On the hardness of the NTRU problem

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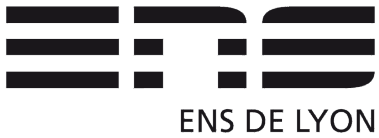
https://eprint.iacr.org/2021/821.pdf



NTRU

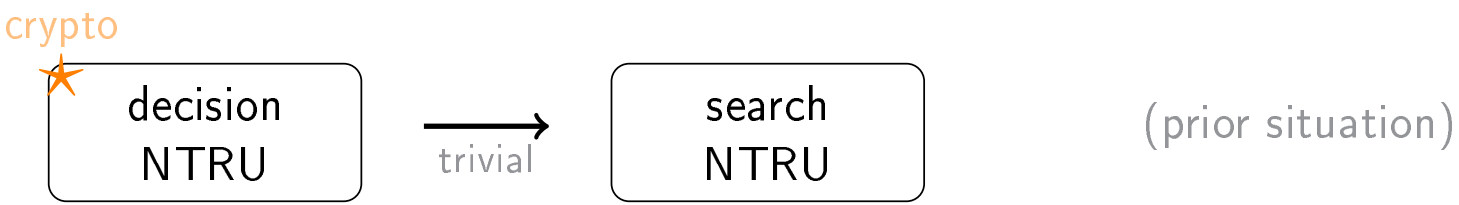
Algorithmic problem based on lattices

- post-quantum
- efficient
- used in Falcon and NTRU / NTRUPrime (NIST finalists)
- old (for lattice-based crypto): introduced in 1996

[HPS98] Hoffstein, Pipher, and Silverman. NTRU: a ring based public key cryptosystem. ANTS.

Our result

We prove reductions between NTRU variants and other lattice problems



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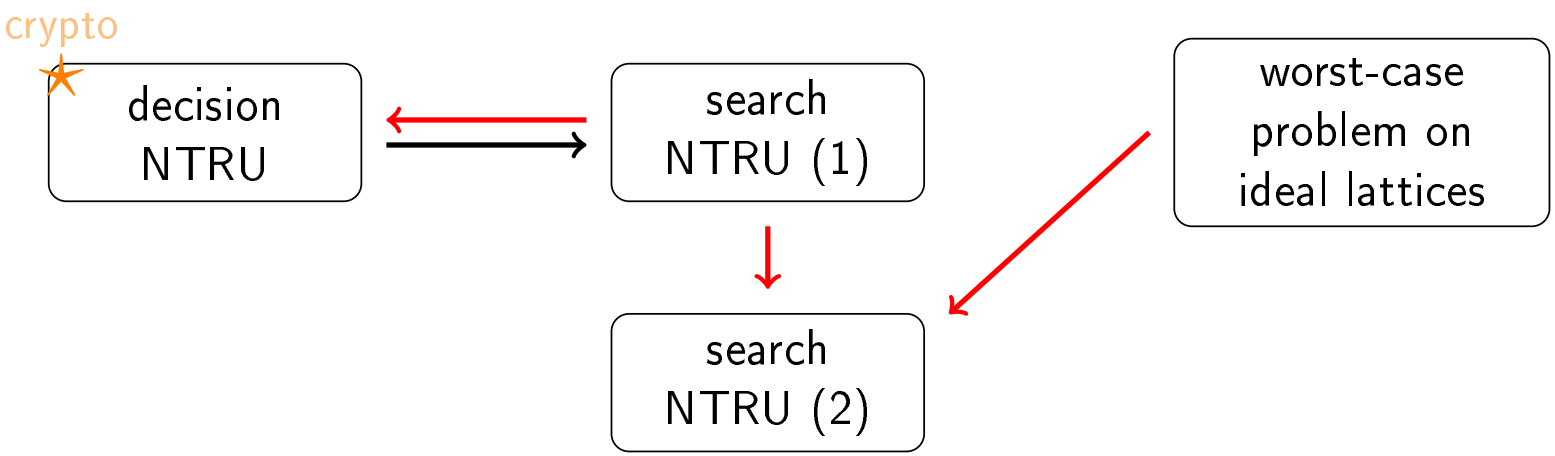

The different NTRU problems

NTRU instances

$$R = \mathbb{Z}[X]/(X^n + 1), \quad K = \mathbb{Q}[X]/(X^n + 1), \quad n = 2^k, \quad R_q = R/(qR)$$

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NTRU instance

A (γ, q)-NTRU instance is $h \in R_q$ s.t.

- $h = f/g \bmod q$ (or $gh = f \bmod q$)
- $\|f\|, \|g\| \leq \frac{\sqrt{q}}{\gamma}$ (if $y = \sum_{i=0}^{n-1} y_i X^i \in R$, then $\|y\| = \sqrt{\sum_i y_i^2}$)

The pair (f, g) is a trapdoor for h.

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The pair (f, g) is a trapdoor for h.

Claim: if (f, g) and (f', g') are two trapdoors for the same h,

$$\frac{f'}{g'} = \frac{f}{g} =: h_K \in K \quad \text{(division performed in K)}$$

Decisional NTRU problem

dNTRU

The (γ, q)-decisional NTRU problem ((γ, q)-dNTRU) asks, given $h \in R_q$, to decide whether

- $h \leftarrow \mathcal{D}$ where $\mathcal{D}$ is a distribution over (γ, q)-NTRU instances
- $h \leftarrow \mathcal{U}(R_q)$

Search NTRU problems

$\mathrm{NTRU}_{\mathrm{vec}}$

The (γ, q)-search NTRU vector problem ((γ, q)-$\mathrm{NTRU}_{\mathrm{vec}}$) asks, given a (γ, q)-NTRU instance h, to recover $(f, g) \in R^2$ s.t.

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$\Leftrightarrow$ recover $(\alpha f, \alpha g)$ for any $\alpha \in K$

(Both problems exist in worst-case and average-case variants)

NTRU is a (module) lattice problem

NTRU lattice: For $h \in R$, define Λ_h the (module) lattice with basis

$$B_h = \begin{pmatrix} 1 & 0 \\ h & q \end{pmatrix} \quad \text{(in columns)}$$

$$\Lambda_h = \{(g, f)^T \in R^2 \mid g \cdot h = f \bmod q\}$$

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$\mathrm{NTRU}_{\mathrm{mod}}$: recover $\alpha \cdot (f, g) \leftrightarrow$ find the direction where Λ_h is dense

What we know about NTRU

Previous works

Reductions:

[SS11, WW18] If $f, g \leftarrow D_{R,\sigma}$ with $\sigma \geq \mathrm{poly}(n) \cdot \sqrt{q}$ then $f/g \approx \mathcal{U}(R_q)$ (cyclotomic fields)

- dNTRU is provably hard when $\gamma \leq \frac{1}{\mathrm{poly}(n)}$

[SS11] Stehlé and Steinfeld. Making NTRU as secure as worst-case problems over ideal lattices. Eurocrypt.

[WW18] Wang and Wang. Provably secure NTRUEncrypt over any cyclotomic field. SAC.

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[Pei16] Peikert. A decade of lattice cryptography. Foundations and Trends in TCS.

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Attacks: (polynomial time)

[LLL82] dNTRU, $\mathrm{NTRU}_{\mathrm{mod}}$ broken if $\gamma \geq 2^n$

[LLL82] Lenstra, Lenstra, Lovász. Factoring polynomials with rational coefficients. Mathematische Annalen.

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[ABD16, CJL16] dNTRU, $\mathrm{NTRU}_{\mathrm{mod}}$ broken if $(\log q)^2 \geq n \cdot \log \frac{\sqrt{q}}{\gamma}$

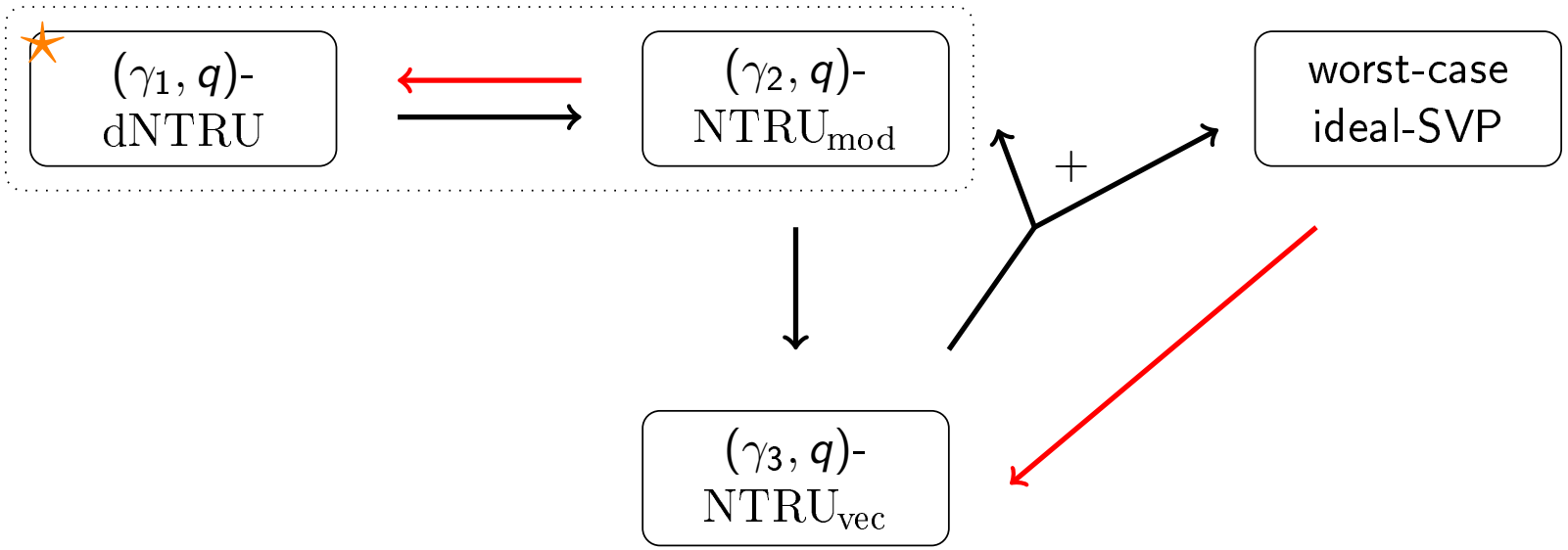
[KF17] (e.g., $q \approx 2^{\sqrt{n}}$ and $\gamma = \sqrt{q}/\mathrm{poly}(n)$)

[ABD16] Albrecht, Bai, and Ducas. A subfield lattice attack on overstretched NTRU assumptions. Crypto.

[CJL16] Cheon, Jeong, and Lee. An algorithm for NTRU problems. LMS J Comput Math.

[KF17] Kirchner and Fouque. Revisiting lattice attacks on overstretched NTRU parameters. Eurocrypt

Our results (with more details)



⚠ the reductions only work for certain distributions of NTRU instances ⚠

(the arrows may not compose)

Worst-case ideal-SVP: given any ideal lattice $I \subset R$, find $v \in I \setminus \{0\}$ such that $\|v\| \leq \min_{w \in I \setminus \{0\}} \|w\|$.

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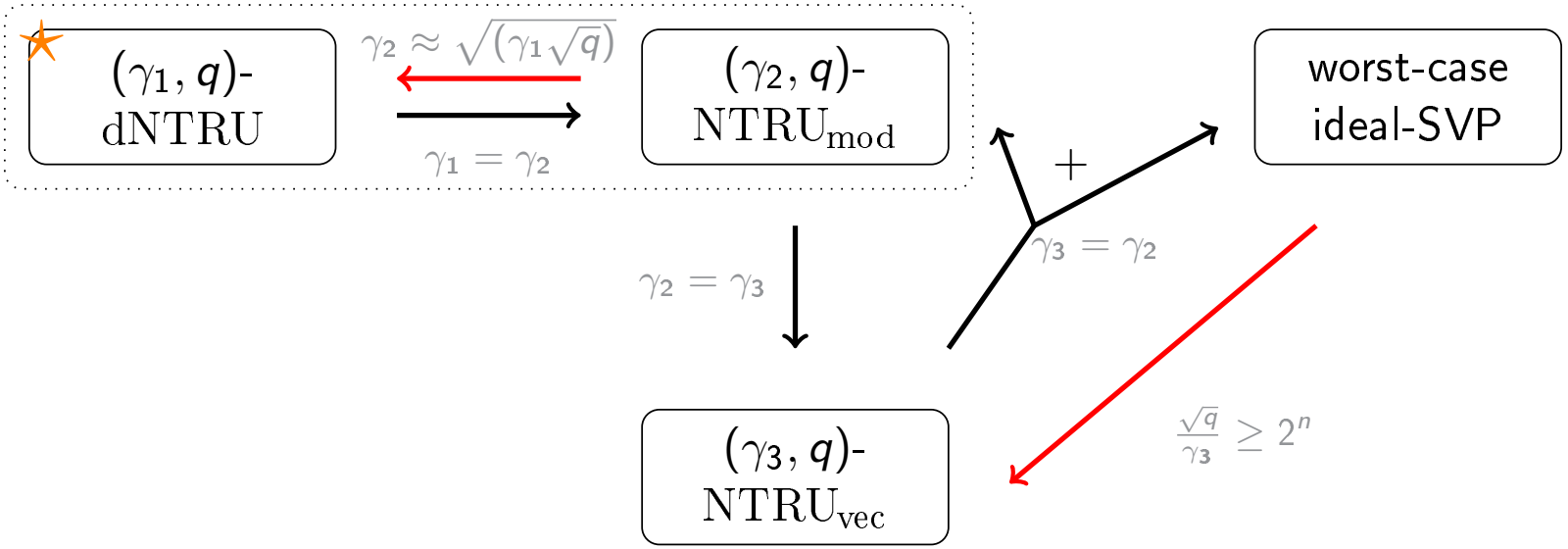

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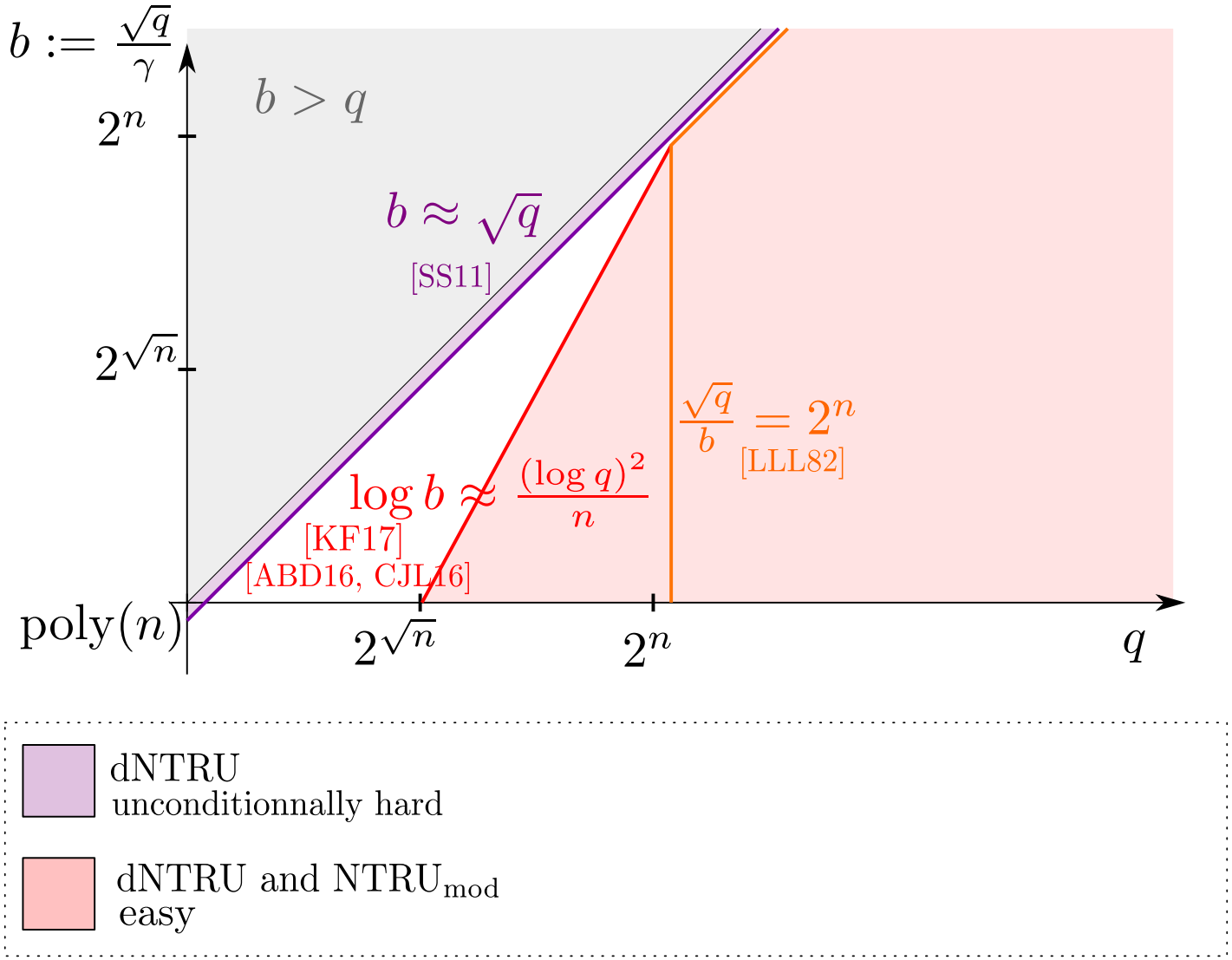
One big picture: poly time attacks and reductions (cyclotomics)



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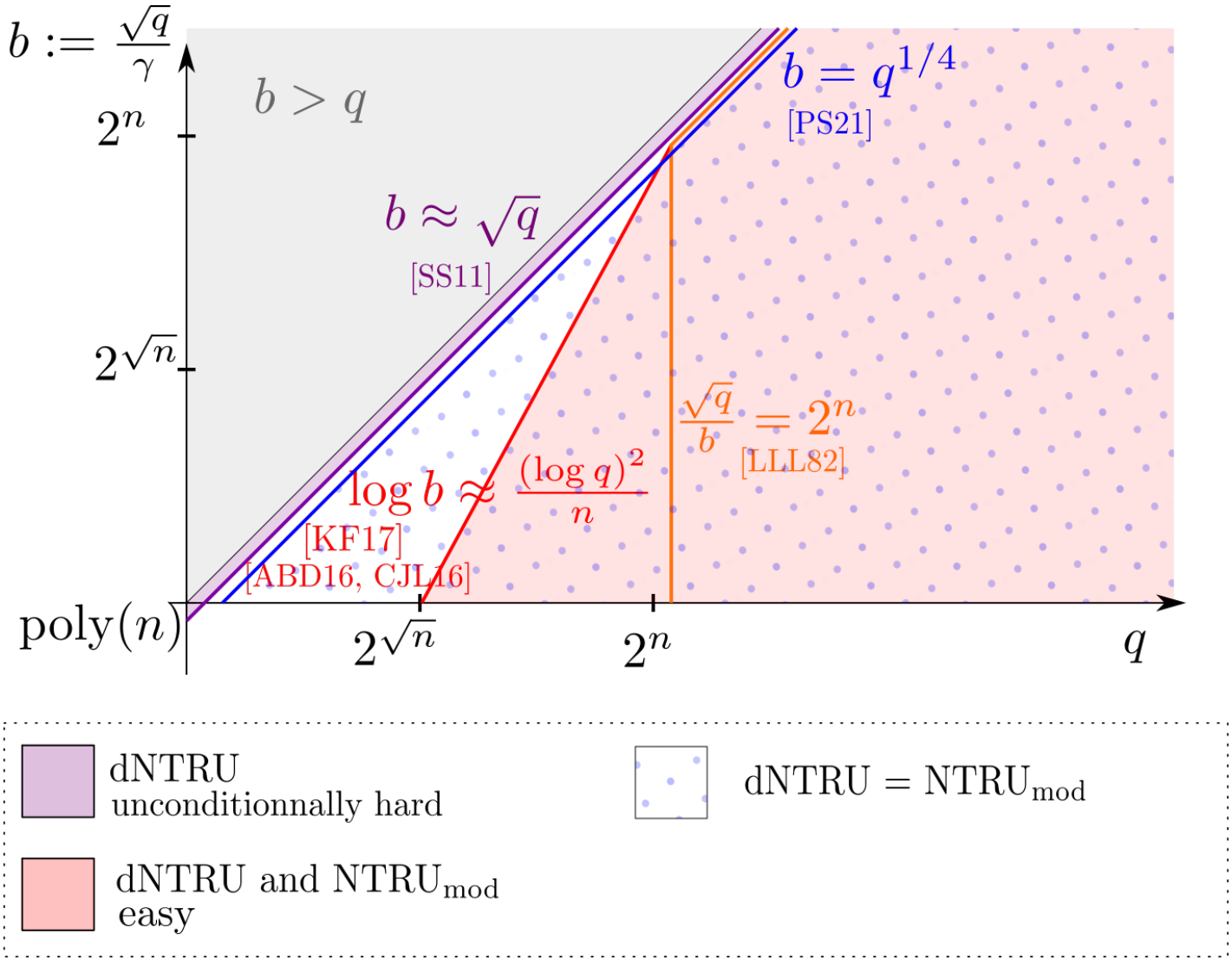

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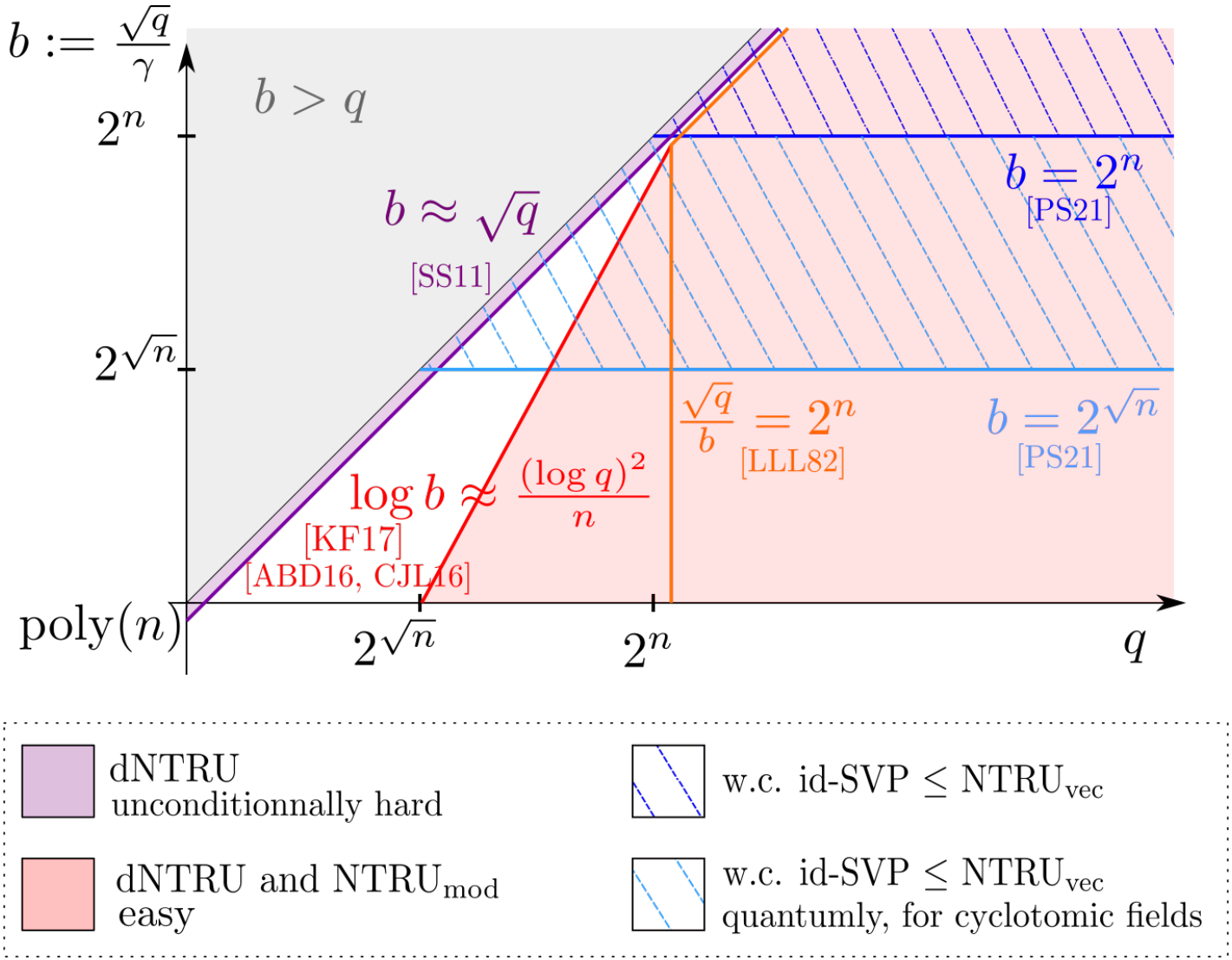
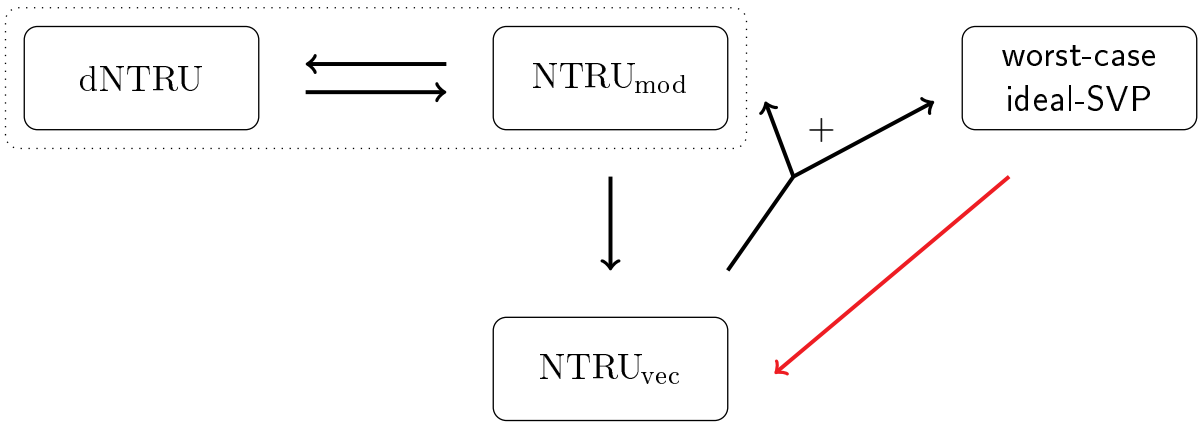


Techniques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worst-case
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From ideal-SVP to $\mathrm{NTRU}_{\mathrm{vec}}$

Objective: Transform an ideal I into an NTRU instance h

- $I = \langle z \rangle = \{z \cdot r \,|\, r \in R\}$ (assume for simplicity that I is principal)
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- $h = q/z$, $f = 0$
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⚠ Not an NTRU instance ($h \in K$ is not in R_q)

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This is an NTRU instance

From ideal-SVP to $\mathrm{NTRU}_{\mathrm{vec}}$ (2)

Summing up: If $I = \langle z \rangle = \{z \cdot r \mid r \in R\}$ and z known

- we can construct an NTRU instance h from I
  - any short $g \in I$ provides a trapdoor (f, g) for h

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What we need to conclude the reduction:

- any trapdoor (f', g') for h is such that $g' \in I$
  - g' solution to ideal-SVP in I

Extending the reduction

Non principal ideals:

- $I = R \cap \langle z \rangle$ with z easily computed
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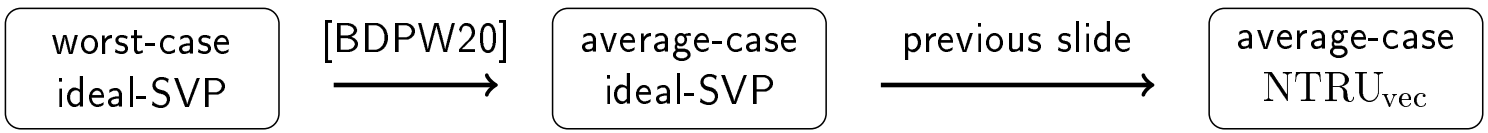
Worst-case to average-case reduction:



[BDPW20] de Boer, Ducas, Pellet-Mary, and Wesolowski. Random Self-reducibility of Ideal-SVP via Arakelov Random Walks. Crypto.

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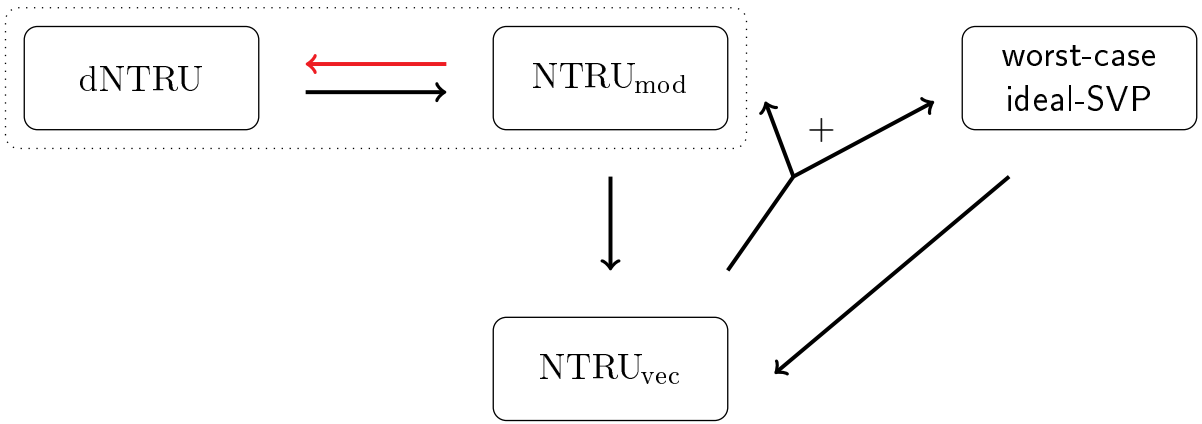
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Idea:

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$\Rightarrow$ we can choose x and y

$\Rightarrow$ we can modify the coordinates one by one

From $\mathrm{NTRU}_{\mathrm{mod}}$ to dNTRU (2)

Simplified problem

$f, g \in \mathbb{R}$ secret, $B \geq 0$ unknown.
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Algorithm:

- Find x_0, y_0 such that $x_0 f + y_0 g = B$
  - (Fix $x_0 \ll B/|f|$ and increase y_0 until the oracle says NO)
- Find x_1, y_1 such that $x_1 \neq x_0$ and $x_1 f + y_1 g = B$

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- Solve for f/g

More technical details

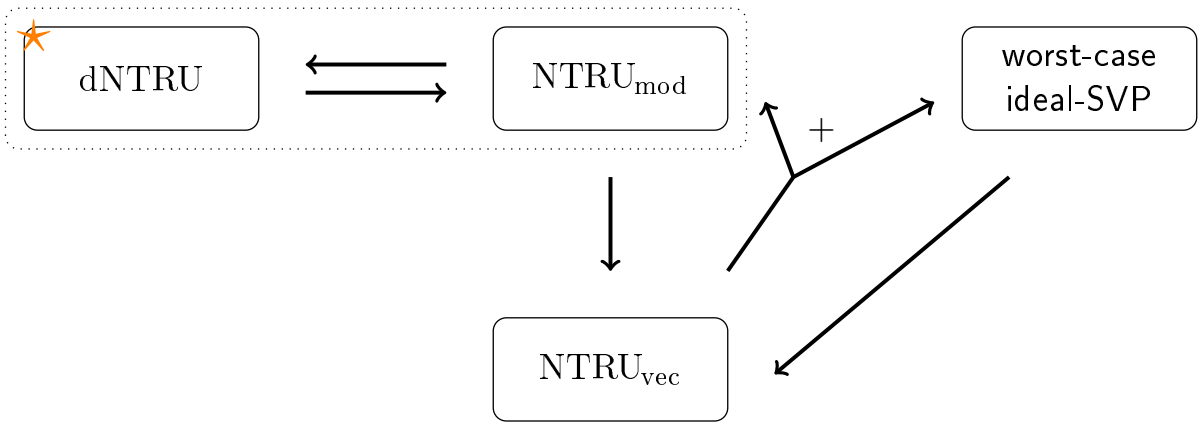
If the oracle is not perfect:

- need to handle distributions
- use the "oracle hidden center" framework [PRS17]

[PRS17] Peikert, Regev, and Stephens-Davidowitz. Pseudorandomness of ring-LWE for any ring and modulus. STOC.

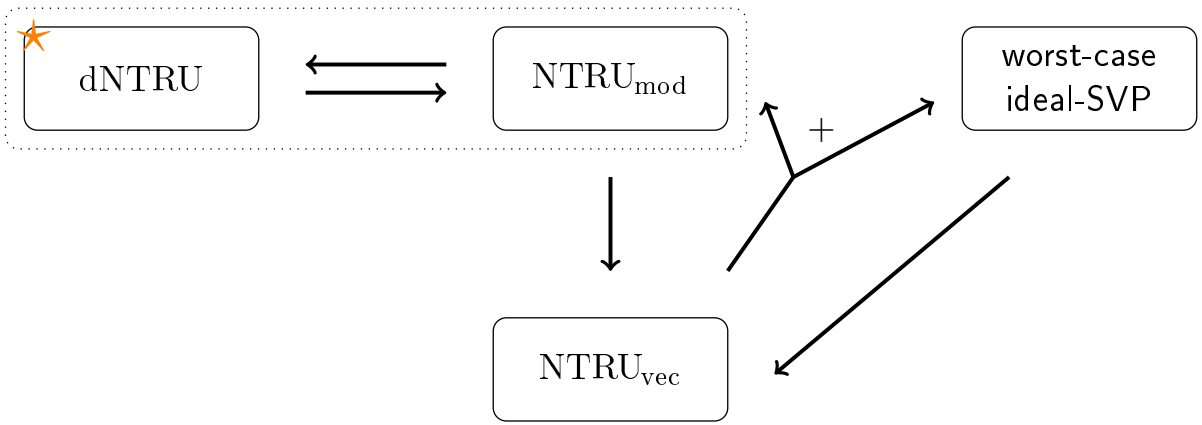
Conclusion

Conclusion and open problems



- Can we make the distributions of the reductions match?
- Can we relate $\mathrm{NTRU}_{\mathrm{mod}}$ and ideal-SVP?
  - maybe not since any "natural reduction" would provide new attacks
- Can we prove reduction from module problems with rank ≥ 2?

Conclusion and open problems



- Can we make the distributions of the reductions match?
- Can we relate $\mathrm{NTRU}_{\mathrm{mod}}$ and ideal-SVP?
  - maybe not since any "natural reduction" would provide new attacks
- Can we prove reduction from module problems with rank ≥ 2?

Thank you