Luby-Rackoff Backwards with More Users and More Security

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Presentation Outline

Luby-Rackoff Backwards: Motivation

Pseudorandom function (PRF) : Important cryptographic primitive.

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[Luby and Rackoff, 1988]: PRF to PRP

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Constructions

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Construction: Sum of Permutations

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[Lucks, 2000, Mennink and Preneel, 2015]: Security same as XORP $\Rightarrow$ Secure up to $O(2^n)$ queries

Sum of Permutations (Contd.)

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Our focus $k = 3$.

Application

CENC [Iwata, 2006, Bhattacharya and Nandi, 2018b], PMAC_Plus [Yasuda, 2011], ZMAC [Iwata et al., 2017].

Multi-user PRF-Security of XORP[k]: A Concern

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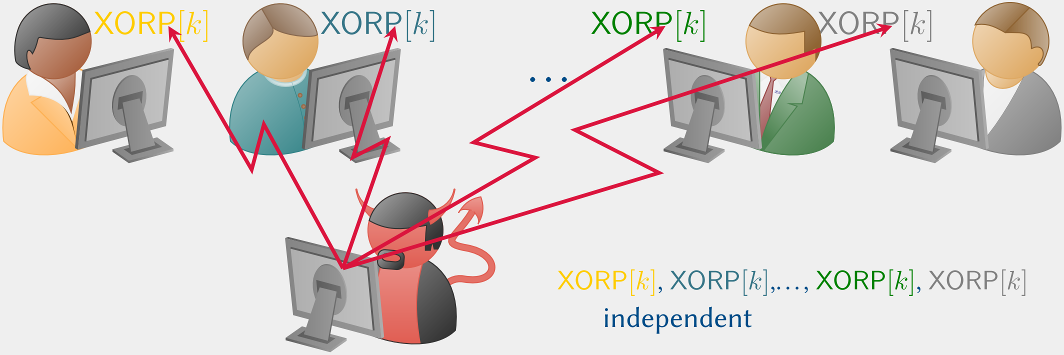

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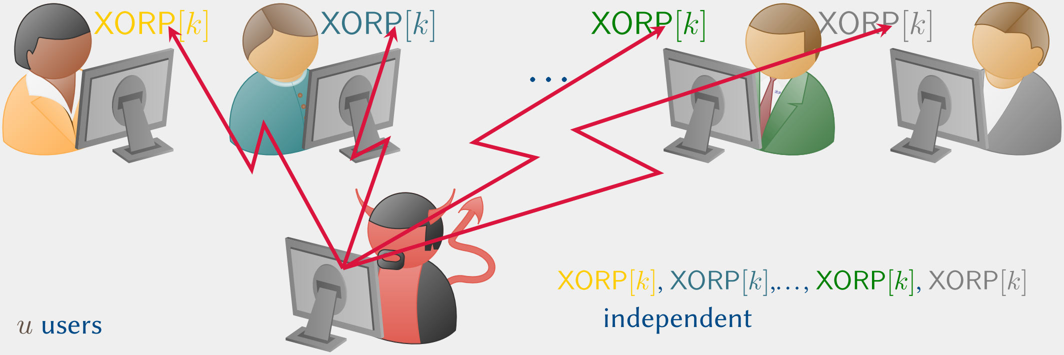

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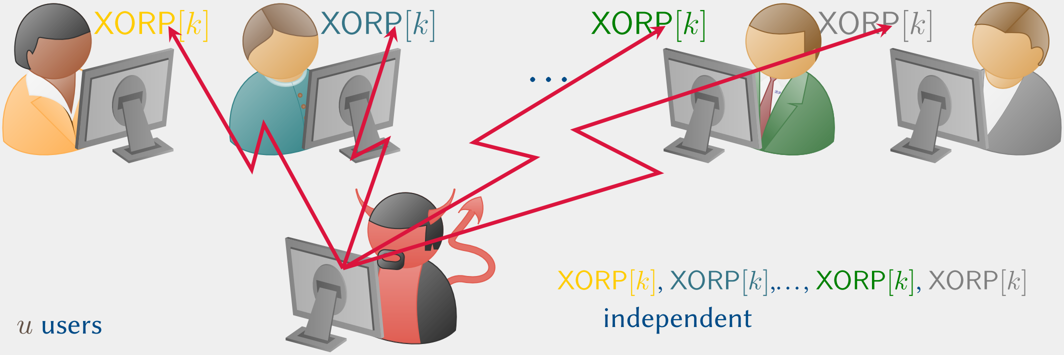

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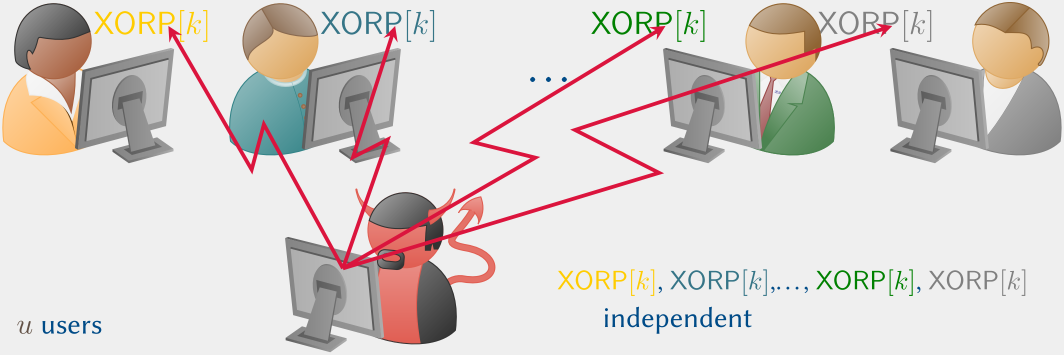

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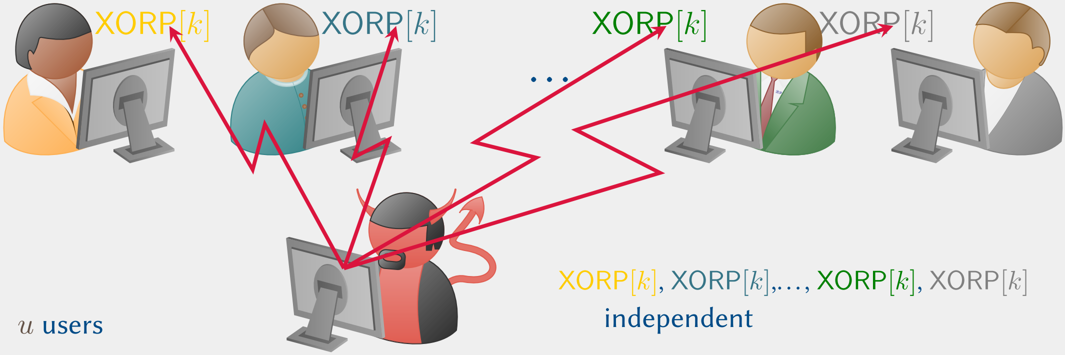

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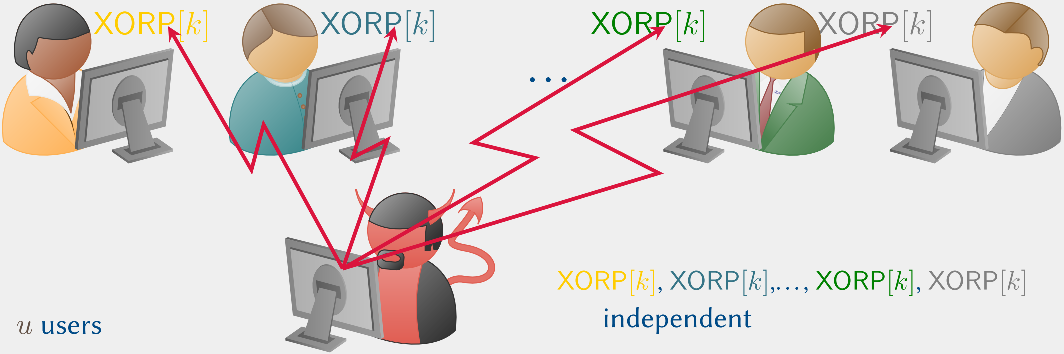

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- Possible fix: Increase the block length of the cipher
- Block ciphers like AES come with fixed block length

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- XORP$'[3]$ provides same level of security

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Func_n: All functions from $\{0,1\}^{n-2}$ to $\{0,1\}^n$.
Perm_n: All permutations from $\{0,1\}^n$ to $\{0,1\}^n$.

Security Notion: Indistinguishability

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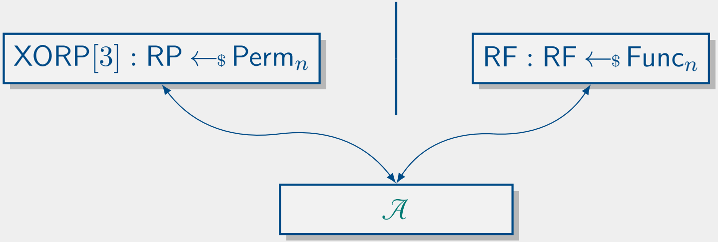

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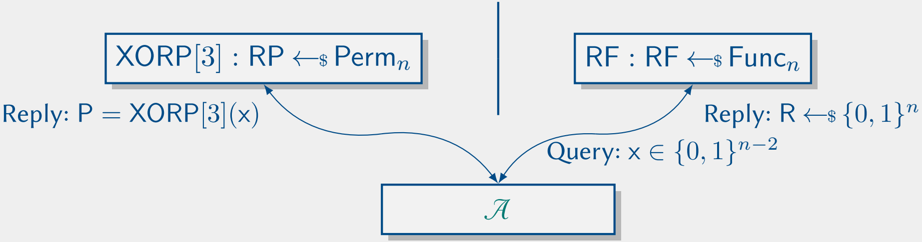

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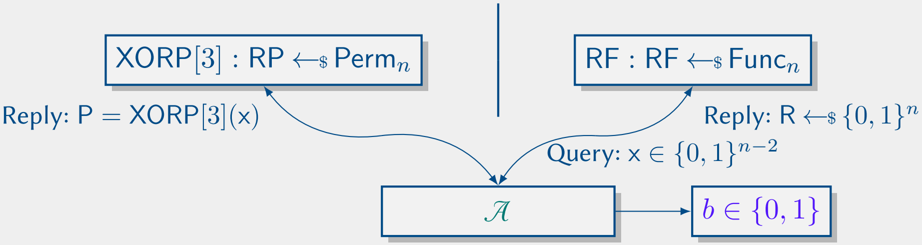

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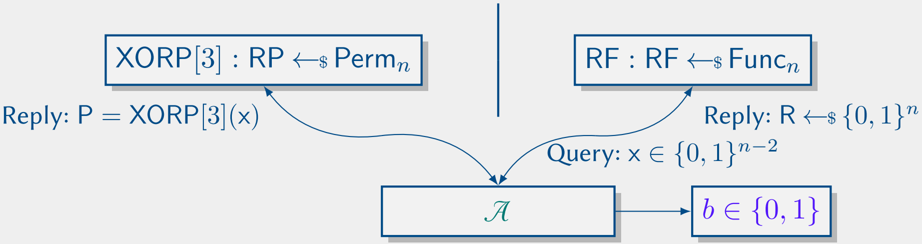

Quantifying Security: Advantage

$$\mathsf{Adv}^{\mathrm{prf}}_{\mathsf{XORP}[3]}(\mathscr{A}) := |\mathbf{Pr}[\mathscr{A}^{\mathsf{XORP}[3]} \rightarrow 1] - \mathbf{Pr}[\mathscr{A}^{\mathsf{RF}} \rightarrow 1]|$$

- Focus on information theoretic security of XORP[3].
  - $\mathcal{A}$ comuptationally unbounded $\Rightarrow$ $\mathcal{A}$ is deterministic (runs with best coins)
- Restrict $\mathcal{A}$ to q queries.
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$\mathsf{XORP}[3]$ transcript $\mathsf{P} := (\mathsf{P}_1, \mathsf{P}_2, \ldots, \mathsf{P}_q)$; RF transcript $\mathsf{R} := (\mathsf{R}_1, \mathsf{R}_2, \ldots, \mathsf{R}_q)$

$$\mathsf{Adv}^{\mathrm{prf}}_{\mathsf{XORP}[3]}(\mathcal{A}) \leq \|\mathbf{Pr}_{\mathsf{P}} - \mathbf{Pr}_{\mathsf{R}}\|$$

Setting

$\mathsf{Func}_n^u := \{f | f : [u] \times \{0,1\}^{n-2} \mapsto \{0,1\}^n\}$, $\mathsf{RF} \leftarrow_{\$} \mathsf{Func}_n^u$
$\mathsf{RP}_1, \mathsf{RP}_2, \ldots, \mathsf{RP}_u \leftarrow_{\$} \mathsf{Perm}_n$

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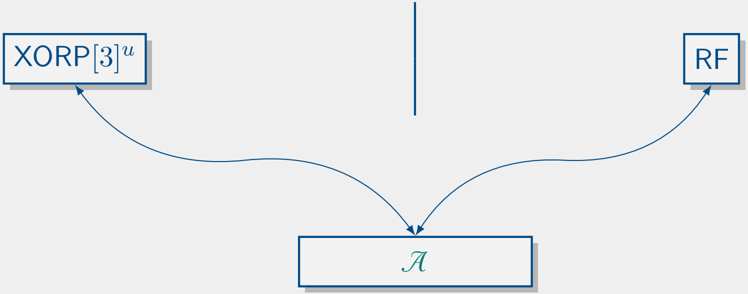

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$= \mathsf{RP}_i(\mathsf{x}\|00) \oplus \mathsf{RP}_i(\mathsf{x}\|01) \oplus \mathsf{RP}_i(\mathsf{x}\|10)$

$\mathsf{XORP}[3]^u$

RF

Reply: $\mathsf{P} = \mathsf{XORP}_{\mathsf{RP}_i}[3](\mathsf{x})$

Reply: $\mathsf{R} \leftarrow_\$ \{0,1\}^n$

Query: $(i, \mathsf{x}) \in [u] \times \{0,1\}^{n-2}$

$\mathcal{A}$

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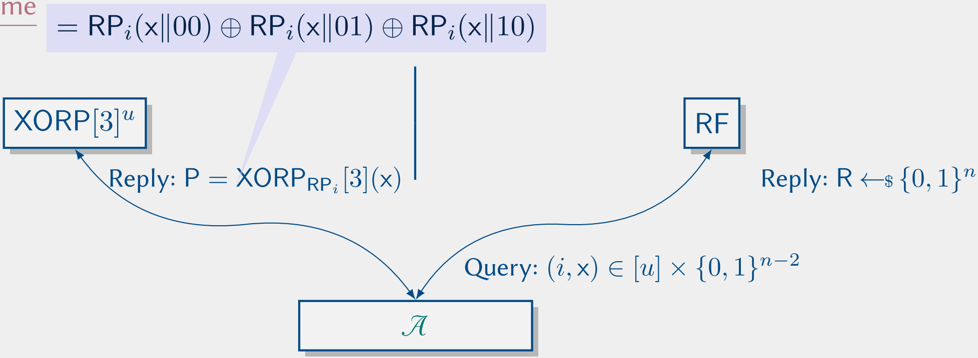

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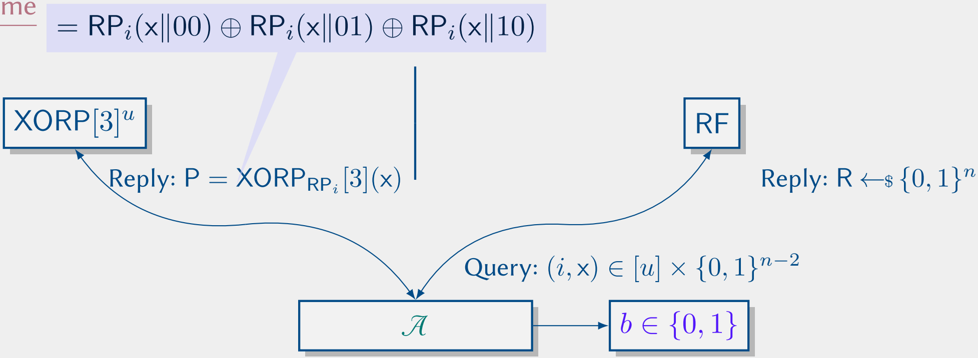

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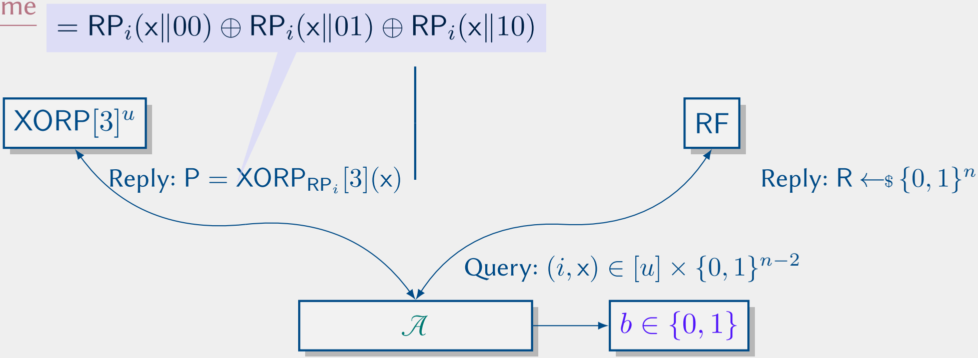

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Our Contribution (Formal) and Application

- $\mathsf{Adv}^{\mathrm{mu\_prf}}_{\mathsf{XORP[3]}}(\mathcal{A}) \leq 20\sqrt{uq_{max}}/2^n \quad (q_{max} \leq 2^n/12)$

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  - For single user, adversary's advantage is $O\left(\frac{1}{\sqrt{2^n}}\right)$ even after making $O(2^n)$ queries.

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Counter-mode encryption using $\mathsf{XORP}[3]$

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Application

Counter-mode encryption using $\mathsf{XORP}[3]$

- Multi-user security similar to $\mathsf{XORP}[3]$ (when instantiated with a good block cipher)
- [Bellare et al., 1999]: Parity-method encryption
  - similar security, but requires additional randomness

Our Technique: χ^2-method

- $\mathsf{X}^q := (\mathsf{X}_1, \ldots, \mathsf{X}_q) \sim \mathrm{Pr}_{\mathsf{X}}$, and $\mathsf{Z}^q := (\mathsf{Z}_1, \ldots, \mathsf{Z}_q) \sim \mathrm{Pr}_{\mathsf{Z}}$ over $\Omega \times \cdots \times \Omega$.
- $\mathbf{Pr}_{\mathsf{X}|\mathsf{x}^{i-1}}(\mathsf{x}_i) := \mathbf{Pr}[\mathsf{X}_i = \mathsf{x}_i \mid \mathsf{X}_1 = \mathsf{x}_1, \ldots, \mathsf{X}_{i-1} = \mathsf{x}_{i-1}]$,
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Theorem ([Dai et al., 2017])

If $\mathbf{Pr}_{\mathsf{X}|\mathsf{x}^{i-1}}$ is contained within the support of the distribution $\mathbf{Pr}_{\mathsf{Z}|\mathsf{x}^{i-1}}$ for all x^{i-1}, then

$$\|\mathbf{Pr}_{\mathsf{X}} - \mathbf{Pr}_{\mathsf{Z}}\| \leq \left(\frac{1}{2} \sum_{i=1}^{q} \mathbf{Ex}[\chi^2(\mathsf{X}^{i-1})] \right)^{\frac{1}{2}}. \tag{1}$$

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Effectively applied in [Bhattacharya and Nandi, 2018b, Bhattacharya and Nandi, 2018a, Choi et al., 2019, Mennink, 2019, Gunsing and Mennink, 2020].

Presentation Outline

Applying the χ^2-method: Problem Due to Adaptive Choice

$$\mathsf{Adv}^{\mathrm{mu\_prf}}_{\mathsf{XORP}[3]}(\mathscr{A}) \leq \|\mathbf{Pr}_{\mathsf{P}} - \mathbf{Pr}_{\mathsf{R}}\|$$

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Can we apply the χ^2-method to upper bound $\|\mathbf{Pr}_{\mathsf{P}} - \mathbf{Pr}_{\mathsf{R}}\|$?

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User of the i-th query

$\mathcal{A}$ chooses user U_i adaptively $\Rightarrow$ U_i potentially depends on all the previous replies (from all the users)

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Is there a way round?

Reorder (permute) the transcript (P to S and R to U)

Club the replies from the same user together

More precisely ...

A Solution: Reordering the Transcript

Random Experiment for U

$\mathsf{U} := (\mathsf{U}_i : i \in [q]) \xleftarrow{\text{wr}} \mathcal{G}$
return U

Random Experiment for S

for $1 \leq i \leq u$
  $\widehat{\mathsf{T}}_i := (\mathsf{T}_{j,k} : j \in [\mathrm{I}_i], k \in [3]) \xleftarrow{\text{wor}} \mathcal{G}$
/ $\widehat{\mathsf{T}}_i$ is sampled independent of $\widehat{\mathsf{T}}_j$, $1 \leq j \leq i-1$
for $1 \leq \ell \leq q$
  $\mathsf{S}_\ell = \mathsf{T}_{\ell,1} + \mathsf{T}_{\ell,2} + \mathsf{T}_{\ell,3}$
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Reordering P to S:

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Two Observations

- Distribution of output is independent of input in both worlds.
- $\mathcal{A}$ makes same number ($= q_{max}$) of queries to each user.

Reordering (Contd.)

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Reordering preserves statistical distance: $\|\mathbf{Pr}_\mathsf{S} - \mathbf{Pr}_\mathsf{U}\| = \|\mathbf{Pr}_\mathsf{P} - \mathbf{Pr}_\mathsf{R}\|$

Enough to upper bound $\|\mathbf{Pr}_{\mathsf{S}} - \mathbf{Pr}_{\mathsf{U}}\|$

Extending the Transcripts: Ensuring the Support Condition

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How to extend?

$\mathsf{S}_i = \mathsf{T}_{i,1} + \mathsf{T}_{i,2} + \mathsf{T}_{i,3}$

$$\left.\begin{aligned} \mathsf{X}_i &= (\mathsf{T}_{i,1}, \mathsf{T}_{i,2}, \mathsf{S}_i) \\ \mathsf{Y}_i &= (\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{U}_i) \end{aligned}\right\} \text{ for all } i \in [q]$$

$\mathsf{V}_{i,3} = \mathsf{V}_{i,1} + \mathsf{V}_{i,2} + \mathsf{U}_i$

$(\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3})$, $i \in [q]$ *behaves like* a WOR sample

What are $\mathsf{V}_{i,1}, \mathsf{V}_{i,2}$?

Extending the Transcripts: Details

$$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$$

$U_i = j$

$i = (j-1)q_{max} + k \quad \Rightarrow \quad$ i-th query $=$ j-th user's k-th query

$U_i = j$

$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$

$$n_i = \Bigg\{ (\mathsf{v}_1, \mathsf{v}_2) \Bigg| \; \mathsf{v}_1, \mathsf{v}_2, \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2 \in \mathcal{G} \setminus \bigcup_{i=(j-1)q_{max}+1}^{(j-1)q_{max}+k-1} \{\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3}\}, \; \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2, \mathsf{v}_1, \mathsf{v}_2 \text{ distinct} \Bigg\}$$

$U_i = j$

$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$

All previous samples of the j-th user

$$\mathcal{n}_i = \left\{ (\mathsf{v}_1, \mathsf{v}_2) \;\middle|\; \begin{array}{l} \mathsf{v}_1, \mathsf{v}_2, \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2 \in \mathcal{G} \setminus \bigcup\limits_{i=(j-1)q_{max}+1}^{(j-1)q_{max}+k-1} \{\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3}\}, \\ \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2, \mathsf{v}_1, \mathsf{v}_2 \text{ distinct} \end{array} \right\}$$

$U_i = j$

$$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$$

All previous samples of the j-th user

$$\mathcal{N}_i = \Bigg\{ (\mathsf{v}_1, \mathsf{v}_2) \Bigg| \; \mathsf{v}_1, \mathsf{v}_2, \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2 \in \mathcal{G} \setminus \bigcup_{i=(j-1)q_{max}+1}^{(j-1)q_{max}+k-1} \{\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3}\}, \; \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2, \mathsf{v}_1, \mathsf{v}_2 \text{ distinct} \Bigg\}$$

$$(\mathsf{V}_{i,1}, \mathsf{V}_{i,2}) \leftarrow_{\$} \mathcal{N}_i$$

$U_i = j$

$$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$$

$$\mathcal{n}_i = \left\{ (\mathsf{v}_1, \mathsf{v}_2) \middle| \begin{array}{l} \mathsf{v}_1, \mathsf{v}_2, \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2 \in \mathcal{G} \setminus \bigcup_{i=(j-1)q_{max}+1}^{(j-1)q_{max}+k-1} \{\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3}\}, \\ \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2, \mathsf{v}_1, \mathsf{v}_2 \text{ distinct} \end{array} \right\}$$

All previous samples of the j-th user

$$(\mathsf{V}_{i,1}, \mathsf{V}_{i,2}) \leftarrow_{\$} \mathcal{n}_i$$

After Extension

- Support of X $=$ Support of Y

$U_i = j$

$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$

All previous samples of the j-th user

$$\mathcal{N}_i = \left\{ (\mathsf{v}_1, \mathsf{v}_2) \;\middle|\; \begin{array}{l} \mathsf{v}_1, \mathsf{v}_2, \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2 \in \mathcal{G} \setminus \bigcup_{i=(j-1)q_{max}+1}^{(j-1)q_{max}+k-1} \{\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3}\}, \\ \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2, \mathsf{v}_1, \mathsf{v}_2 \text{ distinct} \end{array} \right\}$$

$$(\mathsf{V}_{i,1}, \mathsf{V}_{i,2}) \leftarrow_{\$} \mathcal{N}_i$$

After Extension

- Support of X $=$ Support of Y
- S and U are marginals of X and Y (resp.) $\Rightarrow$ $\|\mathbf{Pr}_{\mathsf{S}} - \mathbf{Pr}_{\mathsf{U}}\| \leq \|\mathbf{Pr}_{\mathsf{X}} - \mathbf{Pr}_{\mathsf{Y}}\|$

Extending the Transcripts: Details

$U_i = j$

$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$

$$\mathcal{N}_i = \left\{ (\mathsf{v}_1, \mathsf{v}_2) \middle| \begin{array}{l} \mathsf{v}_1, \mathsf{v}_2, \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2 \in \mathcal{G} \setminus \bigcup_{i=(j-1)q_{max}+1}^{(j-1)q_{max}+k-1} \{\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3}\}, \\ \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2, \mathsf{v}_1, \mathsf{v}_2 \text{ distinct} \end{array} \right\}$$

All previous samples of the j-th user

$$(\mathsf{V}_{i,1}, \mathsf{V}_{i,2}) \leftarrow_{\$} \mathcal{N}_i$$

After Extension

- Support of X $=$ Support of Y
- S and U are marginals of X and Y (resp.) $\Rightarrow$ $\|\mathbf{Pr}_{\mathsf{S}} - \mathbf{Pr}_{\mathsf{U}}\| \le \|\mathbf{Pr}_{\mathsf{X}} - \mathbf{Pr}_{\mathsf{Y}}\|$

$U_i = j$

$$i = (j-1)q_{max} + k \quad \Rightarrow \quad i\text{-th query} = j\text{-th user's } k\text{-th query}$$

$$\mathcal{n}_i = \Big\{ (\mathsf{v}_1, \mathsf{v}_2) \mid$$

All previous samples of the j-th user

$$\mathsf{v}_1, \mathsf{v}_2, \mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2 \in \mathcal{G} \setminus \bigcup_{i=(j-1)q_{max}+1}^{(j-1)q_{max}+k-1} \{\mathsf{V}_{i,1}, \mathsf{V}_{i,2}, \mathsf{V}_{i,3}\},$$

$$\mathsf{U}_i + \mathsf{v}_1 + \mathsf{v}_2, \mathsf{v}_1, \mathsf{v}_2 \text{ distinct} \Big\}$$

$$(\mathsf{V}_{i,1}, \mathsf{V}_{i,2}) \leftarrow_{\$} \mathcal{n}_i$$

After Extension

- Support of X $=$ Support of Y
- S and U are marginals of X and Y (resp.) $\Rightarrow$ $\|\mathbf{Pr}_{\mathsf{S}} - \mathbf{Pr}_{\mathsf{U}}\| \leq \|\mathbf{Pr}_{\mathsf{X}} - \mathbf{Pr}_{\mathsf{Y}}\|$

Apply χ^2-method to upper bound $\|\mathbf{Pr}_{\mathsf{X}} - \mathbf{Pr}_{\mathsf{Y}}\|$

Setting

$\mathsf{X}_1, \ldots, \mathsf{X}_q \sim \mathbf{Pr}_{\mathsf{X}}$

$\mathsf{Y}_1, \ldots, \mathsf{Y}_q \sim \mathbf{Pr}_{\mathsf{Y}}$

$i = (j-1)q_{max} + k \Rightarrow$ i-th query $=$ j-th user's k-th query

$\mathsf{X}_i = \widehat{\mathsf{X}}_{j,k}$, $\widehat{\mathsf{X}}_j$ = block of X_i's for the j-th user

$\widehat{\mathsf{X}}_j^{k-1}$ = block of first $k-1$ X_i's for the j-th user.

Setting

$\mathsf{X}_1, \ldots, \mathsf{X}_q \sim \mathbf{Pr}_{\mathsf{X}}$

$\mathsf{Y}_1, \ldots, \mathsf{Y}_q \sim \mathbf{Pr}_{\mathsf{Y}}$

$i = (j-1)q_{max} + k \Rightarrow$ i-th query $= j$-th user's k-th query

$\mathsf{X}_i = \widehat{\mathsf{X}}_{j,k}$, $\widehat{\mathsf{X}}_j =$ block of X_i's for the j-th user

$\widehat{\mathsf{X}}_j^{k-1} =$ block of first $k-1$ X_i's for the j-th user.

Conditional Probabilities Under $\mathbf{Pr}_{\mathsf{X}}$ and $\mathbf{Pr}_{\mathsf{Y}}$

$$\begin{aligned}
\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i \mid \mathsf{x}^{i-1}) &\stackrel{\text{def}}{=} \mathbf{Pr}[\mathsf{X}_i = \mathsf{x}_i \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}, (\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j) = (\mathsf{x}^{i-1} \setminus \widehat{\mathsf{x}}_j^{k-1})] \\
&= \mathbf{Pr}[\widehat{\mathsf{X}}_{j,k} = \widehat{\mathsf{x}}_{j,k} \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}], \\
&\quad \text{since } (\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j) \text{ is independent of } \widehat{\mathsf{X}}_j^{k-1} \text{ and } \widehat{\mathsf{X}}_{j,k} \\
&= \frac{1}{(N - 3(k-1))^3}
\end{aligned}$$

Setting

$\mathsf{X}_1, \ldots, \mathsf{X}_q \sim \mathbf{Pr}_{\mathsf{X}}$

$\mathsf{Y}_1, \ldots, \mathsf{Y}_q \sim \mathbf{Pr}_{\mathsf{Y}}$

$i = (j-1)q_{max} + k \Rightarrow$ i-th query $=$ j-th user's k-th query

$\mathsf{X}_i = \widehat{\mathsf{X}}_{j,k}$, $\widehat{\mathsf{X}}_j =$ block of X_i's for the j-th user

$\widehat{\mathsf{X}}_j^{k-1} =$ block of first $k-1$ X_i's for the j-th user.

Conditional Probabilities Under $\mathbf{Pr}_{\mathsf{X}}$ and $\mathbf{Pr}_{\mathsf{Y}}$

$$\begin{aligned}\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i \mid \mathsf{x}^{i-1}) &\stackrel{\text{def}}{=} \mathbf{Pr}[\mathsf{X}_i = \mathsf{x}_i \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}, (\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j) = (\mathsf{x}^{i-1} \setminus \widehat{\mathsf{x}}_j^{k-1})] \\ &= \mathbf{Pr}[\widehat{\mathsf{X}}_{j,k} = \widehat{\mathsf{x}}_{j,k} \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}],\end{aligned}$$

since $(\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j)$ is independent of $\widehat{\mathsf{X}}_j^{k-1}$ and $\widehat{\mathsf{X}}_{j,k}$

Due to reordering.

$$= \frac{1}{(N - 3(k-1))^3}$$

Setting

$\mathsf{X}_1, \ldots, \mathsf{X}_q \sim \mathbf{Pr}_{\mathsf{X}}$ $\qquad$ $i = (j-1)q_{max} + k \Rightarrow$ i-th query $=$ j-th user's k-th query

$\mathsf{Y}_1, \ldots, \mathsf{Y}_q \sim \mathbf{Pr}_{\mathsf{Y}}$ $\qquad$ $\mathsf{X}_i = \widehat{\mathsf{X}}_{j,k}$, $\quad \widehat{\mathsf{X}}_j =$ block of X_i's for the j-th user

$\widehat{\mathsf{X}}_j^{k-1} =$ block of first $k-1$ X_i's for the j-th user.

Conditional Probabilities Under $\mathbf{Pr}_{\mathsf{X}}$ and $\mathbf{Pr}_{\mathsf{Y}}$

$$\begin{aligned}
\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i \mid \mathsf{x}^{i-1}) &\stackrel{\text{def}}{=} \mathbf{Pr}[\mathsf{X}_i = \mathsf{x}_i \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}, (\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j) = (\mathsf{x}^{i-1} \setminus \widehat{\mathsf{x}}_j^{k-1})] \\
&= \mathbf{Pr}[\widehat{\mathsf{X}}_{j,k} = \widehat{\mathsf{x}}_{j,k} \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}],
\end{aligned}$$

since $(\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j)$ is independent of $\widehat{\mathsf{X}}_j^{k-1}$ and $\widehat{\mathsf{X}}_{j,k}$

Due to reordering.

$$= \frac{1}{(N - 3(k-1))^{\underline{3}}}$$

Falling factorial

χ^2-Method: Steps

Setting

$\mathsf{X}_1, \ldots, \mathsf{X}_q \sim \mathbf{Pr}_{\mathsf{X}}$ $\qquad i = (j-1)q_{max} + k \Rightarrow$ i-th query $= j$-th user's k-th query

$\mathsf{Y}_1, \ldots, \mathsf{Y}_q \sim \mathbf{Pr}_{\mathsf{Y}}$ $\qquad \mathsf{X}_i = \widehat{\mathsf{X}}_{j,k}, \quad \widehat{\mathsf{X}}_j =$ block of X_i's for the j-th user

$\widehat{\mathsf{X}}_j^{k-1} =$ block of first $k-1$ X_i's for the j-th user.

Conditional Probabilities Under $\mathbf{Pr}_{\mathsf{X}}$ and $\mathbf{Pr}_{\mathsf{Y}}$

$$\begin{aligned}
\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i \mid \mathsf{x}^{i-1}) &\stackrel{\text{def}}{=} \mathbf{Pr}[\mathsf{X}_i = \mathsf{x}_i \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}, (\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j) = (\mathsf{x}^{i-1} \setminus \widehat{\mathsf{x}}_j^{k-1})] \\
&= \mathbf{Pr}[\widehat{\mathsf{X}}_{j,k} = \widehat{\mathsf{x}}_{j,k} \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}],
\end{aligned}$$

since $(\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j)$ is independent of $\widehat{\mathsf{X}}_j^{k-1}$ and $\widehat{\mathsf{X}}_{j,k}$

Due to reordering.

$$= \frac{1}{(N-3(k-1))^{\underline{3}}}$$

Falling factorial

Similarly for $\mathbf{Pr}_{\mathsf{Y}}$

$$\mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i \mid \mathsf{x}^{i-1}) = \frac{1}{N} \times \frac{1}{|\mathcal{n}^{\mathsf{u}_i}(\widehat{\mathsf{x}}_j^{k-1})|}$$

Setting

$\mathsf{X}_1, \ldots, \mathsf{X}_q \sim \mathbf{Pr}_{\mathsf{X}}$

$\mathsf{Y}_1, \ldots, \mathsf{Y}_q \sim \mathbf{Pr}_{\mathsf{Y}}$

$i = (j-1)q_{max} + k \Rightarrow$ i-th query $= j$-th user's k-th query

$\mathsf{X}_i = \widehat{\mathsf{X}}_{j,k}$, $\widehat{\mathsf{X}}_j =$ block of X_i's for the j-th user

$\widehat{\mathsf{X}}_j^{k-1} =$ block of first $k-1$ X_i's for the j-th user.

Conditional Probabilities Under $\mathbf{Pr}_{\mathsf{X}}$ and $\mathbf{Pr}_{\mathsf{Y}}$

$$\begin{aligned}
\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i \mid \mathsf{x}^{i-1}) &\stackrel{\text{def}}{=} \mathbf{Pr}[\mathsf{X}_i = \mathsf{x}_i \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}, (\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j) = (\mathsf{x}^{i-1} \setminus \widehat{\mathsf{x}}_j^{k-1})] \\
&= \mathbf{Pr}[\widehat{\mathsf{X}}_{j,k} = \widehat{\mathsf{x}}_{j,k} \mid \widehat{\mathsf{X}}_j^{k-1} = \widehat{\mathsf{x}}_j^{k-1}],
\end{aligned}$$

since $(\mathsf{X}^{i-1} \setminus \widehat{\mathsf{X}}_j)$ is independent of $\widehat{\mathsf{X}}_j^{k-1}$ and $\widehat{\mathsf{X}}_{j,k}$

Due to reordering.

$$= \frac{1}{(N - 3(k-1))_3}$$

Falling factorial

Similarly for $\mathbf{Pr}_{\mathsf{Y}}$

$$\mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i \mid \mathsf{x}^{i-1}) = \frac{1}{N} \times \frac{1}{|\eta^{\mathsf{u}_i}(\widehat{\mathsf{x}}_j^{k-1})|}$$

Same as η_i

χ^2 distance and its expectation

$$\chi^2(\mathsf{x}^{i-1}) := \sum_{\mathsf{x}_i=(\mathsf{v}_{i,1},\mathsf{v}_{i,2},\mathsf{u}_i)} \frac{(\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}) - \mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}))^2}{\mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1})} = C \times \sum_{\mathsf{u}_i\in\{0,1\}^n} (|n^{\mathsf{u}_i}(\widehat{\mathsf{x}}_j^{k-1})| - D)$$

χ^2 distance and its expectation

$C = \frac{N}{((N-3(k-1))\frac{3}{-})^2}$ $\quad$ $D = \frac{(N-3(k-1))\frac{3}{-}}{N}$

$$\chi^2(\mathsf{x}^{i-1}) := \sum_{\mathsf{x}_i=(\mathsf{v}_{i,1},\mathsf{v}_{i,2},\mathsf{u}_i)} \frac{(\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}) - \mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}))^2}{\mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1})} = C \times \sum_{\mathsf{u}_i\in\{0,1\}^n} (|n^{\mathsf{u}_i}(\widehat{\mathsf{x}}_j^{k-1})| - D)$$

χ^2 distance and its expectation

$C = \frac{N}{((N-3(k-1))\frac{3}{})^2}$ $D = \frac{(N-3(k-1))\frac{3}{}}{N}$

$$\chi^2(\mathsf{x}^{i-1}) := \sum_{\mathsf{x}_i=(\mathsf{v}_{i,1},\mathsf{v}_{i,2},\mathsf{u}_i)} \frac{(\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}) - \mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}))^2}{\mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1})} = C \times \sum_{\mathsf{u}_i\in\{0,1\}^n} (|n^{\mathsf{u}_i}(\widehat{\mathsf{x}}_j^{k-1})| - D)$$

$$\Downarrow$$

$$\mathbf{Ex}[\chi^2(\mathsf{X}^{i-1})] \;=\; C \times \sum_{\mathsf{u}_i} \mathbf{Ex}[(|n^{\mathsf{u}_i}(\widehat{\mathsf{X}}_j^{k-1})| - D)^2].$$

χ^2 distance and its expectation

$C = \frac{N}{((N-3(k-1))\frac{3}{})^2}$ $\quad$ $D = \frac{(N-3(k-1))\frac{3}{}}{N}$

$$\chi^2(\mathsf{x}^{i-1}) := \sum_{\mathsf{x}_i=(\mathsf{v}_{i,1},\mathsf{v}_{i,2},\mathsf{u}_i)} \frac{(\mathbf{Pr}_{\mathsf{X}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}) - \mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1}))^2}{\mathbf{Pr}_{\mathsf{Y}}(\mathsf{x}_i|\widehat{\mathsf{x}}_j^{k-1})} = C \times \sum_{\mathsf{u}_i\in\{0,1\}^n} (|n^{\mathsf{u}_i}(\widehat{\mathsf{x}}_j^{k-1})| - D)$$

$$\Downarrow$$

$$\mathbf{Ex}[\chi^2(\mathsf{X}^{i-1})] = C \times \sum_{\mathsf{u}_i} \mathbf{Ex}[(|n^{\mathsf{u}_i}(\widehat{\mathsf{X}}_j^{k-1})| - D)^2].$$

Goal is to compute $\mathbf{Ex}[(|n^{\mathsf{u}_i}(\widehat{\mathsf{X}}_j^{k-1})| - D)^2]$

Finishing the Proof: An Important Lemma

Setting: $\mathcal{G} := \mathbb{F}_{2^n}$ $\quad |\mathcal{G}| = N$ $\quad \mathcal{V}_r$: a random r-set in $\mathcal{G}$

For $\mathsf{u} \in \mathcal{G}$

$$\{(\mathsf{g}_1, \mathsf{g}_2) \in \mathcal{V}_r \times \mathcal{V}_r : \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2 \in \mathcal{V}_r; \mathsf{u}, \mathsf{g}_1, \mathsf{g}_2 \text{ distinct}\}$$

Finishing the Proof: An Important Lemma

Setting: $\mathcal{G} := \mathbb{F}_{2^n}$ $|\mathcal{G}| = N$ $\mathcal{V}_r$: a random r-set in $\mathcal{G}$

For $\mathsf{u} \in \mathcal{G}$

$$\{(\mathsf{g}_1, \mathsf{g}_2) \in \mathcal{V}_r \times \mathcal{V}_r : \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2 \in \mathcal{V}_r; \mathsf{u}, \mathsf{g}_1, \mathsf{g}_2 \text{ distinct}\}|$$

Finishing the Proof: An Important Lemma

Setting: $\mathcal{G} := \mathbb{F}_{2^n}$ $\quad |\mathcal{G}| = N$ $\quad \mathcal{V}_r$: a random r-set in $\mathcal{G}$

For $\mathsf{u} \in \mathcal{G}$

Same as $\mathcal{N}^{\mathsf{u}_i}(\widehat{\mathsf{X}}_j^{k-1})$

$$\{(\mathsf{g}_1, \mathsf{g}_2) \in \mathcal{V}_r \times \mathcal{V}_r : \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2 \in \mathcal{V}_r; \mathsf{u}, \mathsf{g}_1, \mathsf{g}_2 \text{ distinct}\}|$$

Finishing the Proof: An Important Lemma

Setting: $\mathcal{G} := \mathbb{F}_{2^n}$ $\qquad |\mathcal{G}| = N$ $\qquad \mathcal{V}_r$: a random r-set in $\mathcal{G}$

For $\mathsf{u} \in \mathcal{G}$

$$\mathbf{N}_r^{\mathsf{u}} := |\{(\mathsf{g}_1, \mathsf{g}_2) \in \mathcal{V}_r \times \mathcal{V}_r : \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2 \in \mathcal{V}_r; \mathsf{u}, \mathsf{g}_1, \mathsf{g}_2 \text{ distinct}\}|$$

Finishing the Proof: An Important Lemma

Setting: $\mathcal{G} := \mathbb{F}_{2^n}$ $|\mathcal{G}| = N$ $\mathcal{V}_r$: a random r-set in $\mathcal{G}$

For $\mathsf{u} \in \mathcal{G}$

$$\mathbf{N}_r^{\mathsf{u}} := |\{(\mathsf{g}_1, \mathsf{g}_2) \in \mathcal{V}_r \times \mathcal{V}_r : \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2 \in \mathcal{V}_r; \mathsf{u}, \mathsf{g}_1, \mathsf{g}_2 \text{ distinct}\}|$$

To compute $\mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - D)^2]$

Setting: $\mathcal{G} := \mathbb{F}_{2^n}$ $\quad |\mathcal{G}| = N$ $\quad \mathcal{V}_r$: a random r-set in $\mathcal{G}$

For $\mathsf{u} \in \mathcal{G}$

$$\mathbf{N}_r^{\mathsf{u}} := |\{(\mathsf{g}_1, \mathsf{g}_2) \in \mathcal{V}_r \times \mathcal{V}_r : \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2 \in \mathcal{V}_r; \mathsf{u}, \mathsf{g}_1, \mathsf{g}_2 \text{ distinct}\}|$$

To compute $\mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - D)^2]$

Computation of $\mathbf{Ex}[\mathbf{N}_r^{\mathsf{u}}]$:

$$\mathcal{G}_{\mathsf{u}} = \{\mathsf{g} := (\mathsf{g}_1, \mathsf{g}_2) | \mathsf{g}_1 \neq \mathsf{g}_2 \in \mathcal{G} \setminus \{\mathsf{u}\}\}.$$

$$\mathrm{I}_{\mathsf{g}} = \begin{cases} 1 & \text{if } \mathsf{g}_1, \mathsf{g}_2, \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2 \in \mathcal{V}_r, \text{ and } \mathsf{g}_1 \neq \mathsf{u} \neq \mathsf{g}_2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{Ex}[\mathbf{N}_r^{\mathsf{u}}] = \mathbf{Ex}[\sum_{\mathsf{g} \in \mathcal{G}_{\mathsf{u}}} \mathrm{I}_{\mathsf{g}}] = \sum_{\mathsf{g} \in \mathcal{G}_{\mathsf{u}}} \mathbf{Ex}[\mathrm{I}_{\mathsf{g}}] = \mathbf{Pr}[\{\mathsf{g}_1, \mathsf{g}_2, \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2\} \subseteq \mathcal{V}_r] = \frac{r^{\underline{3}}}{N} = D.$$

From expectation to variance:

$$\mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - D)^2] = \mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - \mathbf{Ex}[\mathbf{N}_r^{\mathsf{u}}])^2] = \mathbf{Var}[\mathbf{N}_r^{\mathsf{u}}]$$

From expectation to variance:

$$\mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - D)^2] = \mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - \mathbf{Ex}[\mathbf{N}_r^{\mathsf{u}}])^2] = \mathbf{Var}[\mathbf{N}_r^{\mathsf{u}}]$$

To compute $\mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - D)^2] \quad \Rightarrow \quad$ To compute $\mathbf{Var}[\mathbf{N}_r^{\mathsf{u}}]$

How to compute $\mathbf{Var}[\mathbf{N}_r^{\mathsf{u}}]$?

From expectation to variance:

$$\mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - D)^2] = \mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - \mathbf{Ex}[\mathbf{N}_r^{\mathsf{u}}])^2] = \mathbf{Var}[\mathbf{N}_r^{\mathsf{u}}]$$

To compute $\mathbf{Ex}[(\mathbf{N}_r^{\mathsf{u}} - D)^2] \quad \Rightarrow \quad$ To compute $\mathbf{Var}[\mathbf{N}_r^{\mathsf{u}}]$

How to compute $\mathbf{Var}[\mathbf{N}_r^{\mathsf{u}}]$?

Setting:

$$\text{For } \mathsf{g} = (\mathsf{g}_1, \mathsf{g}_2), \qquad \mathcal{S}_{\mathsf{u}}^{\mathsf{g}} = \{\mathsf{g}_1, \mathsf{g}_2, \mathsf{u} + \mathsf{g}_1 + \mathsf{g}_2\}$$

Observation:

$$w = |\mathcal{S}_{\mathsf{u}}^{\mathsf{g}} \cup \mathcal{S}_{\mathsf{u}}^{\mathsf{g}'}| \in \{3, 5, 6\}$$

Technicalities:

$$\mathbf{Var}[\mathrm{N}_r^{\mathsf{u}}] = \sum_{\mathsf{g}\in\mathcal{G}_{\mathsf{u}}} \mathbf{Var}[\mathrm{I}_{\mathsf{g}}] + \sum_{\mathsf{g}\neq\mathsf{g}'\in\mathcal{G}_{\mathsf{u}}} \mathbf{Cov}(\mathrm{I}_{\mathsf{g}}, \mathrm{I}_{\mathsf{g}'})$$

$$\mathbf{Var}[\mathrm{I}_{\mathsf{g}}] = \mathbf{Ex}[\mathrm{I}_{\mathsf{g}}](1 - \mathbf{Ex}[\mathrm{I}_{\mathsf{g}}])$$

$$\mathbf{Cov}(\mathrm{I}_{\mathsf{g}}, \mathrm{I}_{\mathsf{g}'}) = \mathbf{Ex}[\mathrm{I}_{\mathsf{g}}\mathrm{I}_{\mathsf{g}'}] - \mathbf{Ex}[\mathrm{I}_{\mathsf{g}}]\mathbf{Ex}[\mathrm{I}_{\mathsf{g}'}]$$

$$\boxed{\leq \frac{6r^2(r-1)(r-2)}{N^2}}$$

$w = 3$

$$\boxed{\leq \frac{30r^2(r-1)(r-2)}{N^2}}$$

$w \in \{5, 6\}$

$$\boxed{\leq \frac{4^8(r')^3}{27CN^6}}$$

$$+$$

$$\mathbf{Var}[\mathrm{N}_r^{\mathsf{u}}] \leq \frac{576}{N^3} + \frac{4^8(r')^3}{27CN^6}$$

$$\Downarrow$$

$$\mathbf{Ex}[\chi^2(\mathsf{X}^{i-1})] \leq \frac{576}{N^2} + \frac{4^8(r')^3}{27CN^5} \Rightarrow \|\mathbf{Pr}_{\mathsf{P}} - \mathbf{Pr}_{\mathsf{R}}\| \leq \frac{20\sqrt{uq_{max}}}{N}$$

- Strong PRF security of XORP[3]
  - **Multi-user:**
    - Can be used simultaneously and independently by $O(2^n)$ users
    - Adversary can make $O(2^n)$ queries per user
  - **Single-user:** Adversary's advantage is $O(\frac{1}{\sqrt{2^n}})$ even after $O(2^n)$ queries

- Strong PRF security of XORP$[3]$
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- Strong PRF security of XORP$'[3]$
  - **Multi-user:** Same level of security. Analysis (not shown) similar to XORP$[3]$
  - **Single-user:** Adversary's advantage is $O(\frac{1}{\sqrt{2^n}})$ even after $O(2^n)$ queries

Thank You!

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