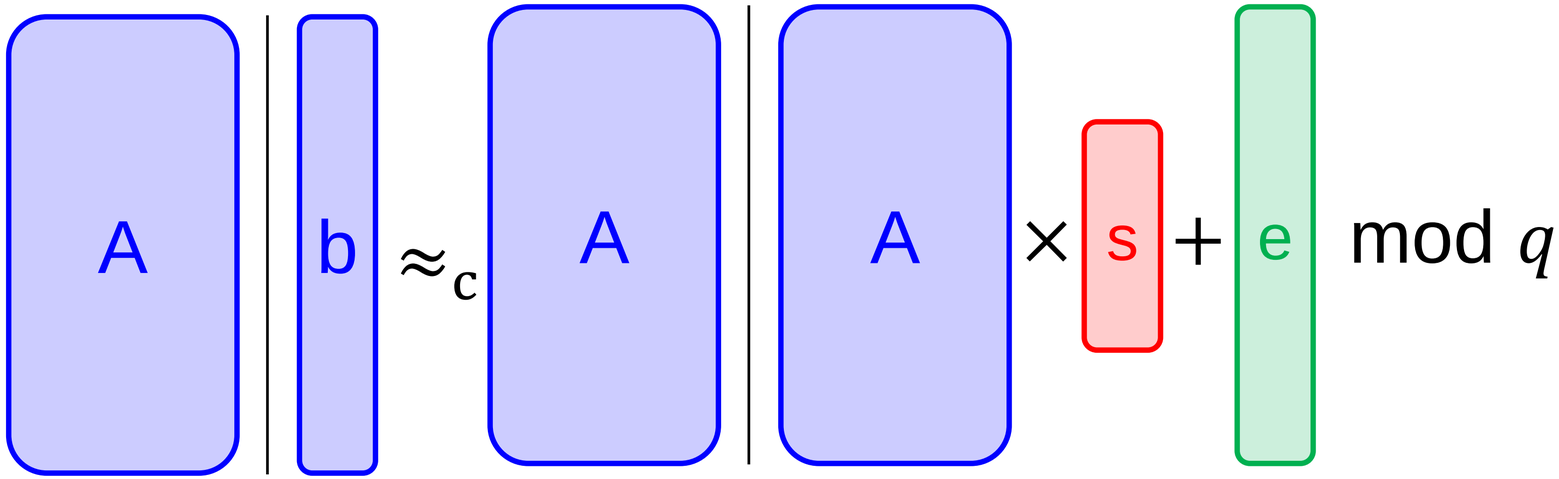

with strong homomorphic properties which enables a lot of different cryptographic primitives:

- Fully Homomorphic Encryption [BV11]
- Lockable Obfuscation [GKW17, WZ17]
- Attribute-Based Encryption [GVW13, BGG+14]

Learning with Errors [Reg05]

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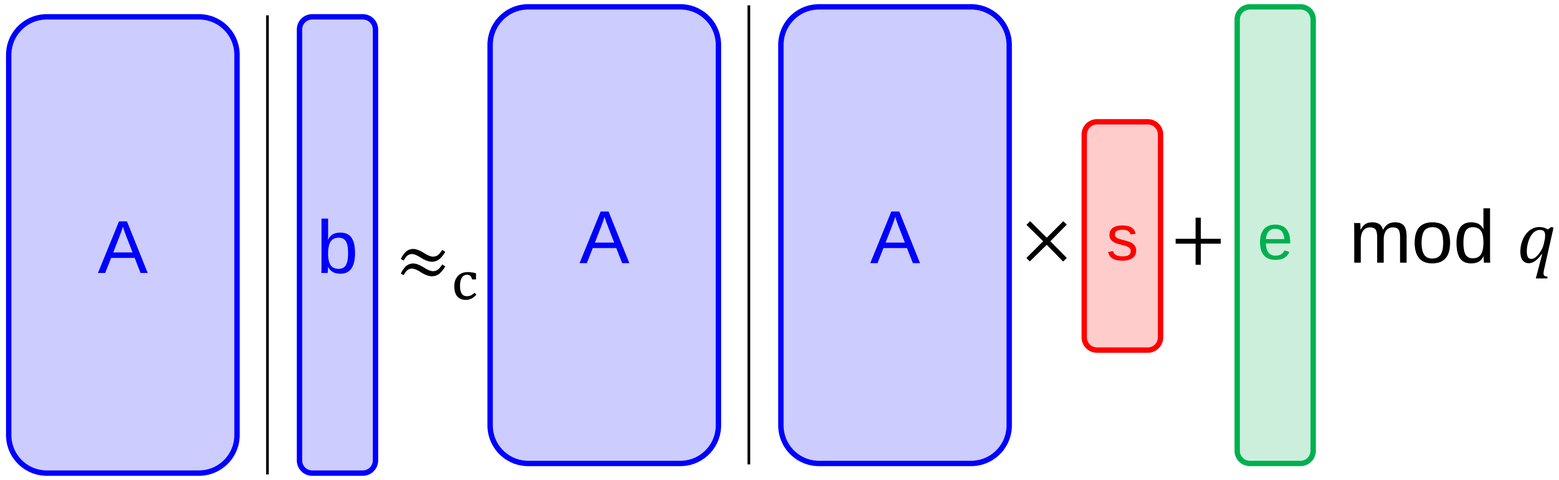

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But what about Functional Encryption?

Comparing Functional Encryption Schemes

	Pairing-Based Schemes	Lattice-Based Schemes
Inner-Product Encryption		✓ [AFV11, ALS16]
Function-Hiding Inner-Product Encryption		
Compact Quadratic FE	✓ [BCFG17]	
Compact Cubic FE		
(Non-Compact Const.-degree FE)	✓	✓

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Function-Hiding Inner-Product Encryption	✓ [BJK15, DDM16, Lin17, ACF+18]	✗
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Compact Cubic FE	This (+ additional assumptions) would imply Indistinguishability Obfuscation. [LT17]	✗
(Non-Compact Const.-degree FE)		

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By Linearization

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Compact Cubic FE	✗	✗
(Non-Compact Const.-degree FE)	✓	✓

(We do not list IBE, ABE and Bounded-Collusion FE here.)

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Question

What hinders us from constructing function-hiding inner-product encryption schemes whose security can be proven solely from the learning with errors assumption?

Maybe fundamental mathematical barriers…

Secret-Key Inner-Product Encryption

- Messages and functions are vectors $x, y \in \mathbb{Z}_p^n$.
- Setup(1^λ) generates a **master secret key** msk.

$$\langle y \mid x \rangle \bmod p$$

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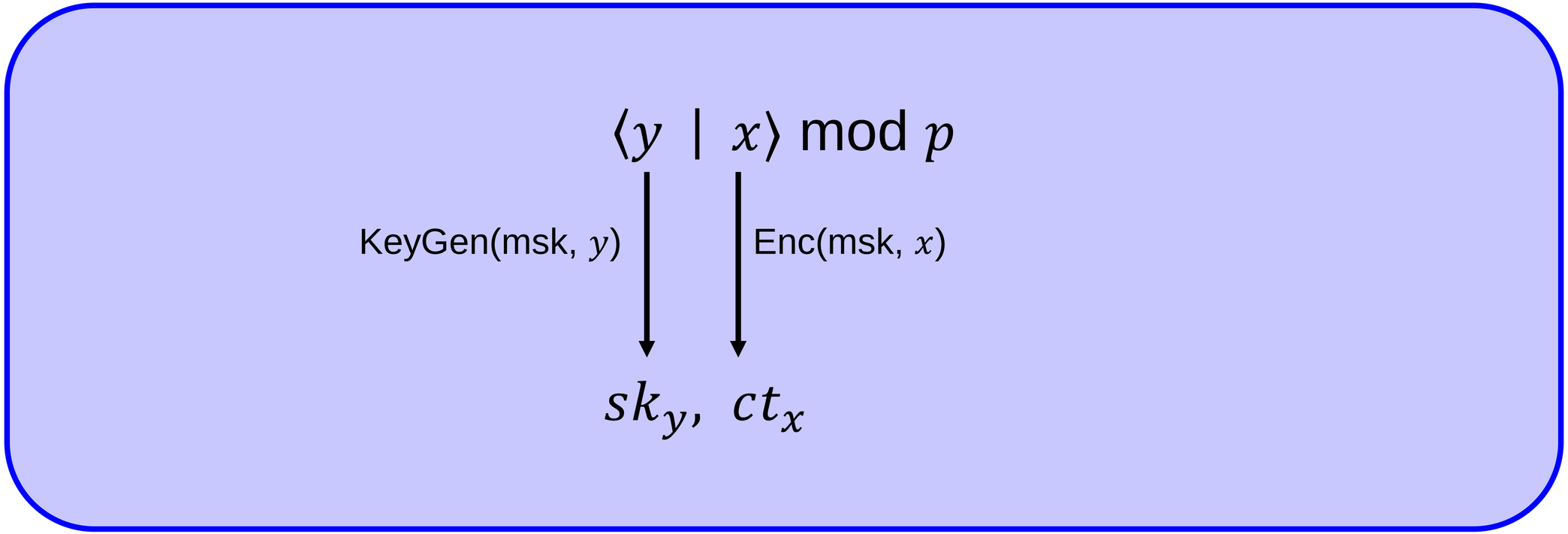

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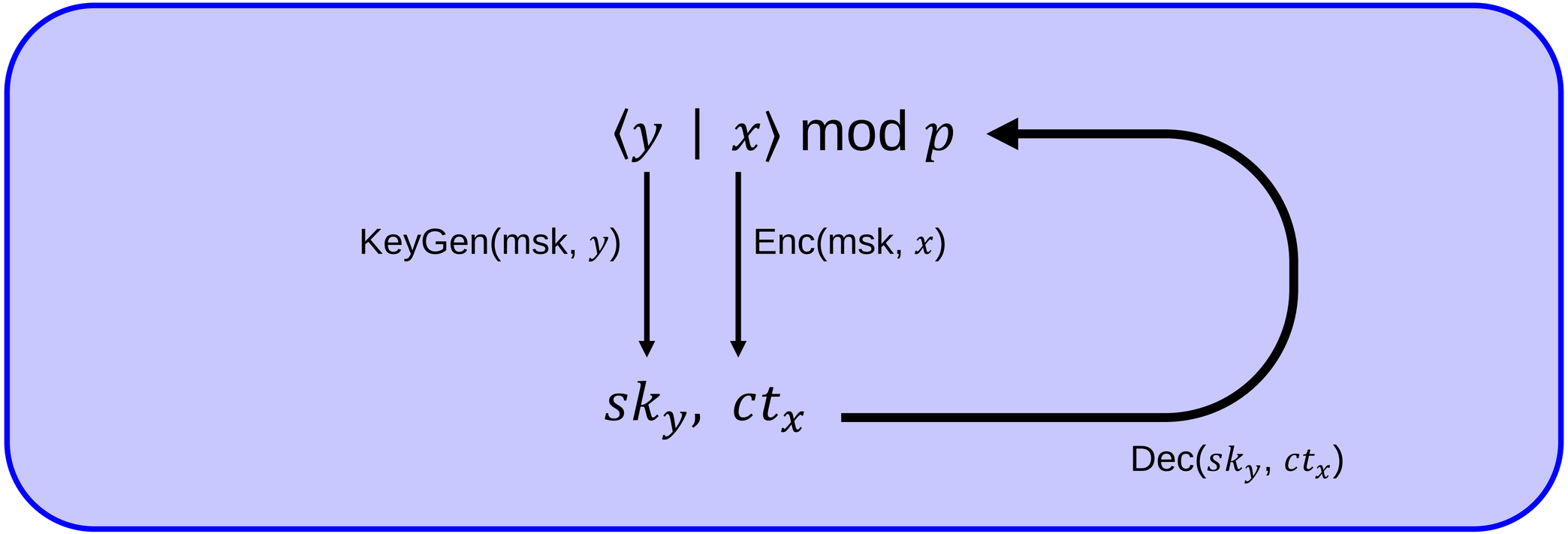
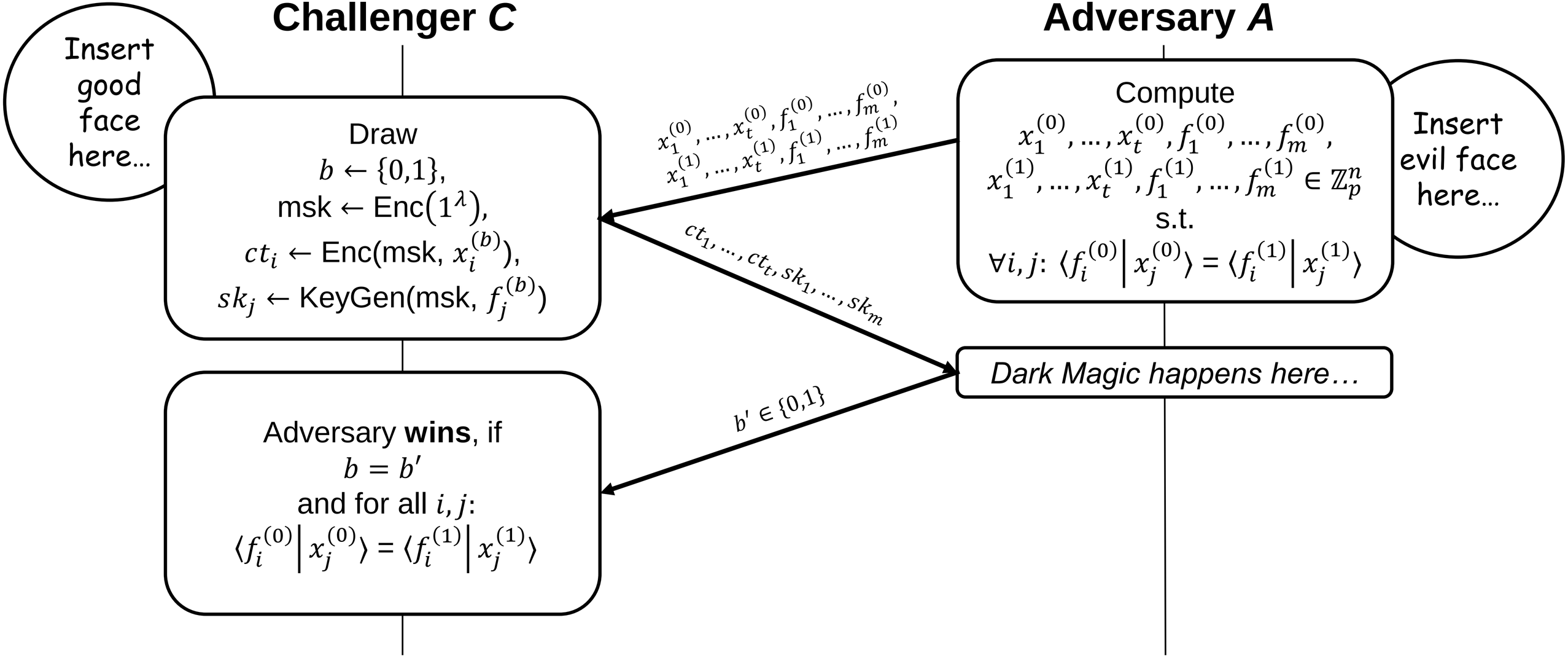


Selective Function-Hiding IND-CPA Security



Selective Function-Hiding IND-CPA Security

- The **advantage** of the adversary A is:

$$\mathrm{Adv}(A)_{m-fh-IND-CPA} := 2 \times \Pr[A \text{ wins}] - 1.$$

- For $m = m(\lambda)$ secret keys, the IPE scheme is called

selectively m-function-hiding IND-CPA secure,

if $\mathrm{Adv}(A)_{m-fh-IND-CPA} \in \mathrm{negl}(\lambda)$ for each ppt A.

- The IPE scheme is called

(unbounded) selectively function-hiding IND-CPA secure,

if it is sel. m-function-hiding IND-CPA secure for each $m \in \mathrm{poly}(\lambda)$.

Contribution

Idealized Impossibility “Theorem”.

There does not exist a lattice-based Inner-Product Encryption scheme which is function-hiding secure.

This really has to mean something!

Idea: Replace „lattice-based“ by common design patterns of lattice-based crypto-schemes.

Common Design Patterns: Linear Decryption

In most cases:

- Ciphertexts ct_x and secret keys sk_f are vectors over $\mathbb{Z}_q$.
- Decryption has the following formula:

$$\mathbf{Dec}(\boldsymbol{sk_f}, \boldsymbol{ct_x}) := \left\lceil \frac{\langle sk_f \mid ct_x \rangle \bmod q}{\left\lfloor \frac{q}{p} \right\rfloor} \right\rfloor \in \mathbb{Z}_p$$

Common Design Patterns: Offline/Online-Encryption [HW14,AR17]

Almost always: Encryption follows an offline/online-pattern.

Offline Phase: compute arbitrary complex randomness without looking at input x.

Online Phase: combine randomness with x in a very simple way (by evaluating const-degree polynomials at x).

Enc(msk, x):

- Compute s multinomial degree-d integer polynomials $(r_1, \dots, r_s) \leftarrow \text{Enc}_{off}(msk)$.
- Compute and output $ct_x := \big(r_1(x), \dots, r_s(x)\big) \bmod q \in \mathbb{Z}_q^s$.

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We call d the **depth** of the encryption algorithm.

Enc(msk, x):

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Contribution

Our Impossibility Theorem.

An Inner-Product Encryption scheme

- *with* ***Linear Decryption***
- *and* ***Offline/Online-Encryption of const. depth***

cannot be selectively function-hiding IND-CPA secure.

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cannot be selectively $m + 1$ *function-hiding IND-CPA secure,*

for some $m \in poly(\lambda)$ *which depends on the scheme.*

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We need:
q is prime, $\frac{q}{p}$ is bounded by a polynomial, p is greater than some constant.

Offline/Online-Encryption

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- Compute s multinomial degree-d integer polynomials $(r_1, \dots, r_s) \leftarrow \text{Enc}_{off}(msk)$.
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Encryption Algorithm of depth d …

Offline/Online-Encryption

Enc(msk, x):

- Compute s multinomial degree-d integer polynomials $(r_1, \dots, r_s) \leftarrow \text{Enc}_{off}(msk)$.
- Compute and output $ct_x := \big(r_1(x), \dots, r_s(x)\big)$ **mod** $q \in \mathbb{Z}_q^s$.

Encryption Algorithm of depth d …

… **over** $\mathbb{Z}_q$.

Offline/Online-Encryption

Enc(msk, x):

- Compute s multinomial degree-d integer polynomials $(r_1, \dots, r_s) \leftarrow \text{Enc}_{off}(msk)$.
- Compute and output $ct_x := (r_1(x), \dots, r_s(x)) \in \mathbb{Z}^s$.

Encryption Algorithm of depth d …

… **over** $\mathbb{Z}$.

Offline/Online-Encryption

Enc : Encryption Algorithm of depth d over $\mathbb{Z}$.

↳ Each ciphertext $ct_x \leftarrow$Enc(msk,x) is an integer vector $ct_x \in \mathbb{Z}^s$.

Enc is **of width** $B = B(\lambda)$, if

$\Pr[ct_x \in \{-B, \dots, B\}^s \mid ct_x \leftarrow \text{Enc}(\text{msk}, x)] \geq 1 - \text{negl}(\lambda)$.

Offline/Online-Encryption

Enc : Encryption Algorithm of depth d over $\mathbb{Z}_q$.

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Under the identification:

$$\mathbb{Z}_q \triangleq \left\{-\frac{q-1}{2}, \dots, 0, \dots, \frac{q-1}{2}\right\} \subset \mathbb{Z}$$

Contribution

Our Impossibility Theorem.

Let q prime, $\frac{q}{p}$ bounded by a polynomial, p greater than some constant.

An Inner-Product Encryption scheme

- *with **Linear Decryption***
- *and **Offline/Online-Encryption of const. depth***

<u>cannot</u> be selectively $m + 1$ function-hiding IND-CPA secure.

Technical Overview

$m + 1$ function-hiding IND-CPA secure IPE scheme of constant depth over $\mathbb{Z}_q$ with linear decryption

… transformed by adversary to …

IND-CPA secure SKE scheme of width $\frac{q}{p}$ and constant depth over $\mathbb{Z}_q$

… transformed by adversary to …

IND-CPA secure SKE scheme of polynomial width and constant depth over $\mathbb{Z}$

… broken by general adversary!

Technical Overview: Step 1

$m + 1$ function-hiding IND-CPA secure IPE scheme of constant depth over $\mathbb{Z}_q$ with linear decryption

Trade Off
Function-Hiding *and* **Linear Decryption**
Against
Short Ciphertexts*!*

IND-CPA secure SKE scheme of width $\frac{q}{p}$ and constant depth over $\mathbb{Z}_q$

Technical Overview: Step 1

$m + 1$ function-hiding IND-CPA secure IPE scheme of constant depth over $\mathbb{Z}_q$ with linear decryption

- Adversary draws m keys for the zero function vector $sk_1, \ldots, sk_m \leftarrow$KeyGen(msk, 0)
- Correctness $\Rightarrow$ Dec$(sk_i, ct_x) = \left\lceil \langle sk_i | ct_x \rangle / \left\lfloor \frac{q}{p} \right\rfloor \right\rfloor = 0$

$$\Rightarrow (\langle sk_1 | ct_x \rangle, \ldots, \langle sk_m | ct_x \rangle) \in \left\{ -\left\lfloor \frac{q}{p} \right\rfloor, \ldots, \left\lfloor \frac{q}{p} \right\rfloor \right\}^m$$

Decryption Noises

IND-CPA secure SKE scheme of width $\frac{q}{p}$ and constant depth over $\mathbb{Z}_q$

Technical Overview: Step 1

$m + 1$ function-hiding IND-CPA secure IPE scheme of constant depth over $\mathbb{Z}_q$ with linear decryption

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  $\Rightarrow (\langle sk_1 | ct_x \rangle, \dots, \langle sk_m | ct_x \rangle) \in \left\{ -\left\lfloor \frac{q}{p} \right\rfloor, \dots, \left\lfloor \frac{q}{p} \right\rfloor \right\}^m$
- Function-Hiding ⇒ Pr$[sk_y \in \text{span}_{\mathbb{Z}_q}\{sk_1, \dots, sk_m\}] \notin \text{negl}(\lambda)$

  $\Rightarrow \langle sk_y | ct_x \rangle$ can be reconstructed from $\langle sk_1 | ct_x \rangle, \dots, \langle sk_m | ct_x \rangle$

⇒ Use the vector $(\langle sk_1 | ct_x \rangle, \dots, \langle sk_m | ct_x \rangle) \in \mathbb{Z}_q^m$ as new ciphertext in SKE for x.

IND-CPA secure SKE scheme of width $\frac{q}{p}$ and constant depth over $\mathbb{Z}_q$

Technical Overview: Step 2

IND-CPA secure SKE scheme of width $\frac{q}{p}$ and constant depth over $\mathbb{Z}_q$

Get Rid Of Arithmetic Reduction in Online Part!

IND-CPA secure SKE scheme of polynomial width and constant depth over $\mathbb{Z}$

Technical Overview: Step 2

IND-CPA secure SKE scheme of width $\frac{q}{p}$ and constant depth over $\mathbb{Z}_q$

Very rough Idea:

- $(r_1, \ldots, r_s) \leftarrow \text{Enc}_{off}(msk)$
- Each r_i has small output values

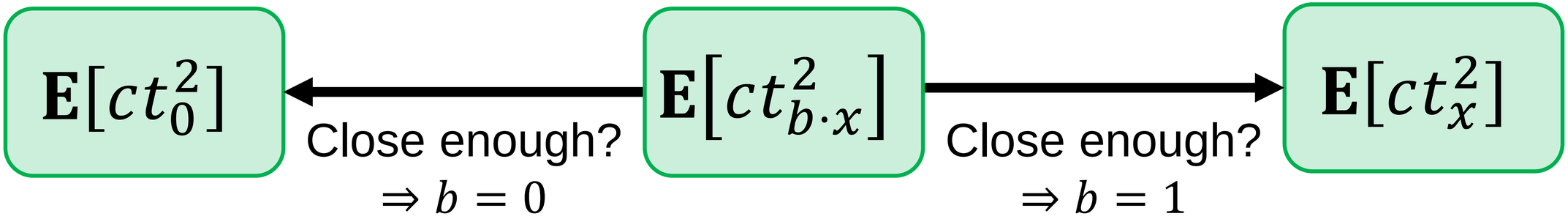
  $\Rightarrow$ Each r_i has small coefficients
- When we evaluate a polynomial with small coefficients on a small input, then the result is in $\left\{-\frac{q-1}{2}, \ldots, 0, \ldots, \frac{q-1}{2}\right\}$, even without applying arithmetic reduction modulo q.

IND-CPA secure SKE scheme of polynomial width and constant depth over $\mathbb{Z}$

Technical Overview: Step 3

IND-CPA secure SKE scheme of polynomial width and const. depth over $\mathbb{Z}$

- Adversary submits messages 0, $b \cdot x$ and x.
- He estimates $\mathbf{E}[ct_0^2]$, $\mathbf{E}[ct_{b\cdot x}^2]$ and $\mathbf{E}[ct_x^2]$.



- Rest of proof is just Mathematics.

Conclusion

A lattice-based FE scheme which uses popular design choices for encryption (*online/offline-encryption*) and decryption (*linear decryption*) cannot be function-hiding IND-CPA secure.

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