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Lower bounds on lattice sieving and information set decoding

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  - Parameter selection: Conditional security guarantees.

Hash-based model

- **Closest pairs problem**:
  Let (M, d) be a bounded metric space, and let $r \geq 0$ be a given target distance.
  Let $L \subset M$ be a subset of M, with elements drawn uniformly at random from M.
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- **Locality-sensitive hash functions**: Functions h satisfying:

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  - **Nearest neighbor literature**: Various (M, d), focus on $|L| = 2^{o(d)}$.

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- Usually $h^{-1}(n)$ has similar shapes for all n.

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- Usually $h^{-1}(n)$ has similar shapes for all n.
- **Problem**: For fixed $\sigma(A)$, find $A \subset \mathcal{S}^{d-1}$ which maximizes:

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Lemma (Baernstein–Taylor inequality for $\mathcal{S}^{d-1}$ [BT76])

Let $f, g : \mathcal{S}^{d-1} \to \mathbb{R}$ *be arbitrary Lebesgue-integrable functions. Let* $h : [-1, 1] \to \mathbb{R}$ *be non-decreasing, bounded, and measurable. Let* $f^*, g^* : \mathcal{S}^{d-1} \to \mathbb{R}$ *be the symmetric non-decreasing rearrangements of* f, g*. Then:*

$$\iint_{\mathcal{S}^{d-1} \times \mathcal{S}^{d-1}} f(\mathbf{x}) g(\mathbf{y}) h(\mathbf{x} \cdot \mathbf{y}) \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y}) \leq \iint_{\mathcal{S}^{d-1} \times \mathcal{S}^{d-1}} f^*(\mathbf{x}) g^*(\mathbf{y}) h(\mathbf{x} \cdot \mathbf{y}) \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y}).$$

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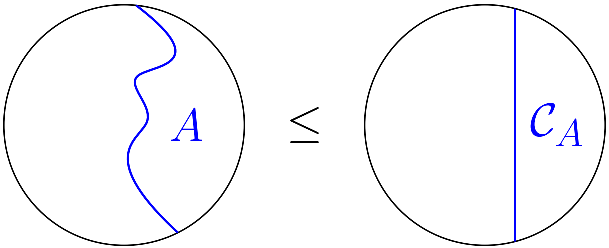

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- **Solution**: Performance maximized for spherical caps!

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- **Tuple sieving**: results from [HKL18] are conditionally optimal.

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Thank you for watching!