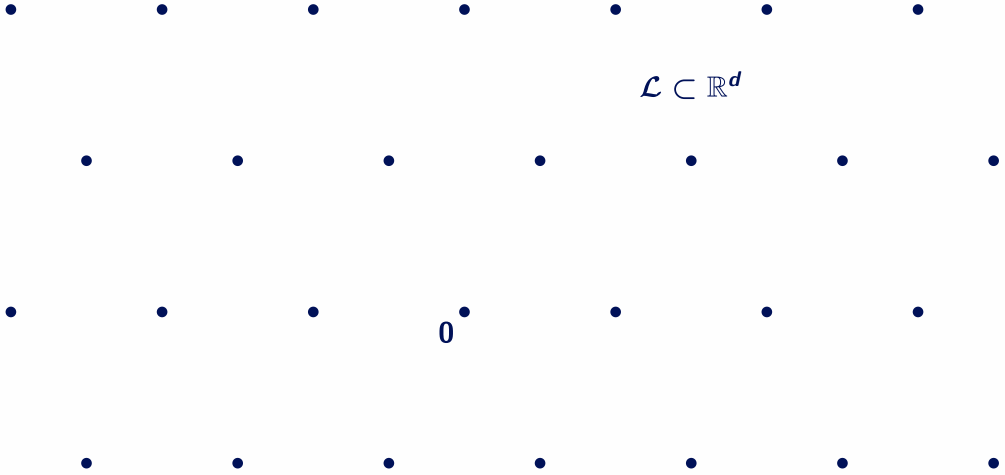
The randomized slicer for CVPP: sharper, faster, smaller, batchier

Leo Ducas, Thijs Laarhoven and **Wessel van Woerden.**

Centrum Wiskunde & Informatica

Technische Universiteit Eindhoven
University of Technology

Lattice



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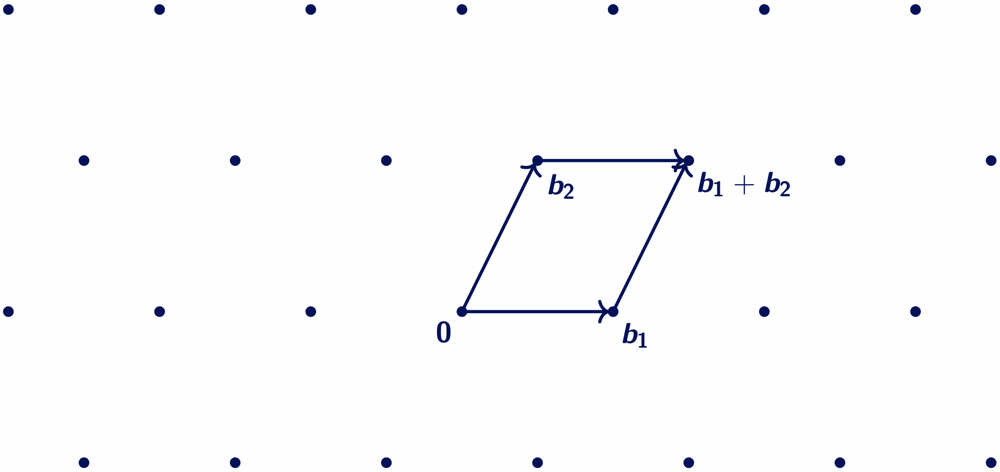
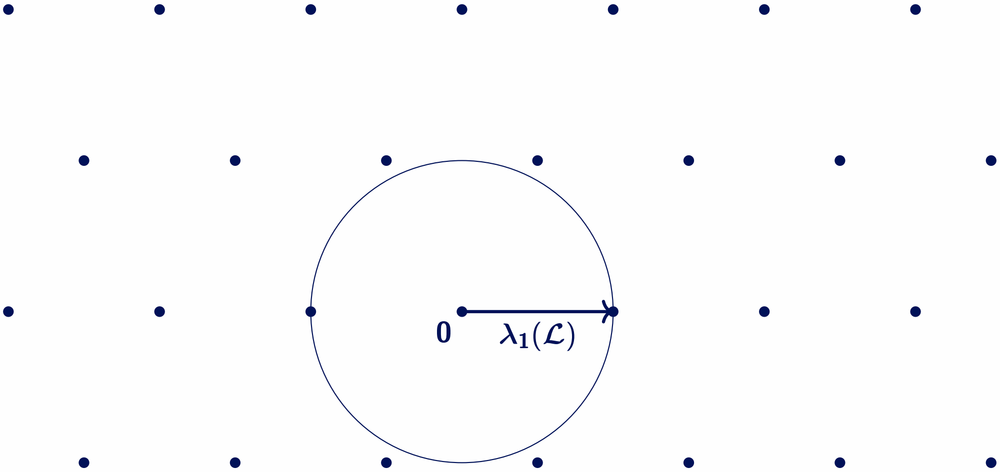


Shortest Vector Problem



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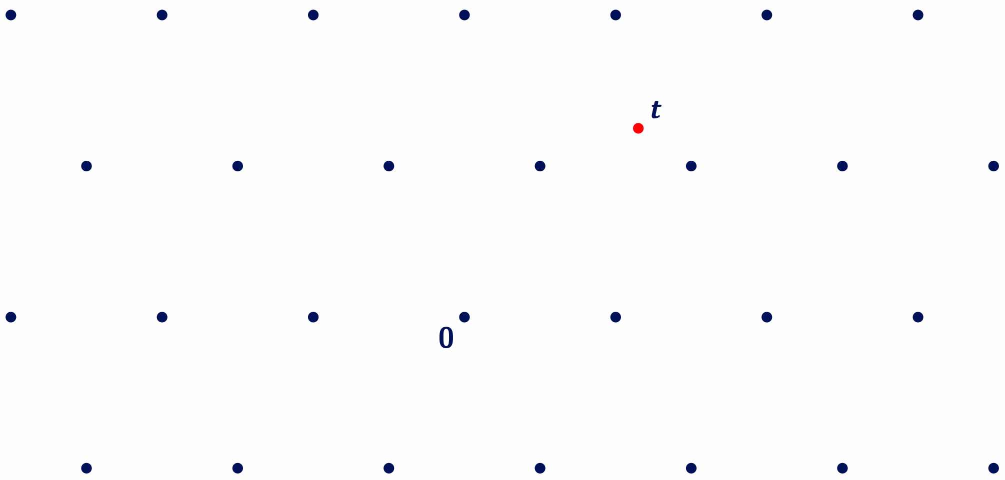

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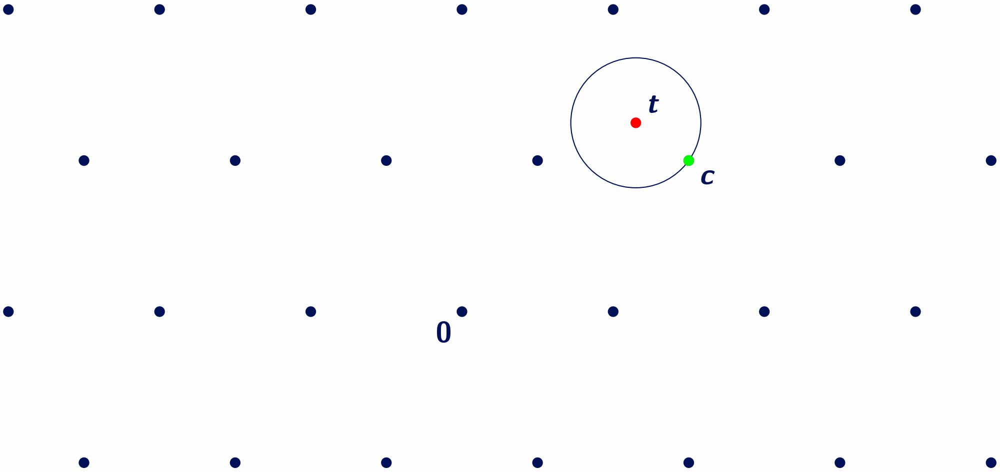

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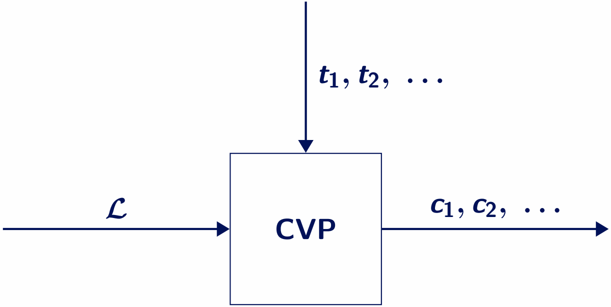
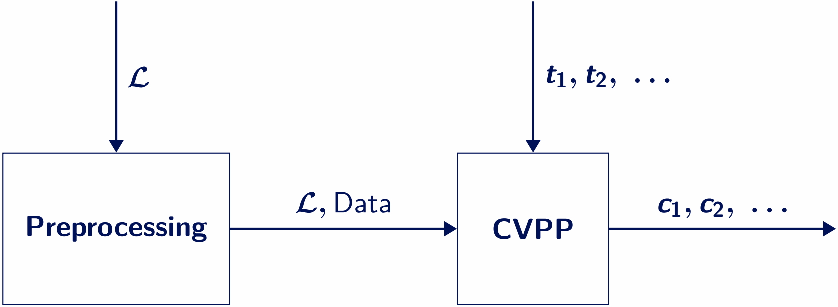


Closest Vector Problem with Preprocessing



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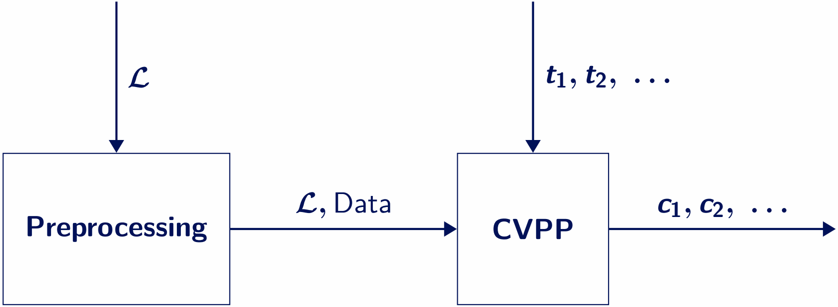

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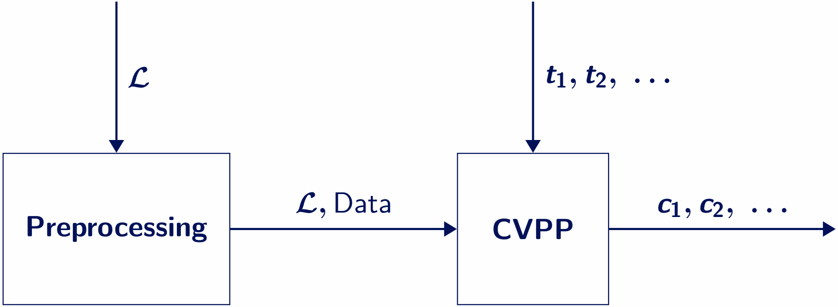

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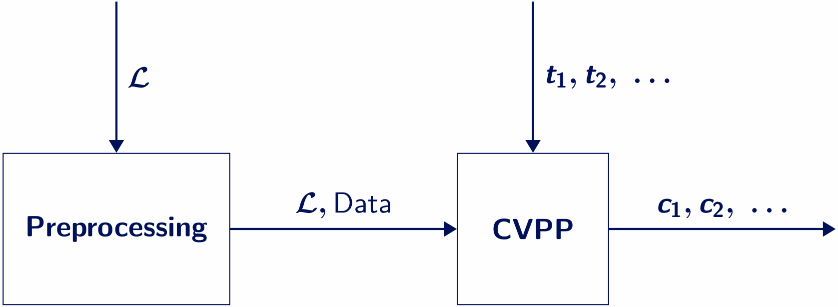

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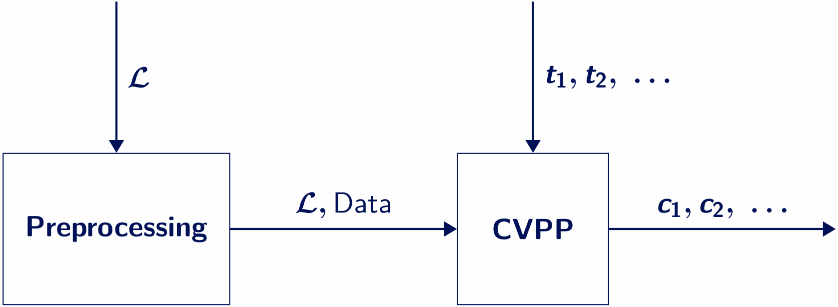

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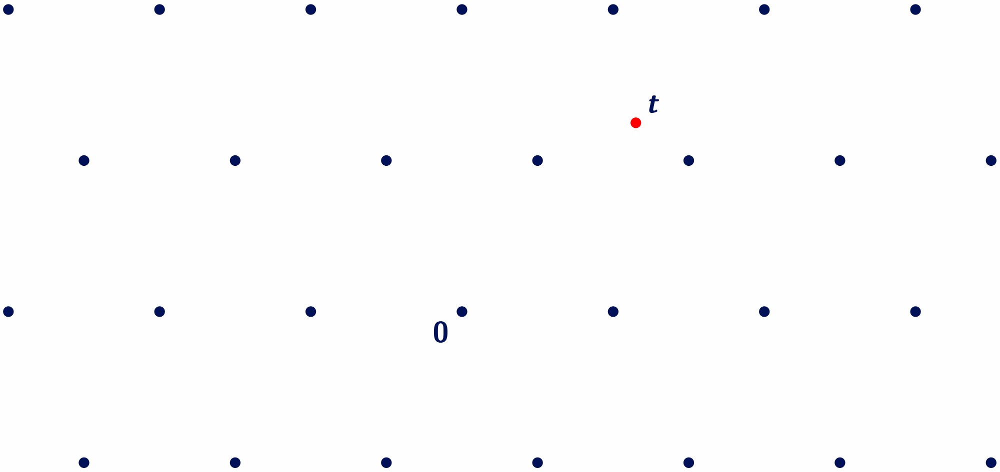
- Preprocessing can be started before any target is known.

Iterative Slicer [SFS09]



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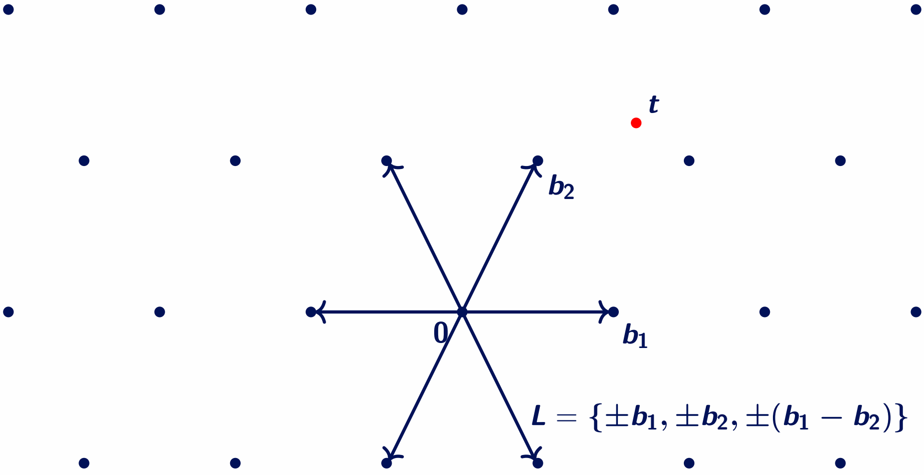

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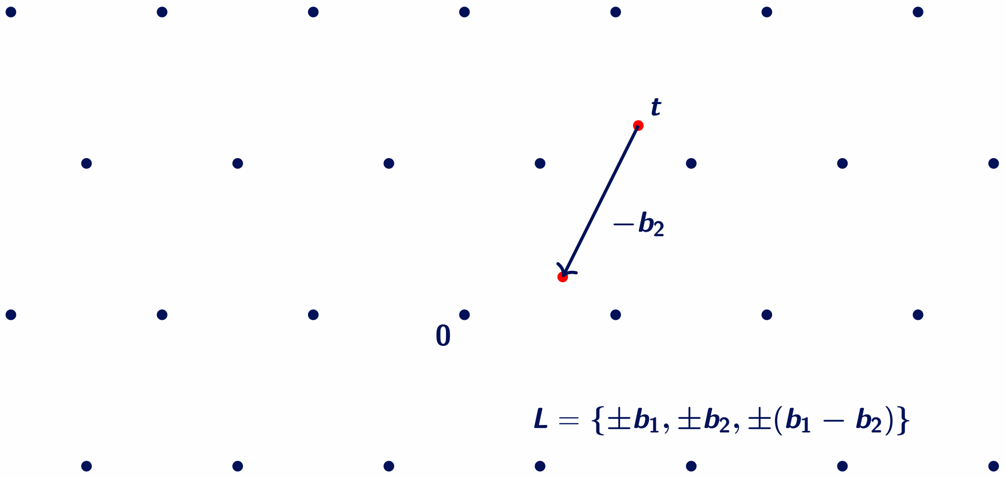

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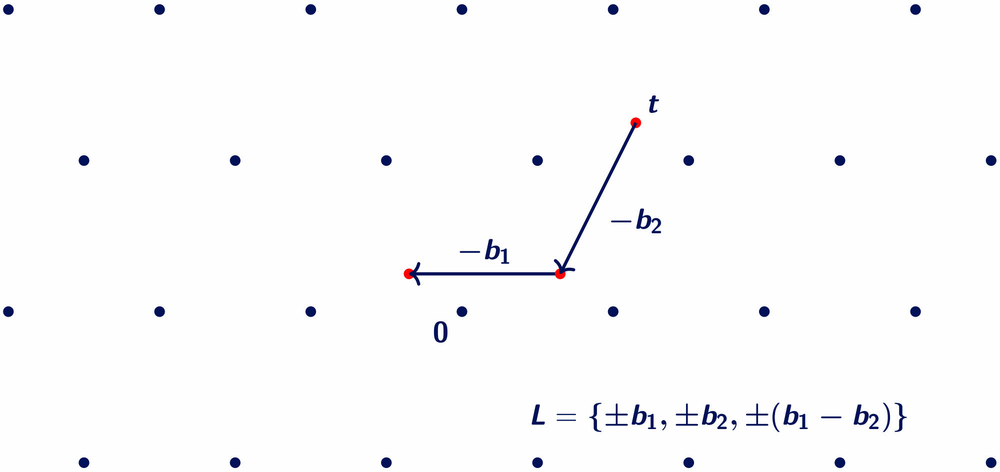

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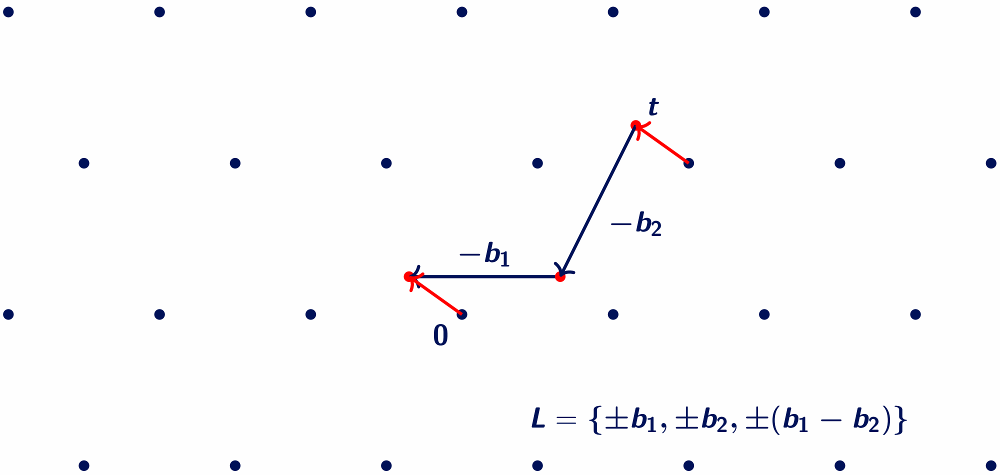

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Provably succeeds when
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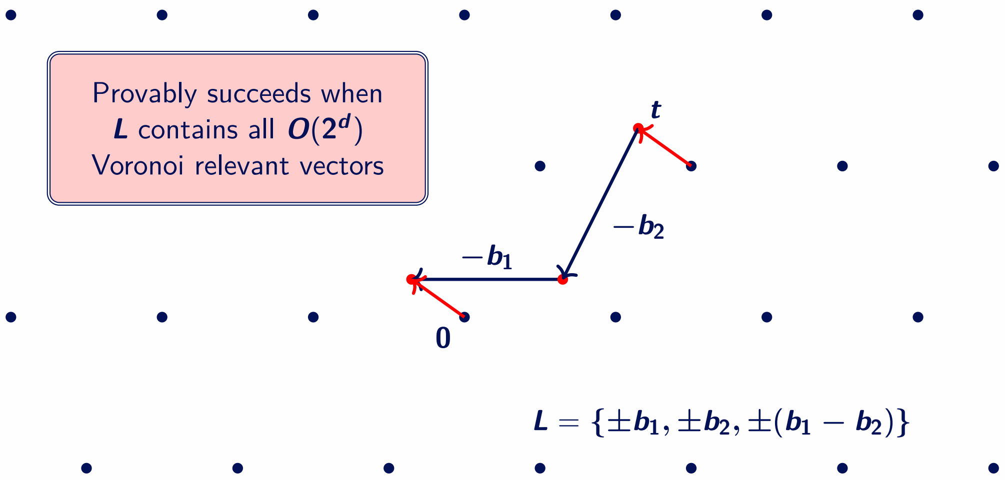

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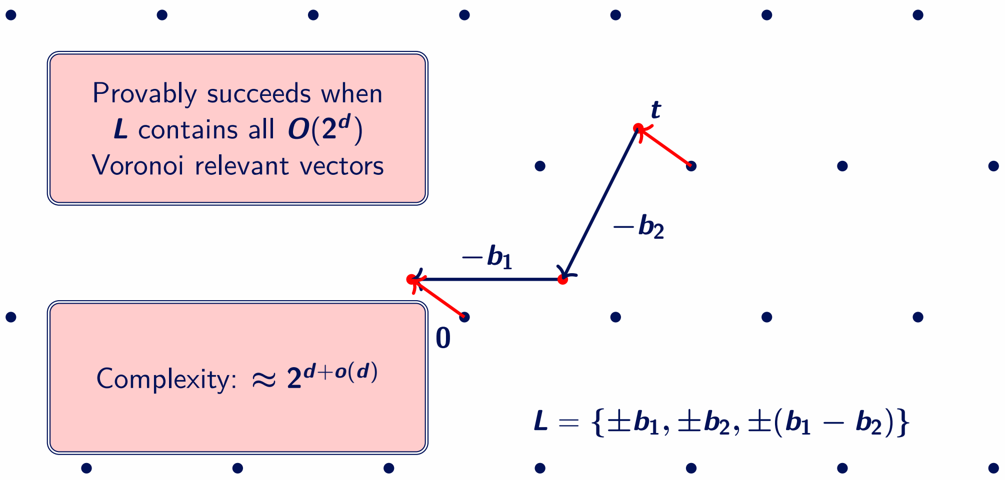

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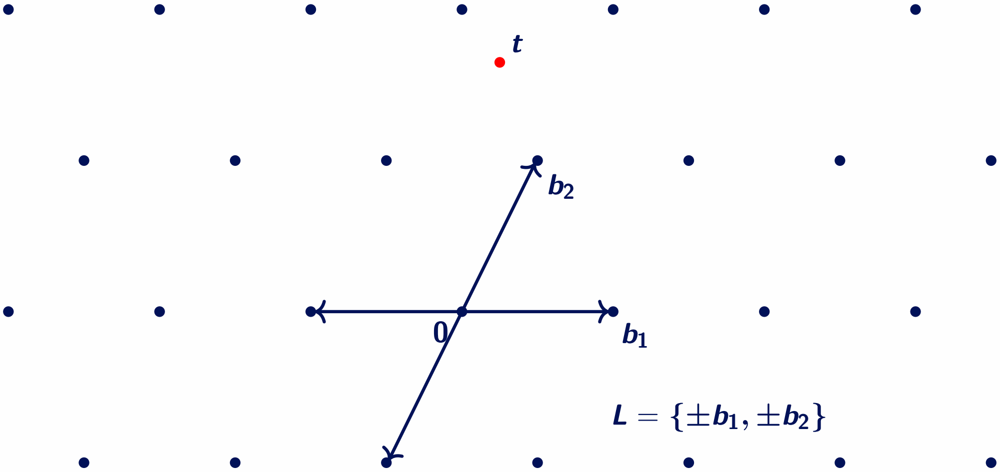

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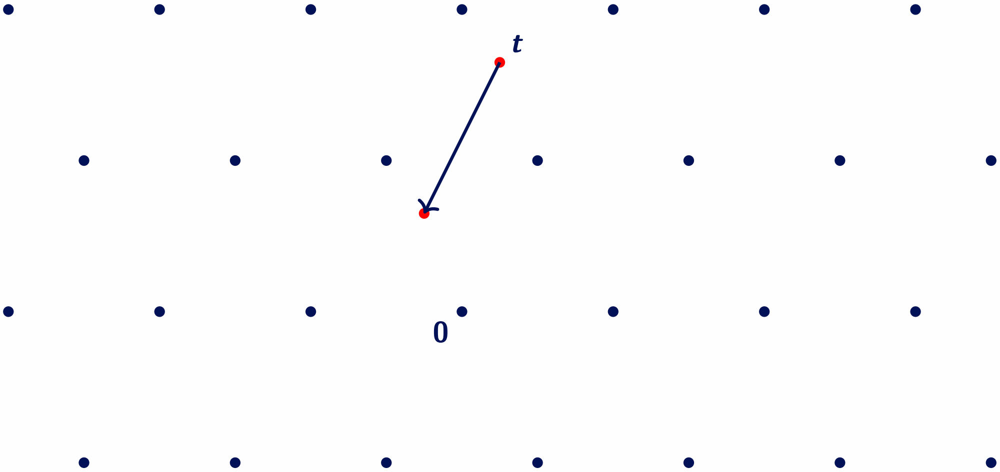

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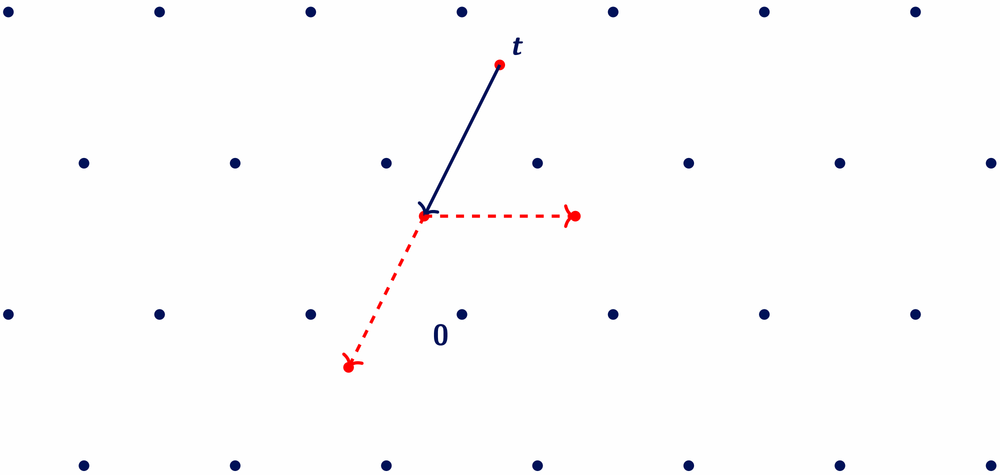

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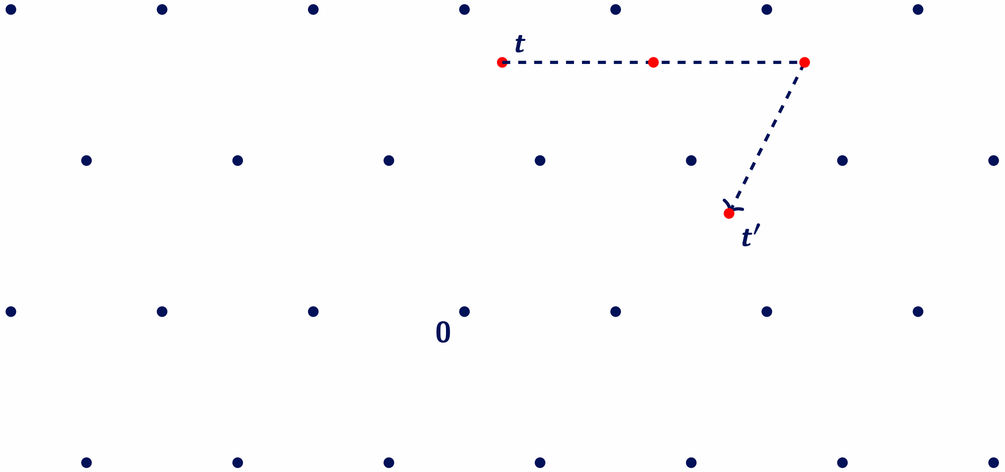

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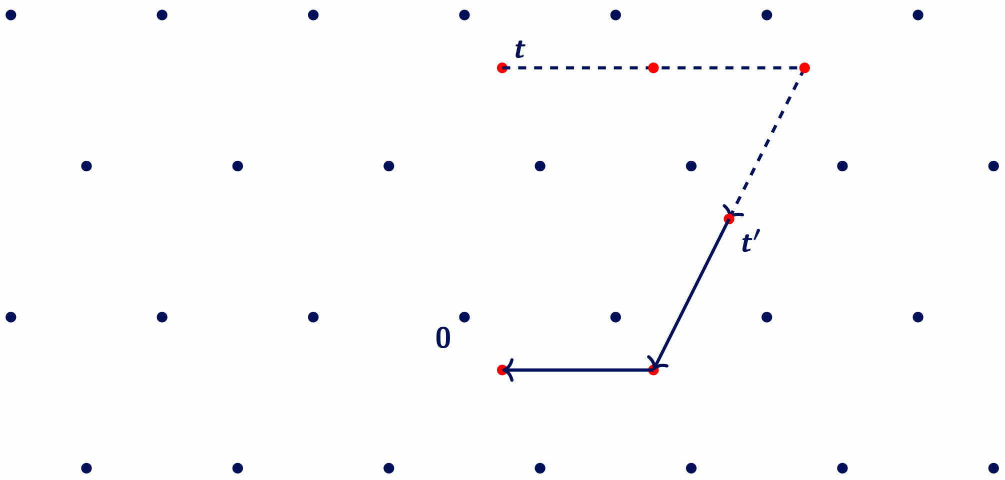

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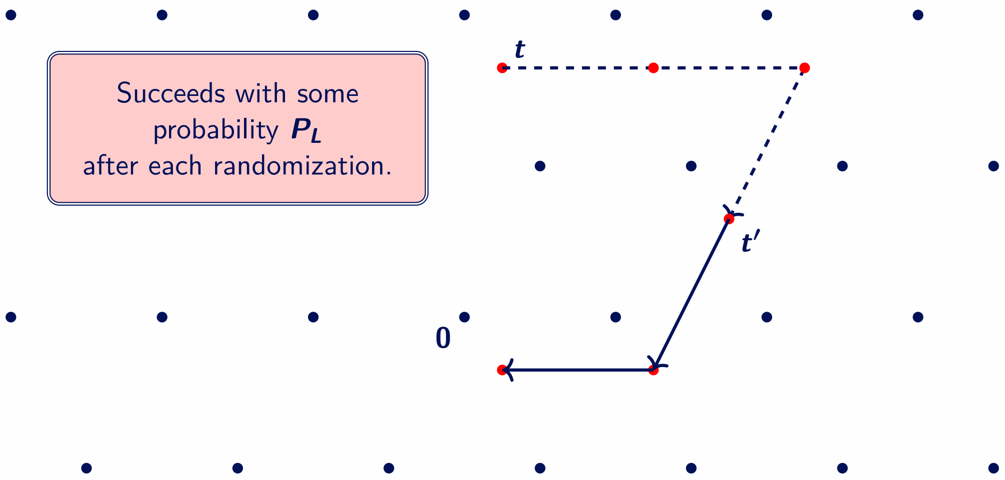

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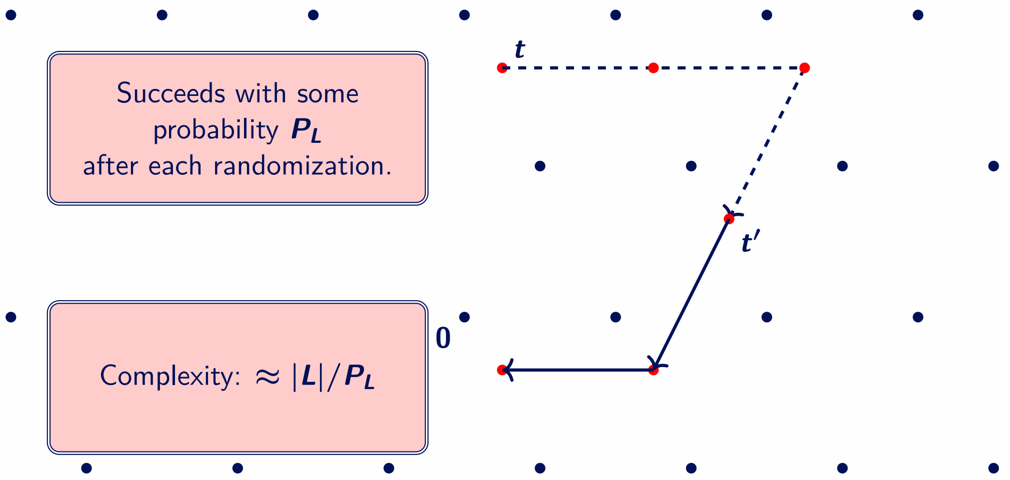

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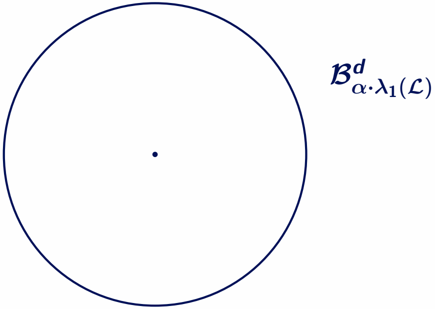

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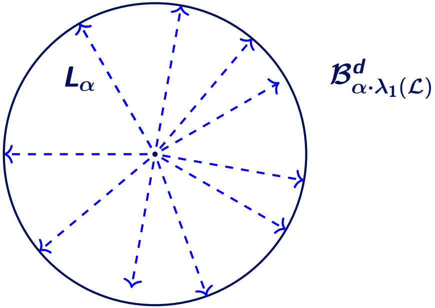
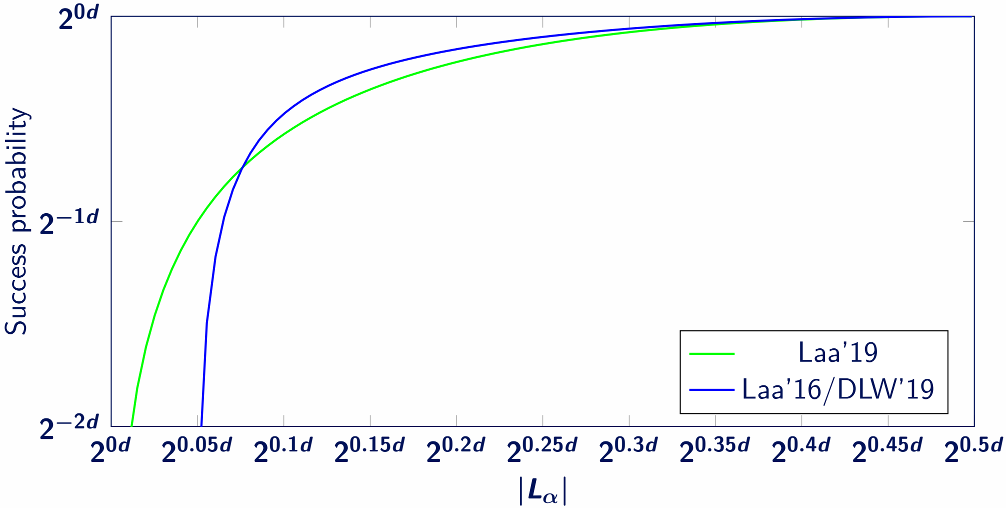
Uniformly distributed



Success Probability - Lower bounds



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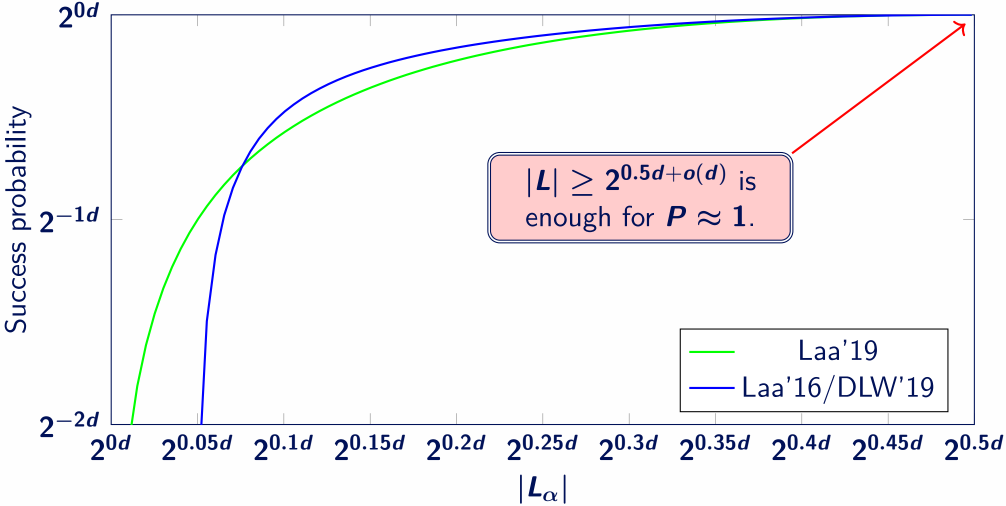

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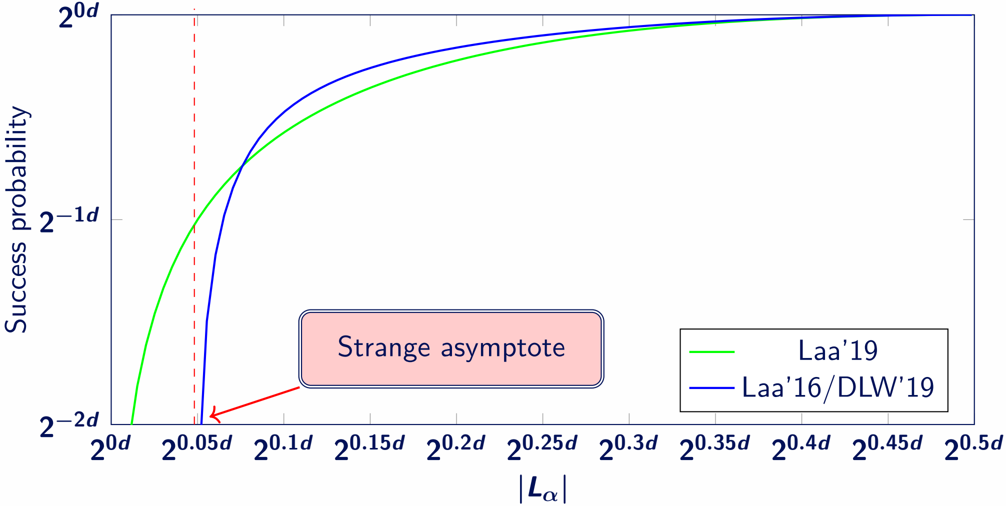

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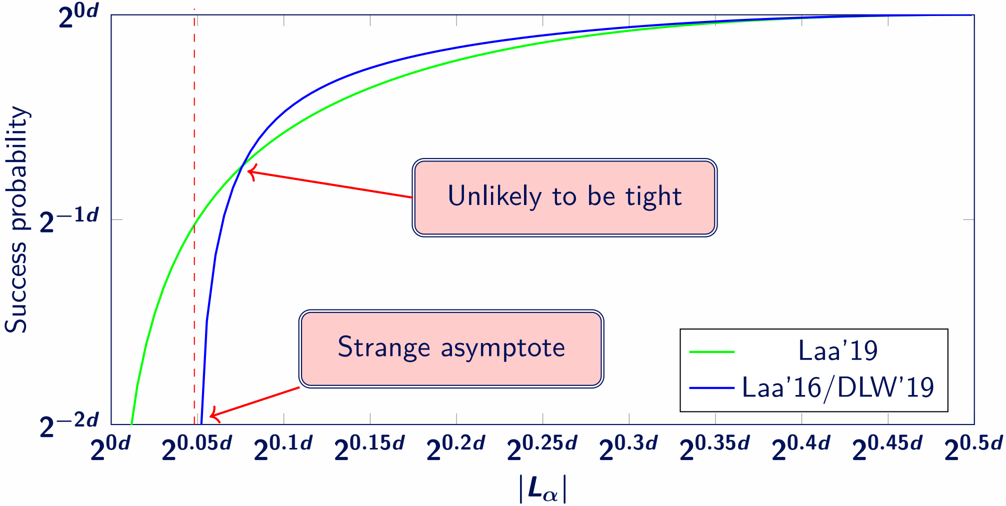

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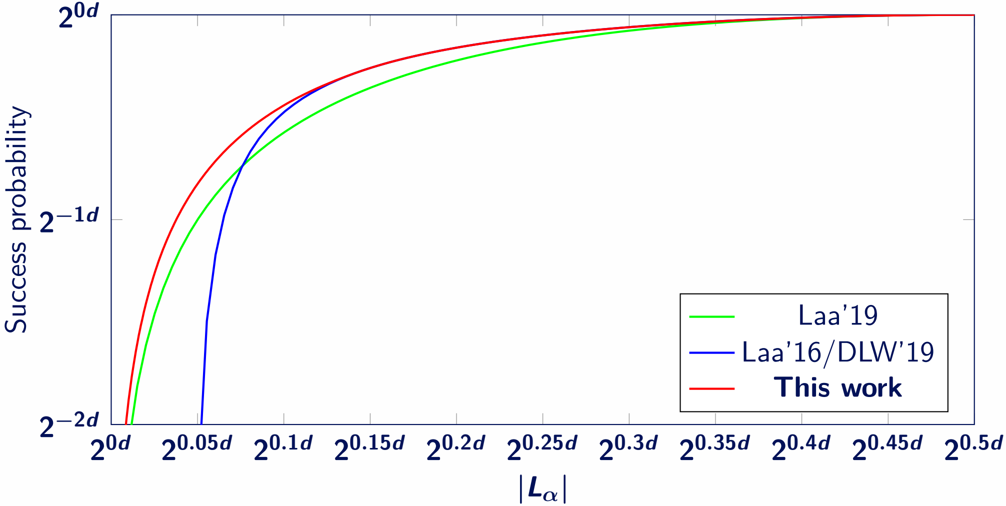

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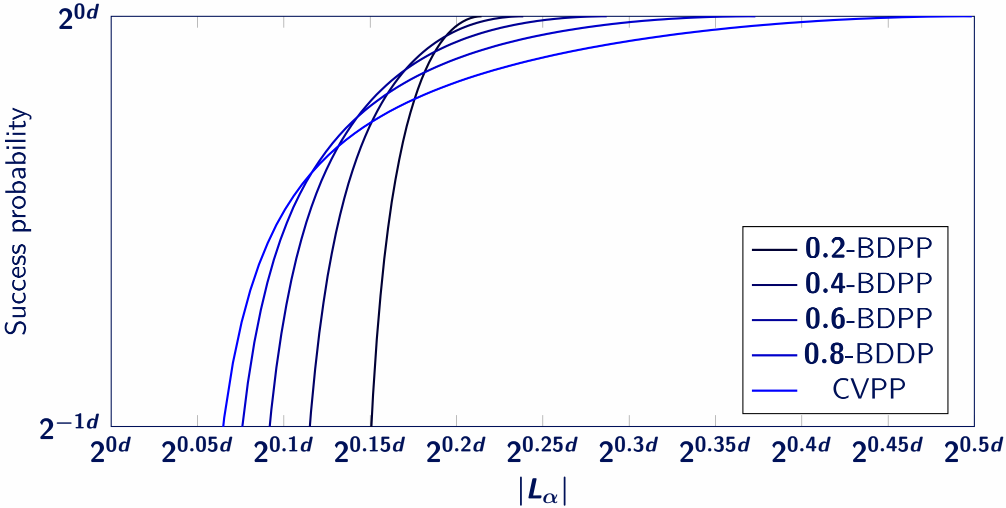

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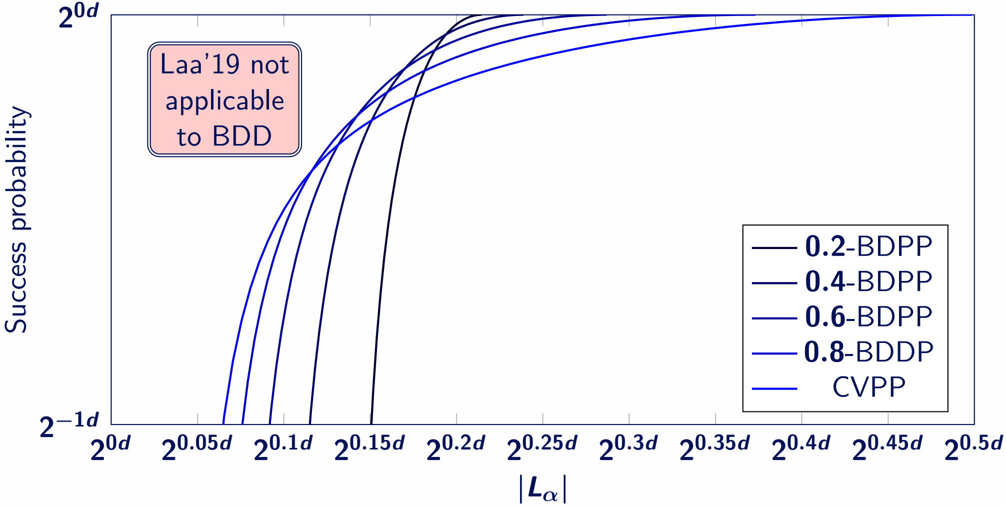

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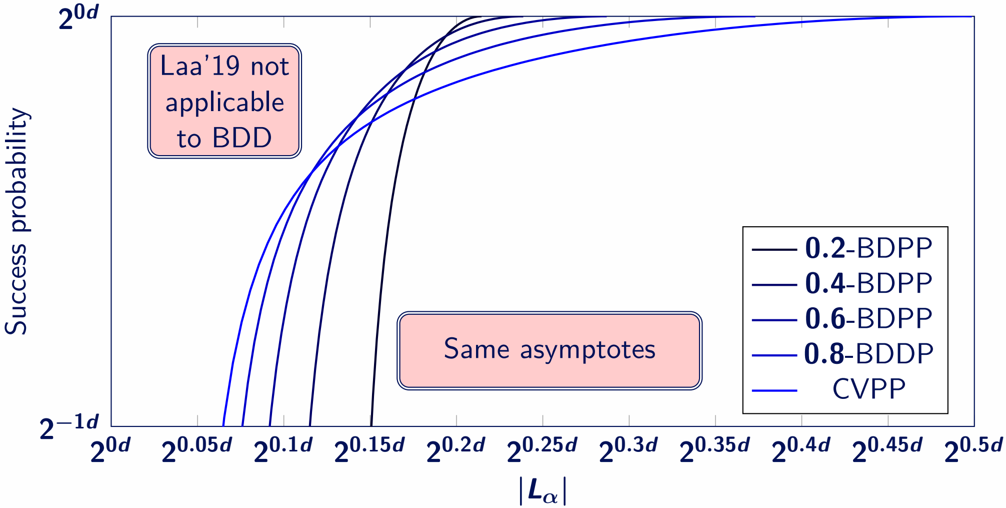

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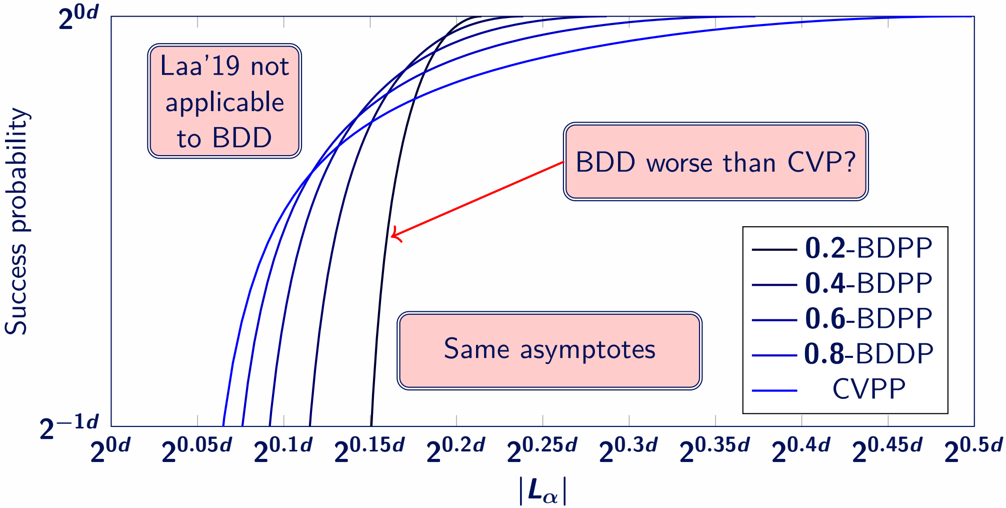

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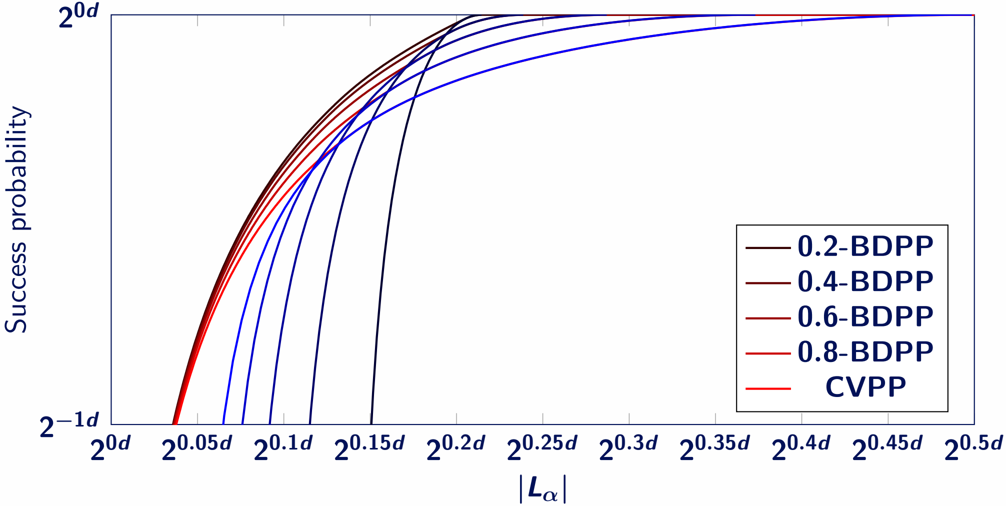

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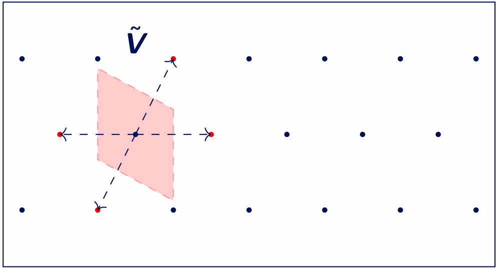

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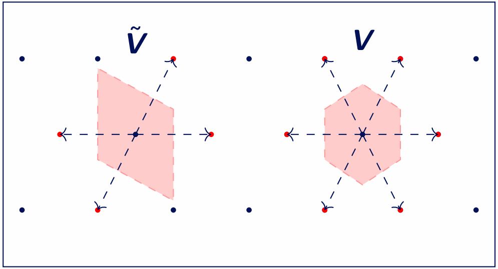

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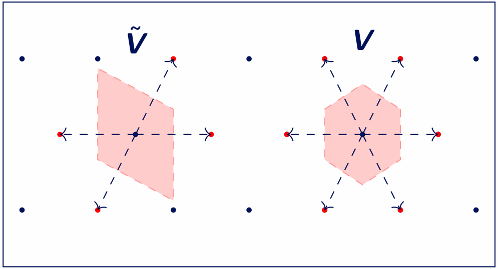

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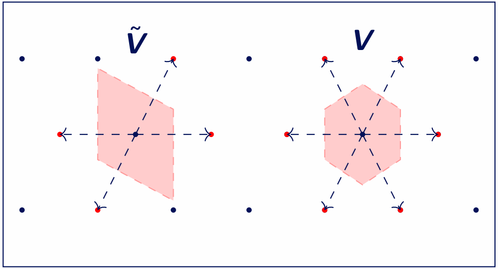

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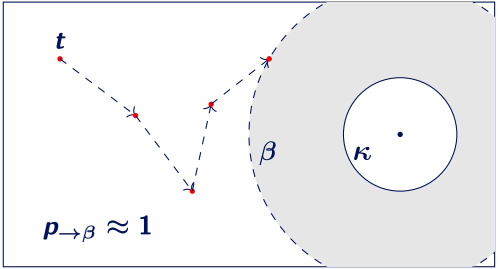

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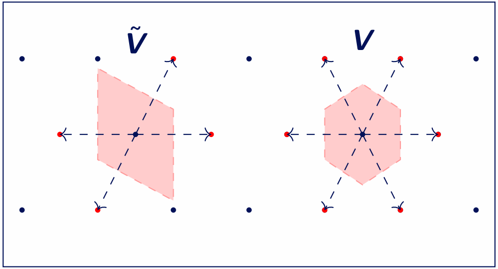

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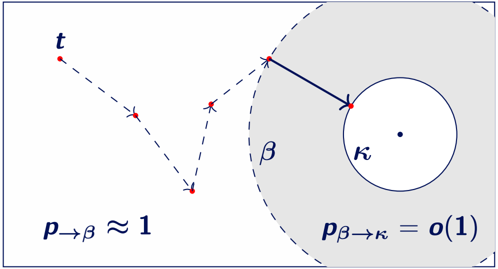

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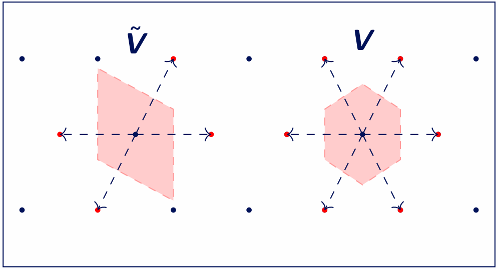

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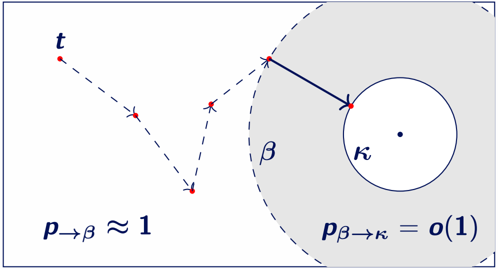
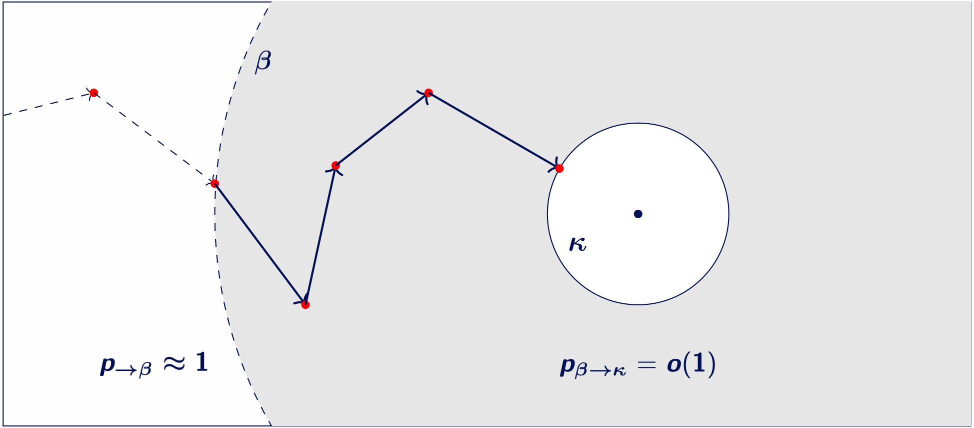


$$P_\alpha \geq p_{\to\beta} \cdot p_{\beta\to\kappa} \approx p_{\beta\to\kappa}$$

Multi-step analysis [This work]

- Model "all" events $\implies$ Random Walk.



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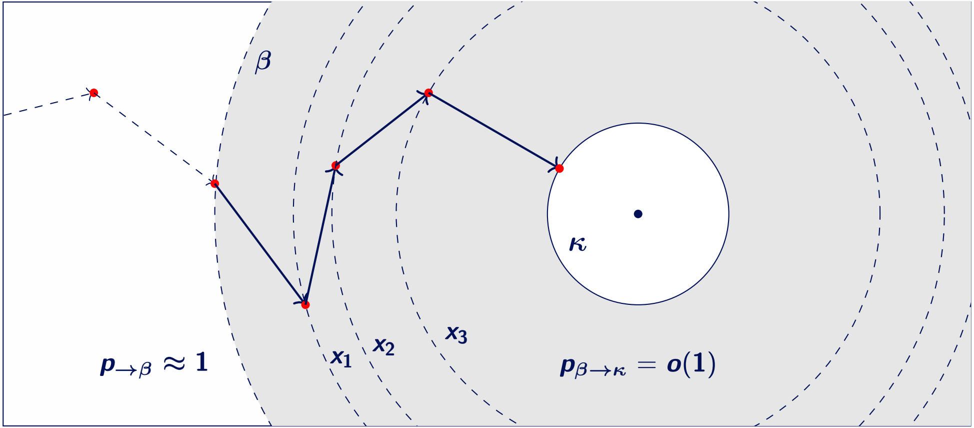

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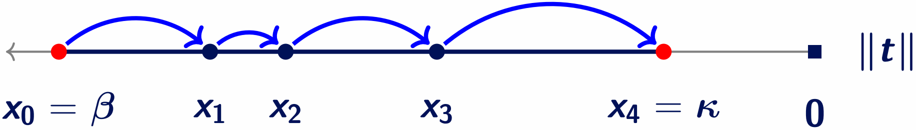

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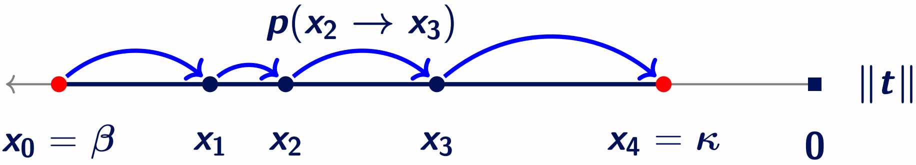

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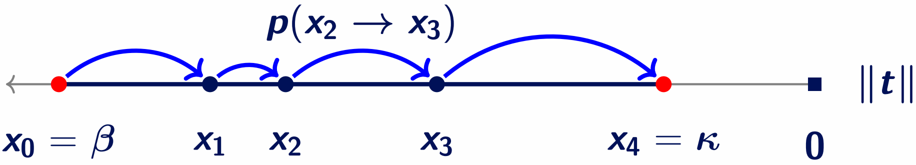

- This gives a lower bound

$$P \geq \prod_{i=1}^{4} p(x_{i-1} \to x_i).$$

Optimization Problem

- Each path $\boldsymbol{X}$: $\beta = \boldsymbol{x}_0 \rightarrow \boldsymbol{x}_1 \rightarrow \ldots \rightarrow \boldsymbol{x}_s = \boldsymbol{\kappa}$ gives a lower bound

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$$p(y \to z) = 2^{-c(y \to z) \cdot d + o(d)}.$$

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- Formal analysis using densities makes this bound tight (up to $2^{o(d)}$)

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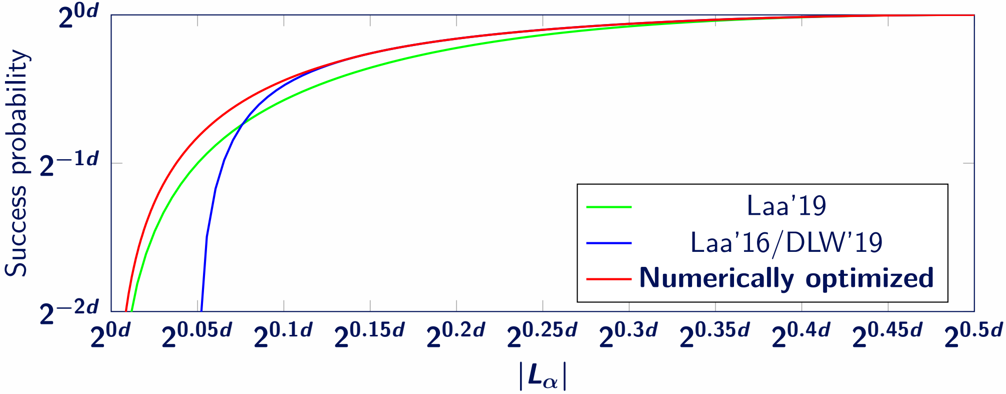

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- Express solution in terms of $\alpha, \boldsymbol{\kappa}$ using symbolic algebra.

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- Use known constraints:

$$\frac{\partial}{\partial x_i} C(X) = 0 \text{ for all } 1 \leq i < s,$$

$$x_0 = \beta, x_s = \kappa.$$

- Express solution in terms of α, κ using symbolic algebra.
- Show which path length s is optimal.

A (surprising) analytic Solution

Theorem (Optimal path)

The path $\boldsymbol{X} : \beta \to x_1 \to \ldots \to x_s = \kappa$ *that minimizes* $\boldsymbol{C}(\boldsymbol{X})$ *consists of*

$$s = \left\lceil -\frac{1}{2} + \frac{1}{2\alpha^2}\sqrt{(4\beta^2 - \alpha^2)^2 - 8(2\beta^2 - \alpha^2)\kappa^2} \right\rceil$$

steps, and is for $s > 1$ *given by* $x_i = \sqrt{u \cdot i^2 + v \cdot i + \beta^2}$, *with*

$$u := \frac{(\beta^2 + \kappa^2 - \alpha^2)s - \sqrt{(\alpha^2 s^2 - (\beta^2 + \kappa^2)) + 4\beta^2\kappa^2(s^2 - 1)}}{s^3 - s}$$

$$v := \frac{(\alpha^2 - 2\beta^2)s^2 + (\beta^2 - \kappa^2) + \sqrt{(\alpha^2 s^2 - (\beta^2 + \kappa^2)) + 4\beta^2\kappa^2(s^2 - 1)}s}{s^3 - s}.$$

Nearest Neighbor Search (NNS)

- We still need to iterate over $|\boldsymbol{L}|$ vectors per reduction step, cost $\tilde{\boldsymbol{O}}(|\boldsymbol{L}|)$.

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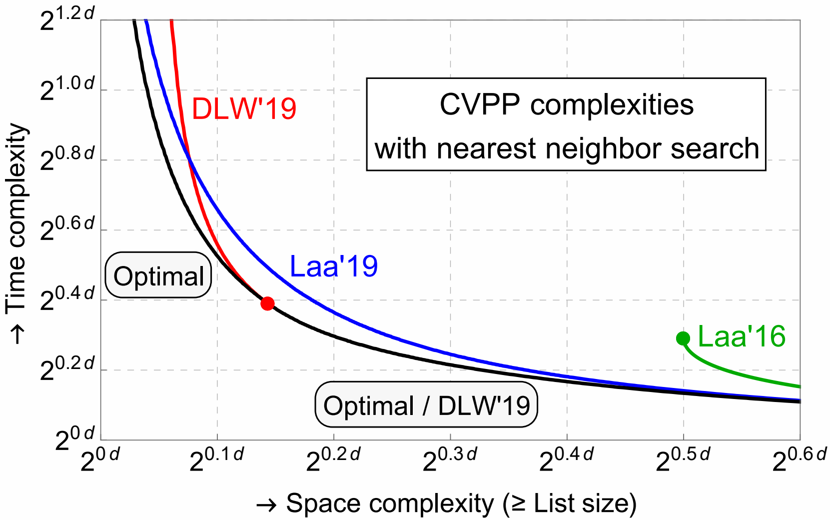
- NNS data structures reduce this, at the cost of more memory.

New CVPP time-memory trade-off



Memoryless NNS for batch CVPP

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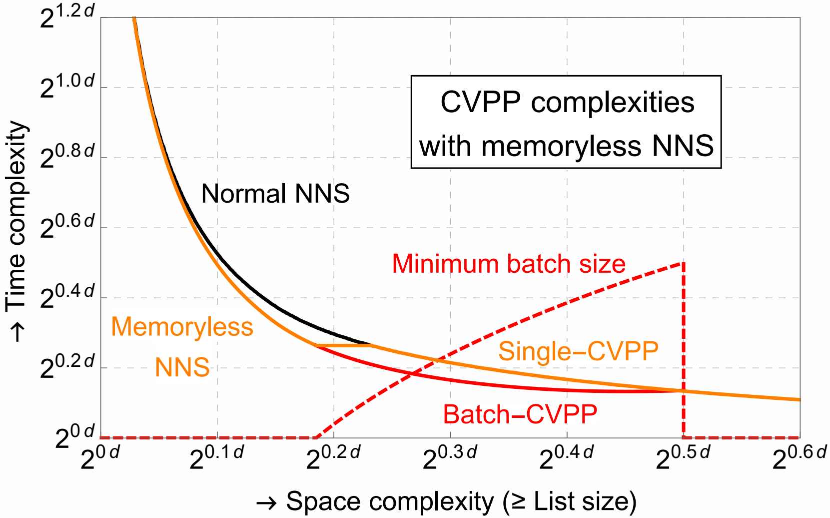
- Batches of size $\min\{1, P \cdot |L|\}$ are enough.

Further improvements using memoryless NNS



Conclusion

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Conclusion

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- **Sharper:** Full understanding of the asymptotic behaviour of the Iterative Slicer, leading to a tight bound on the success probability.
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- **Smaller:** We decrease the memory requirement for NNS, even for a single CVPP instance.

Conclusion

"The randomized slicer for CVPP: sharper, faster, smaller, batchier"

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- **Faster:** We obtain better time-memory trade-offs for CVPP.
- **Smaller:** We decrease the memory requirement for NNS, even for a single CVPP instance.
- **Batchier:** We significantly improve on the per-target time complexities for batch-CVPP.

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Thank you!