The t-wise Independence of Substitution-Permutation Networks

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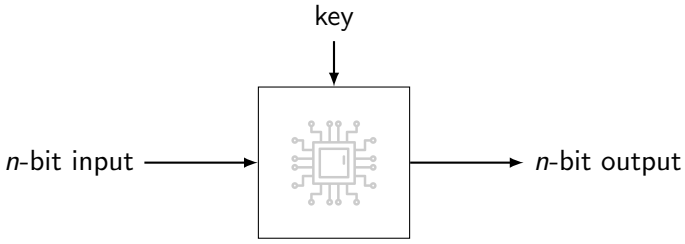
[1]University of Washington, Seattle

[2]MIT, Cambridge

CRYPTO 2021

Random-looking Keyed Permutation

indistinguishable from a random permutation



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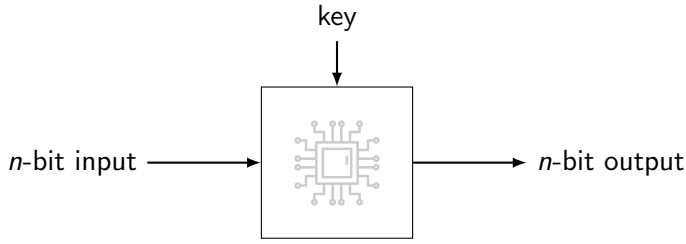

theory

Pseudorandom Permutation

Provable security
based on hardness assumptions

practice

Block Cipher

Heuristic security
resisting known attacks

theory

Pseudorandom Permutation

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provable security based on . . .

Feistel [LR88] plus
- one-way functions [GGM84]
- factoring [NR04,. . .]
- lattice problems [BPR12,. . .]

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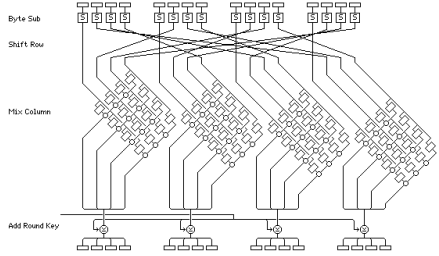
very efficient ciphers (e.g. AES)



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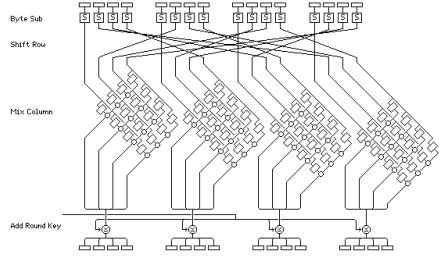

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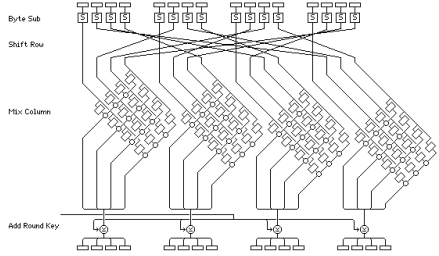

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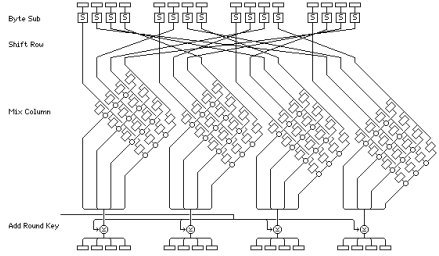

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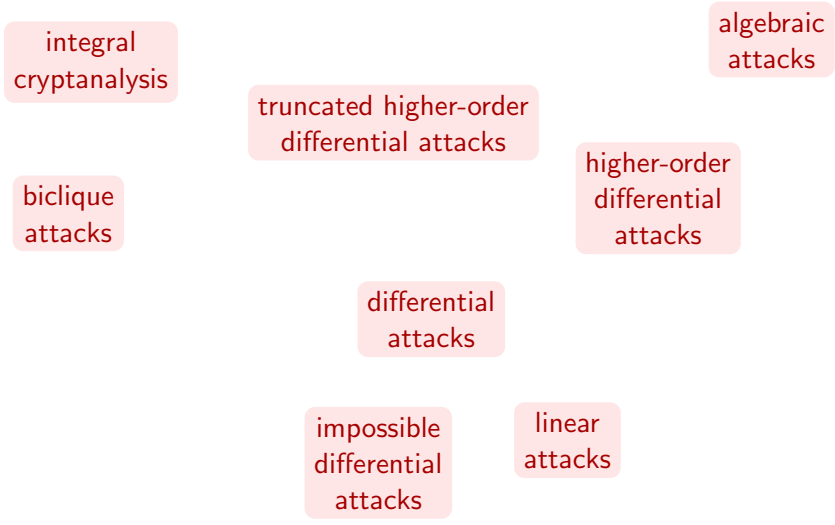
Provable bounds

on the advantage of known attacks

NK95, KMT01, PSC+02, PSLL03, Kel04, KS07, etc

Prove bounds against an attack class



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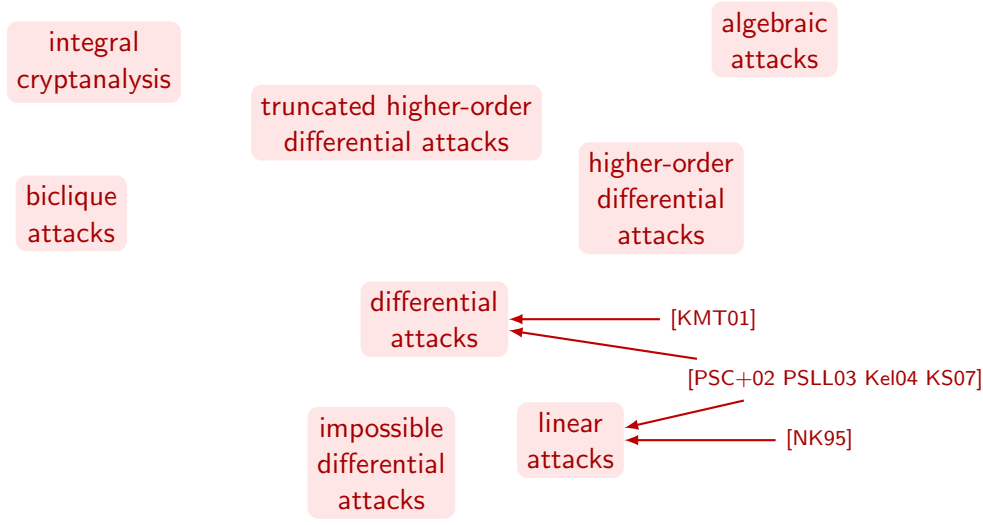

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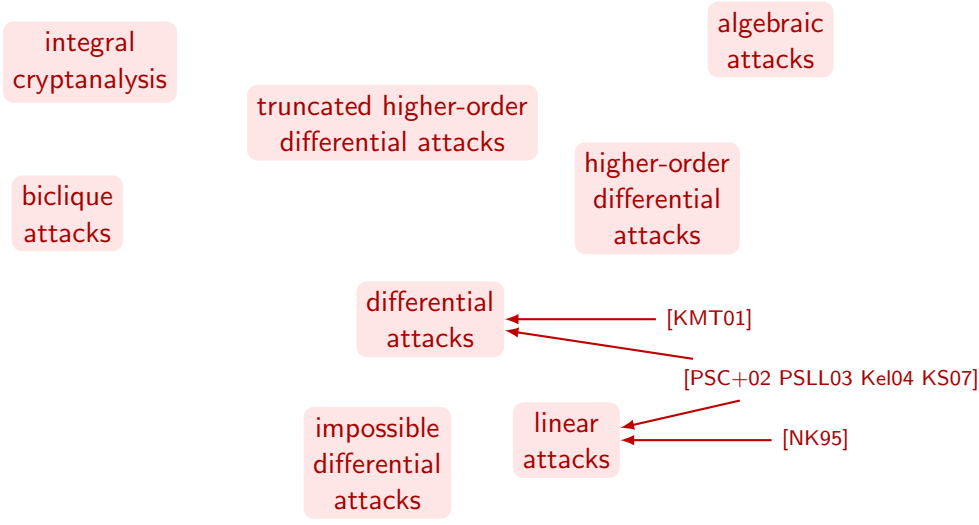

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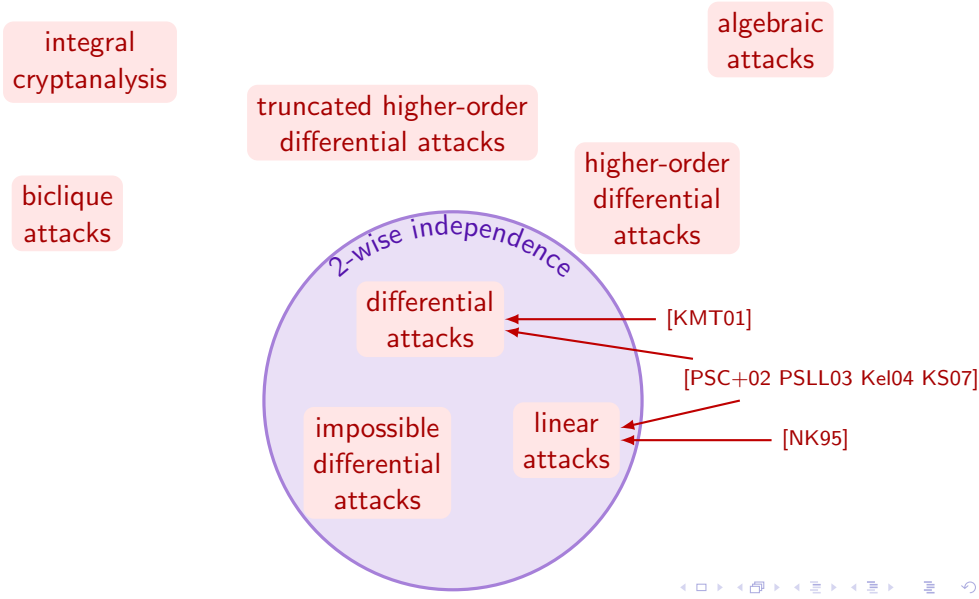

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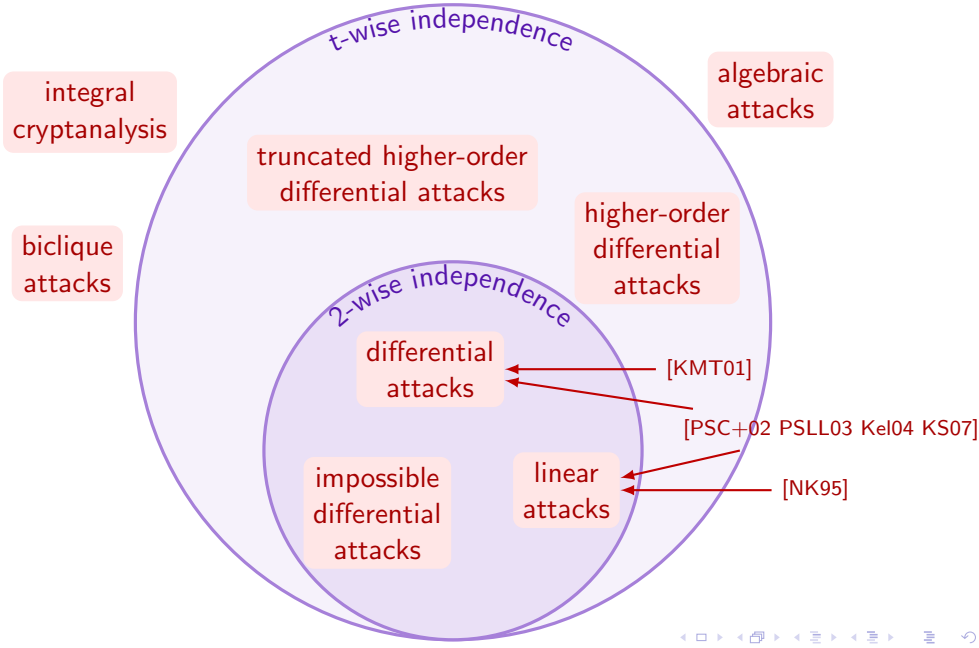
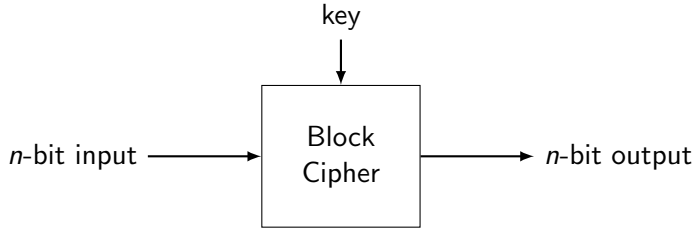
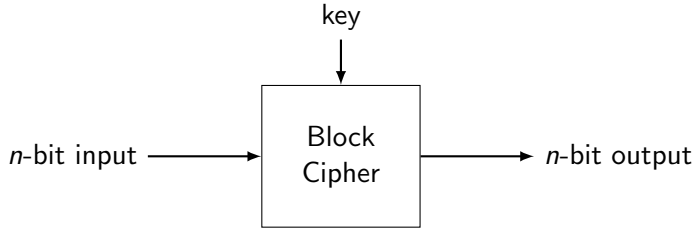




t-wise Independence

$\forall \text{input}_1, \ldots, \text{input}_t$
$\text{output}_1, \ldots, \text{output}_t$ are i.i.d. uniform

used in [HMMR05, KNR05, BH08, AL13]



ε-close to t-wise Independence

$$\forall \text{input}_1, \ldots, \text{input}_t$$
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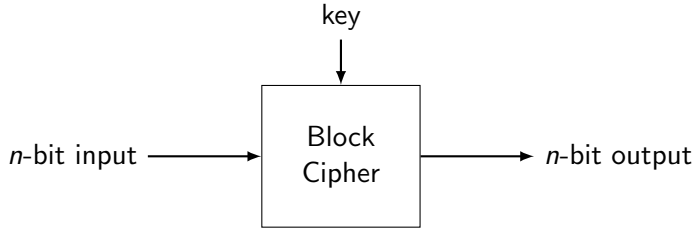

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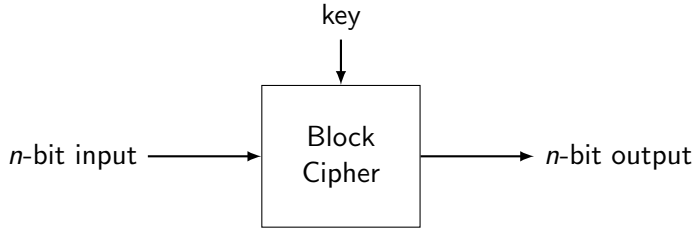

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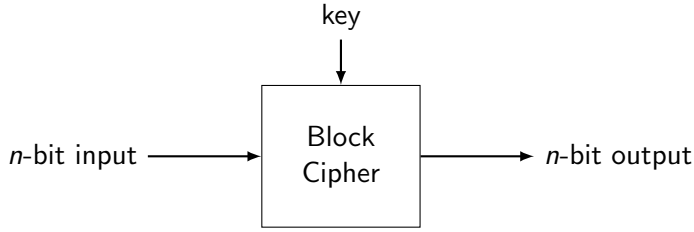

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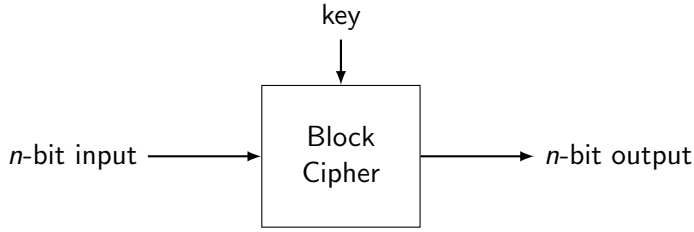

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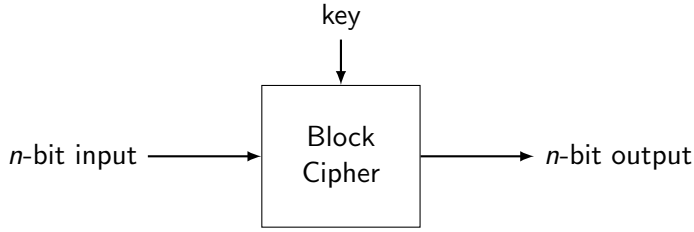

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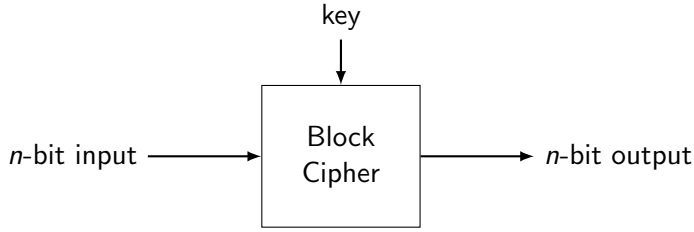
- 2^d non-adaptive queries **order-d differential attacks**



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$$\varepsilon\text{-close to 2-wise indp} \Longrightarrow \begin{cases} \text{MEDP} \leq \varepsilon + \frac{1}{2^{n-1}} & \text{(differential attack)} \\ \text{CORR} \leq 8\varepsilon + \frac{4}{2^n} & \text{(linear attack)} \end{cases}$$

Key-Alternating Cipher (KAC)

Substitution-Permutation Network (SPN)

Advanced Encryption Standard (AES)

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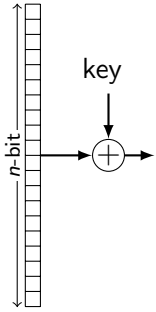

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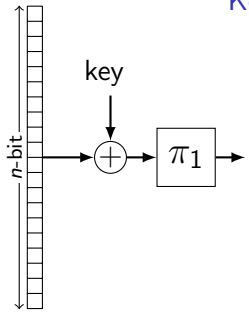

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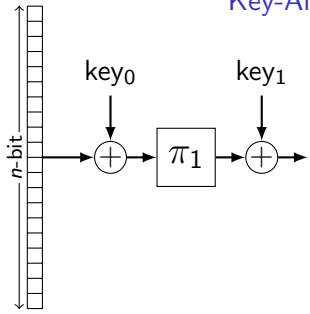

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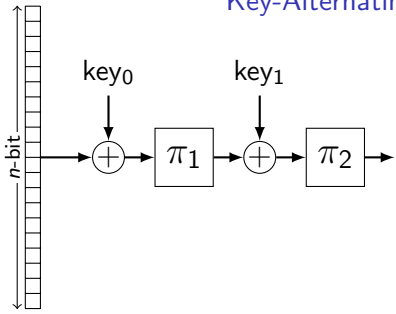

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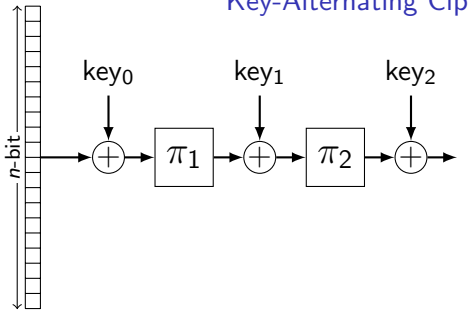

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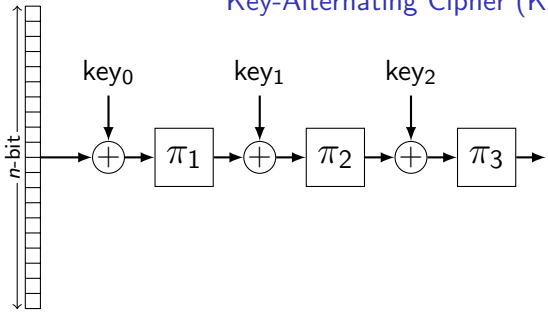

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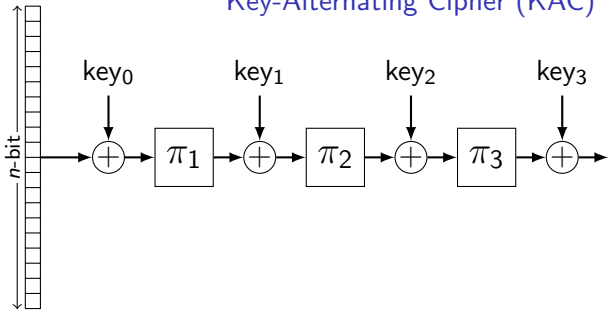

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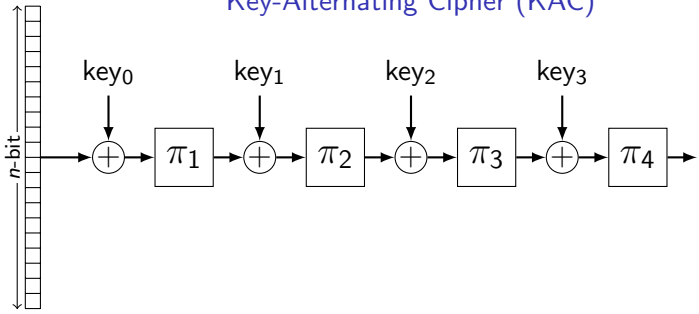

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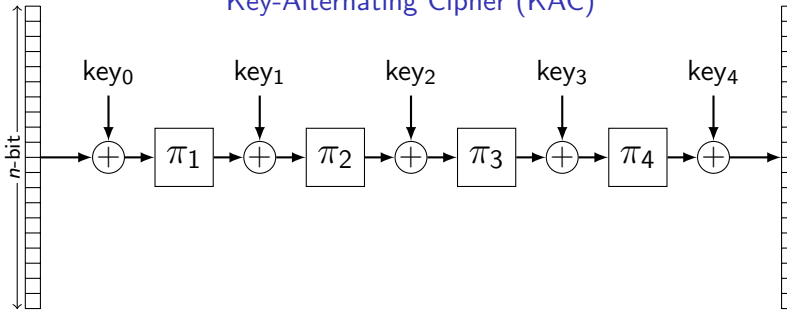

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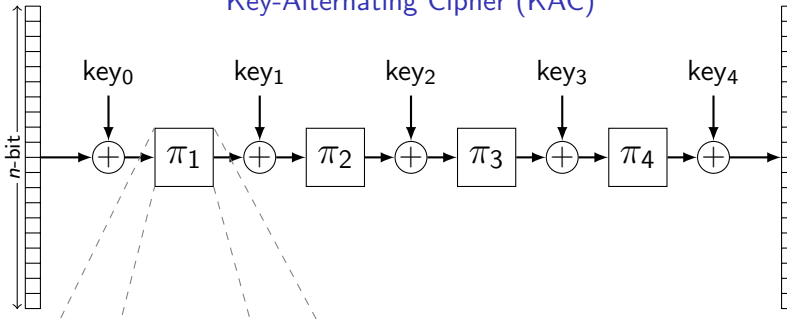

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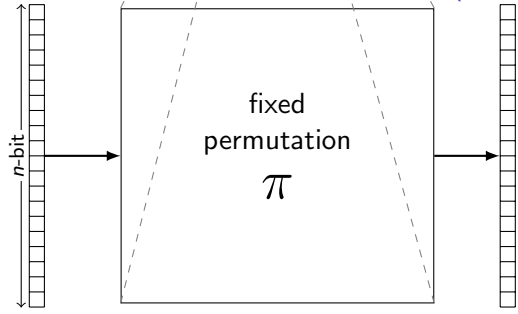

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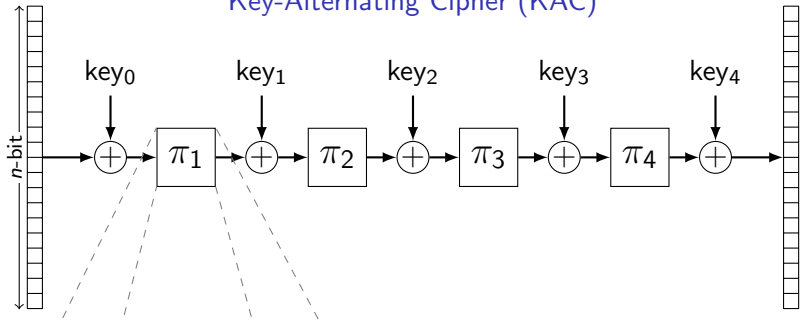

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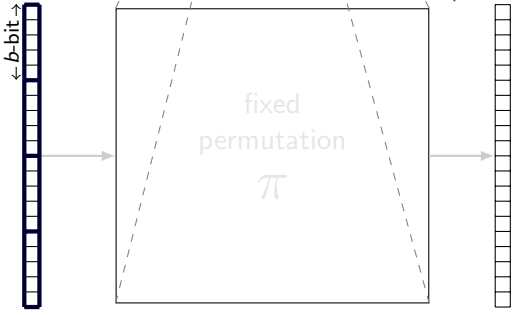

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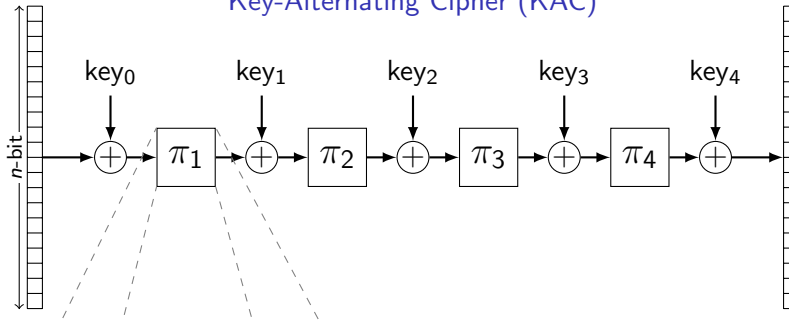

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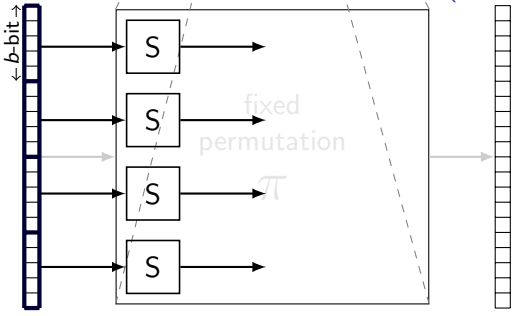

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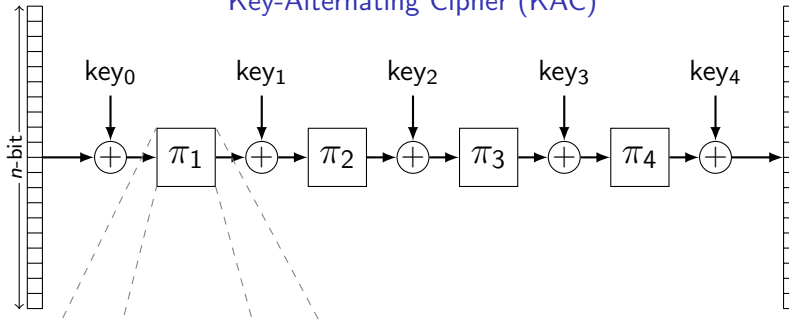

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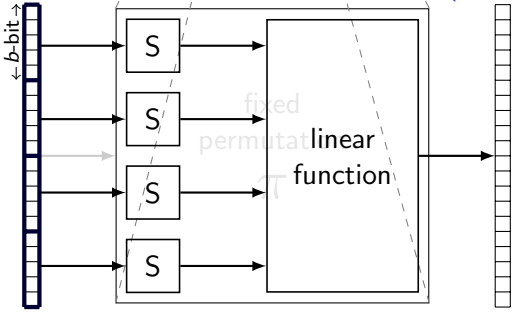

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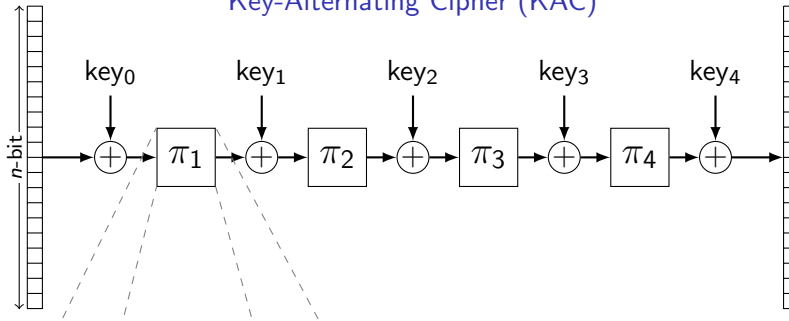

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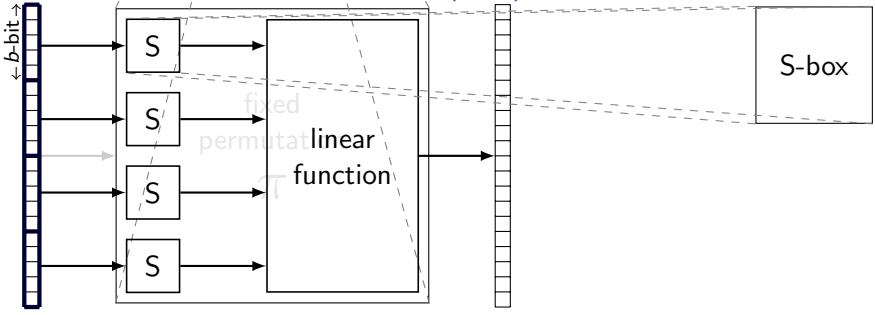

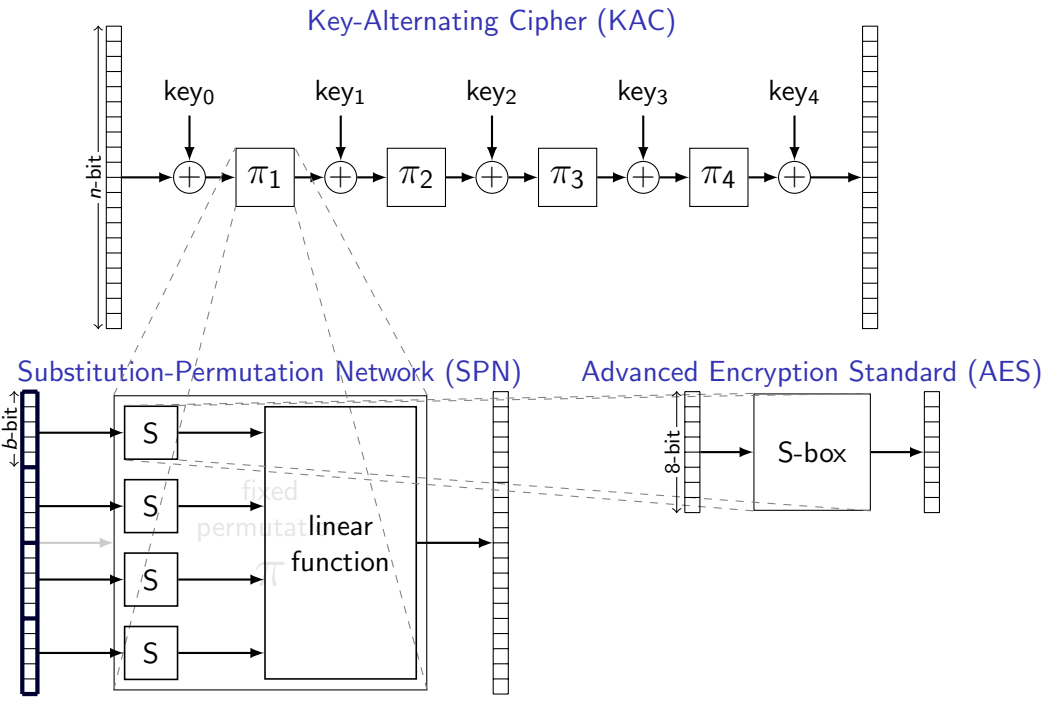
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key1
key2
key3
key4
n-bit
π1
π2
π3
π4
Substitution-Permutation Network (SPN)
Advanced Encryption Standard (AES)
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S
S
S
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fixed
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function
8-bit
S-box

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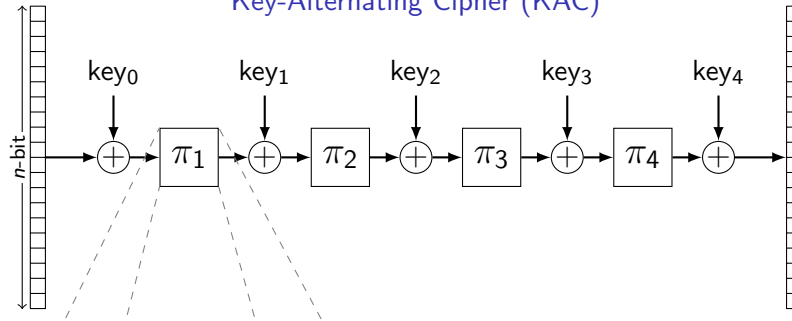
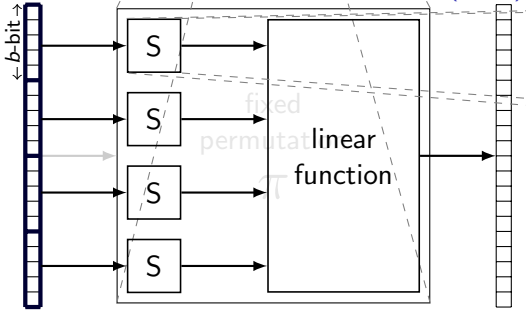
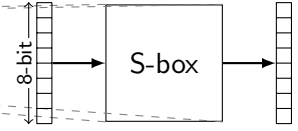


Substitution-Permutation Network (SPN)



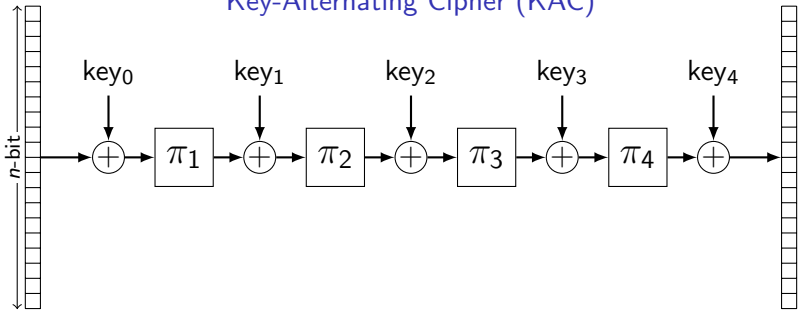
Advanced Encryption Standard (AES)



$$S(x) = \begin{cases} x^{-1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

*ignoring linear transformation

Key-Alternating Cipher (KAC)



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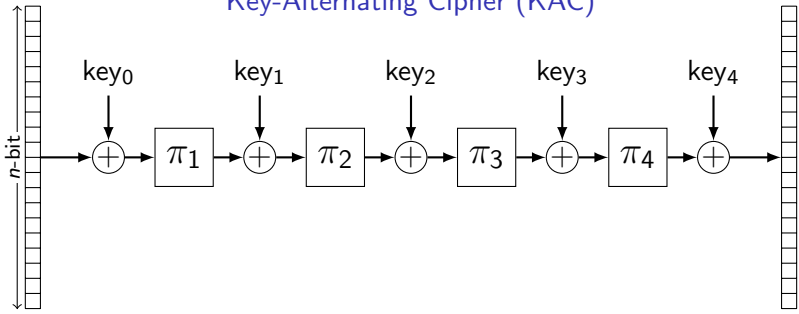
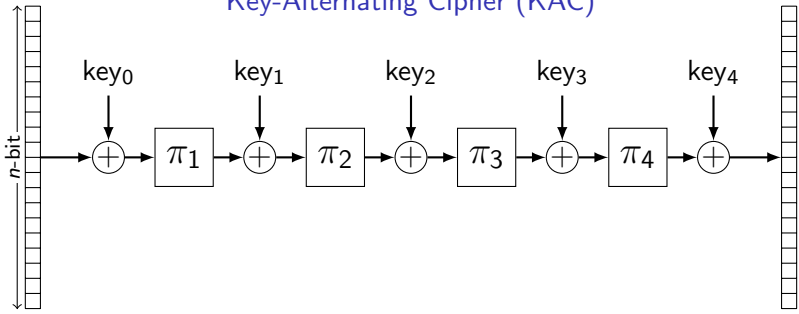


r-round KAC$(\pi_1, \ldots, \pi_r)$ is not $(r+2)$-wise independent

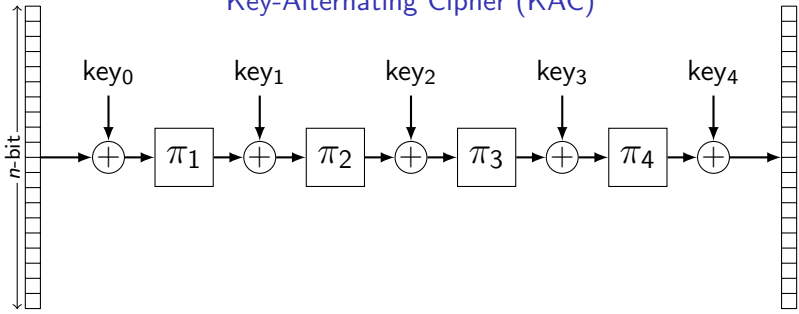
Key-Alternating Cipher (KAC)



Our Results (KAC)

r-round KAC$(\pi_1, \ldots, \pi_r)$ is close to
$(r - o(r))$-wise independent
for most $\pi_1, \ldots, \pi_r$

Key-Alternating Cipher (KAC)



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*existential result & probabilistic method

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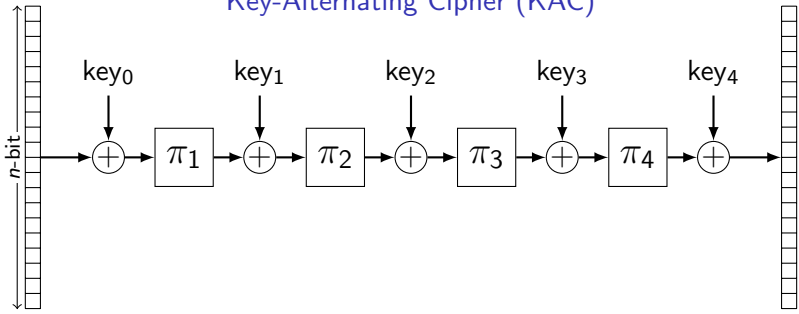

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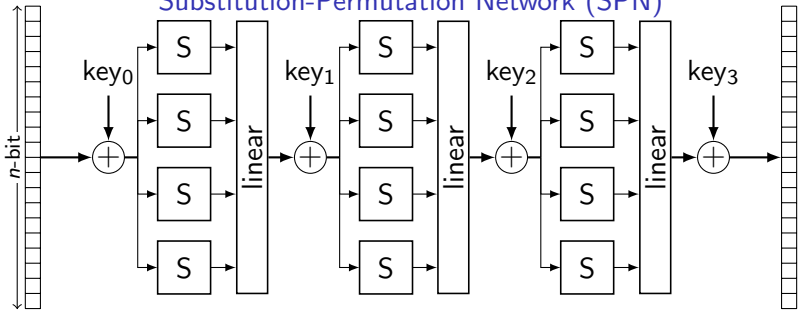
*existential result & probabilistic method

*unlike ideal model results, $\pi_1, \ldots, \pi_r$ are completely known to adv

Substitution-Permutation Network (SPN)



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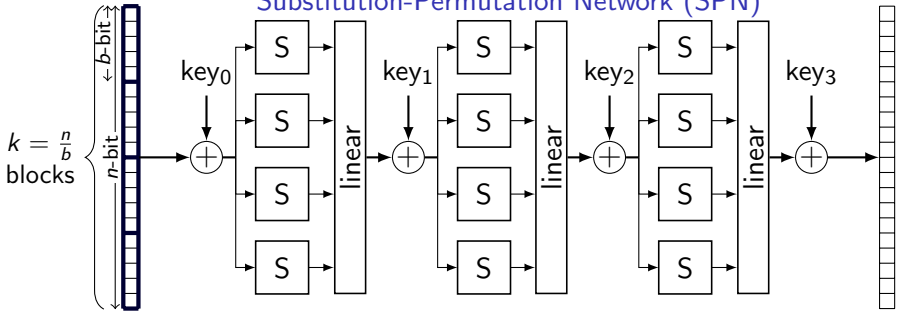

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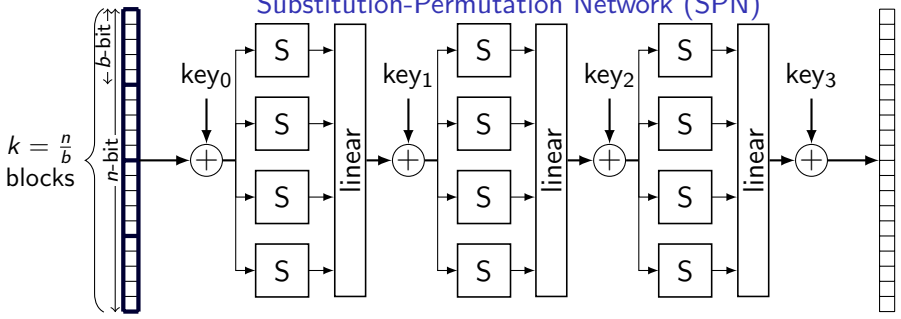
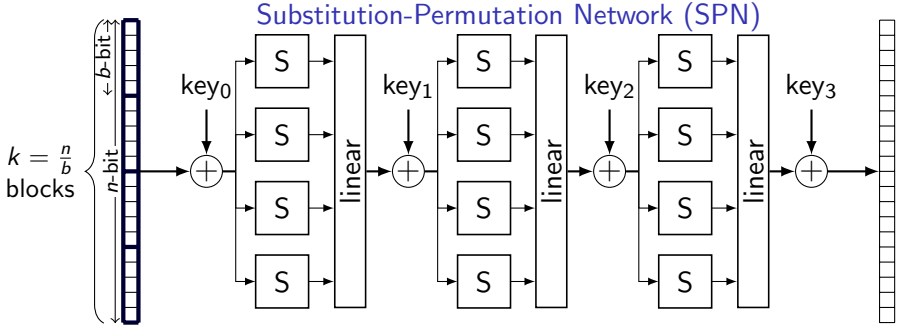


$S(x) = x^{-1}$ (used by AES) *or* $S(x) = x^3$ (used by MiMC)



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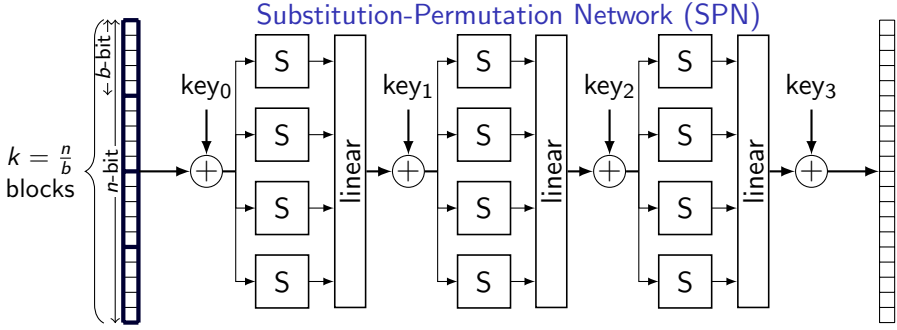
Our Results

2-round SPN is $(\frac{4k}{2^b} + \sqrt{\frac{2^k}{2^b}})$-close to 2-wise independent.

3-round SPN is $(\frac{8k}{2^b} + \sqrt{\frac{k}{2^b}})$-close to 2-wise independent.



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6-round AES is 0.472-close to 2-wise independent.

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State of the art [Park-Sung-Lee-Lim 03]

4-round AES is pointwise 2^{17}-close to 2-wise independent.

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State of the art [Park-Sung-Lee-Lim 03]

4-round AES is pointwise 2^{17}-close to 2-wise independent.

def pointwise ε-close to uniform

$$1 - \varepsilon \leq \frac{\Pr[X \leftarrow \text{distribution}; X = v]}{\Pr[X \leftarrow \text{uniform}; X = v]} \leq 1 + \varepsilon$$

Our Results (SPN & AES)

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MPR Amplification Lemma [Maurer-Pietrzak-Renner 07]

$\left.\begin{array}{l} \mathcal{F} \text{ is } \varepsilon\text{-close to 2-wise indp.} \\ \mathcal{G} \text{ is } \delta\text{-close to 2-wise indp.} \end{array}\right\} \Longrightarrow \mathcal{F} \circ \mathcal{G}$ is $2\varepsilon\delta$-close to 2-wise indp.

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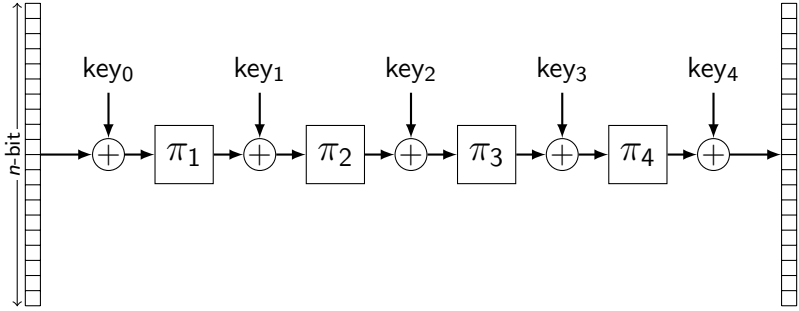
Amplifying Our Results

$6r$-round AES is $(2^{r-1}0.472^r)$-close to 2-wise independent.

Proof Overview (KAC)



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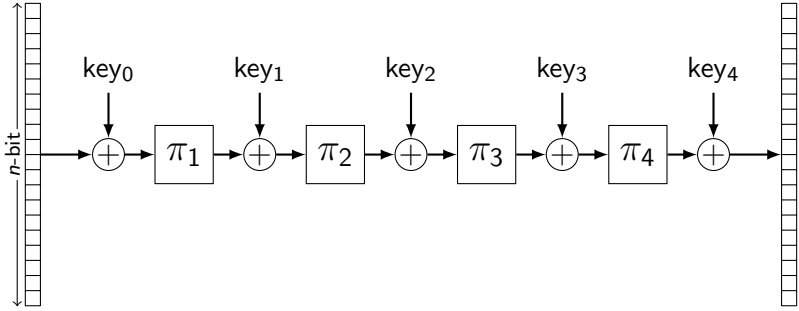
Proof Overview (KAC)



Our Results (KAC)

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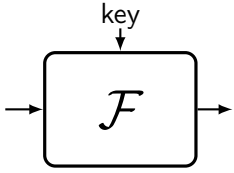
Prove by induction

Proof Overview (KAC)

$\mathcal{F}$ is t-wise indp.



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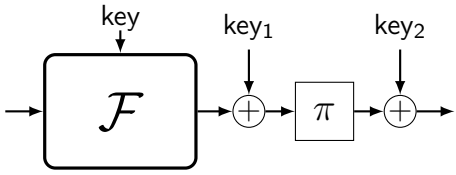
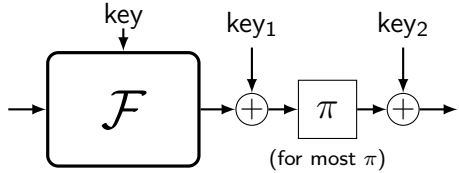


Proof Overview (KAC)

$\mathcal{F}$ is t-wise indp. $\Longrightarrow$

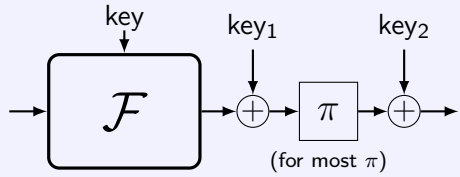


is pointwise $O(t^2 n)$-close to $(t+1)$-wise indp.

Proof Overview (KAC)

Independence Amplification Lemma

$\mathcal{F}$ is t-wise indp. $\Longrightarrow$



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*existential result & probabilistic method on π

Proof Overview (KAC)

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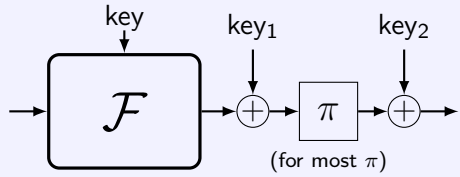

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pointwise ε-close to t-wise independence

$\forall \text{input}_1, \ldots, \text{input}_t, \text{output}_1, \ldots, \text{output}_t$

$$\frac{1-\varepsilon}{2^{tn}} \leq \Pr[\text{output}_1, \ldots, \text{output}_t] \leq \frac{1+\varepsilon}{2^{tn}}$$

Proof Overview (KAC)

Independence Amplification Lemma

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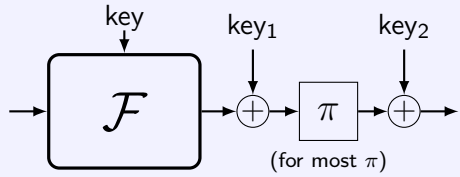

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*meaningful even if $\varepsilon \gg 1$

Proof Overview (KAC)

Independence Amplification Lemma

$\mathcal{F}$ is pointwise
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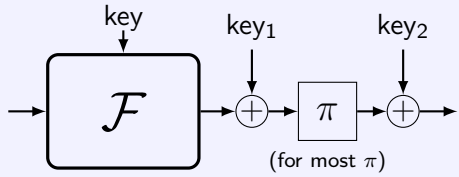
$\Longrightarrow$



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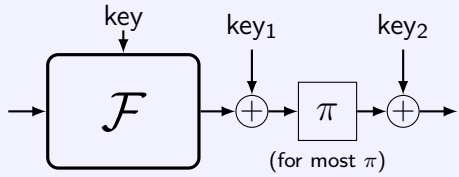

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0-round KAC (=one-time pad) is 1-wise indp.

Proof Overview (KAC)

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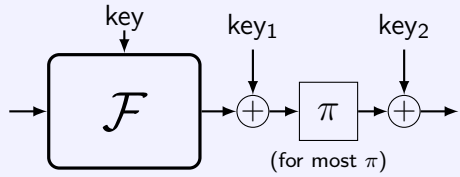

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$\Downarrow$

1-round KAC is pointwise $O(n)$-close to 2-wise indp.

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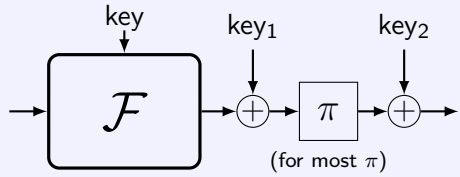
$\Longrightarrow$



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$\Downarrow$

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$\Downarrow$

2-round KAC is pointwise $O(n^2)$-close to 3-wise indp.

Proof Overview (KAC)

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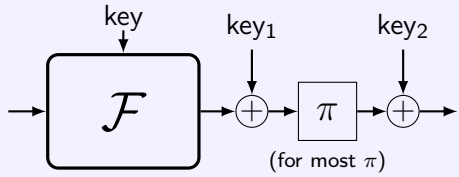

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$\Downarrow$

1-round KAC is pointwise $O(n)$-close to 2-wise indp.

$\Downarrow$

2-round KAC is pointwise $O(n^2)$-close to 3-wise indp.

$\Downarrow$

r-round KAC is pointwise $n^r r^{O(r)}$-close to $(r+1)$-wise indp.

Proof Overview (KAC)

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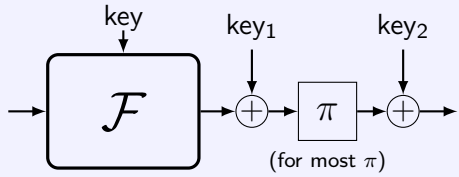
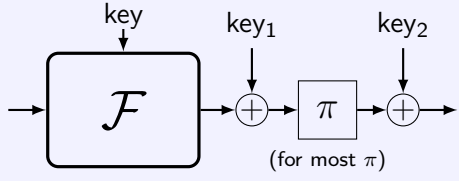


is pointwise $O((1+\varepsilon)t^2 n)$-close to $(t+1)$-wise indp.

Distance Amplification Lemma

$\mathcal{F}$ is pointwise **very** close to t-wise indp. & pointwise **somewhat** close to $(t+1)$-wise indp. $\Longrightarrow$



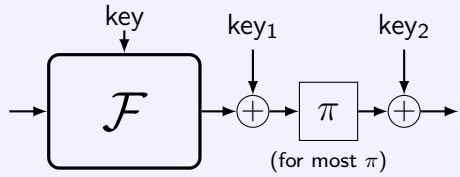
is pointwise **very** close to $(t+1)$-wise indp.

Proof Overview (KAC)

Independence Amplification Lemma

$\mathcal{F}$ is pointwise ε-close to t-wise indp.

$\Longrightarrow$



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Distance Amplification Lemma

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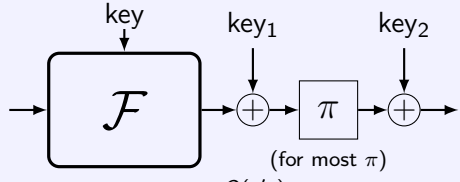
$\Longrightarrow$



is pointwise $(\varepsilon + \frac{O(\varepsilon' t)}{2^{n/3}})$-close to $(t+1)$-wise indp.

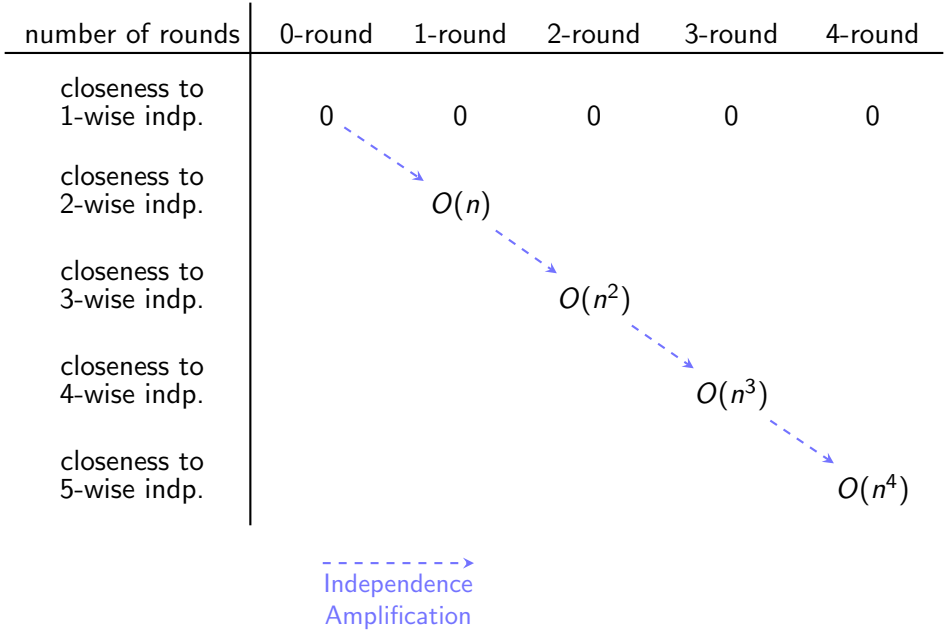
Proof Overview (KAC)

number of rounds	0-round	1-round	2-round	3-round	4-round
closeness to 1-wise indp.					
closeness to 2-wise indp.					
closeness to 3-wise indp.					
closeness to 4-wise indp.					
closeness to 5-wise indp.					

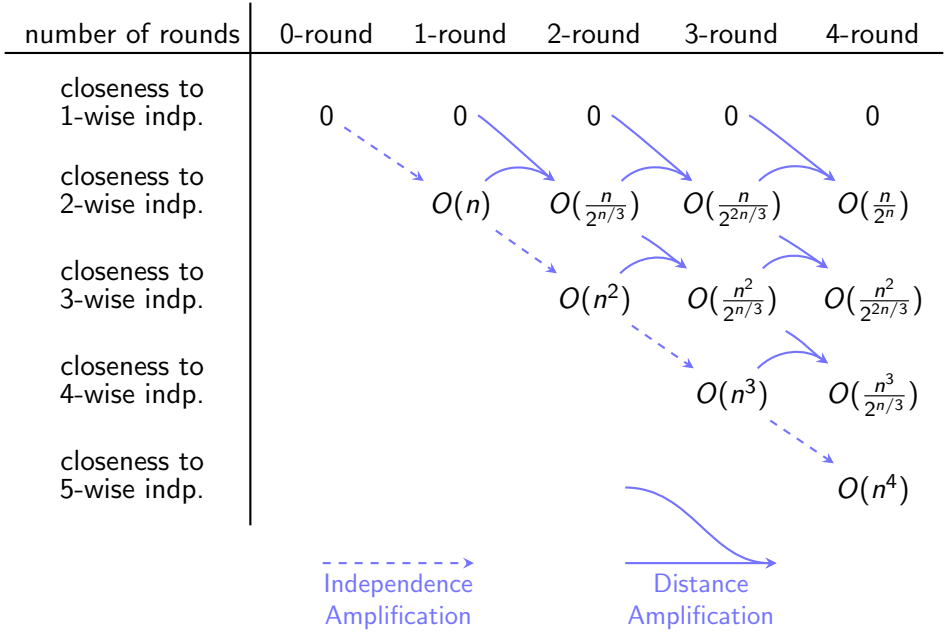
Proof Overview (KAC)

number of rounds	0-round	1-round	2-round	3-round	4-round
closeness to 1-wise indp.	0	0	0	0	0
closeness to 2-wise indp.					
closeness to 3-wise indp.					
closeness to 4-wise indp.					
closeness to 5-wise indp.					

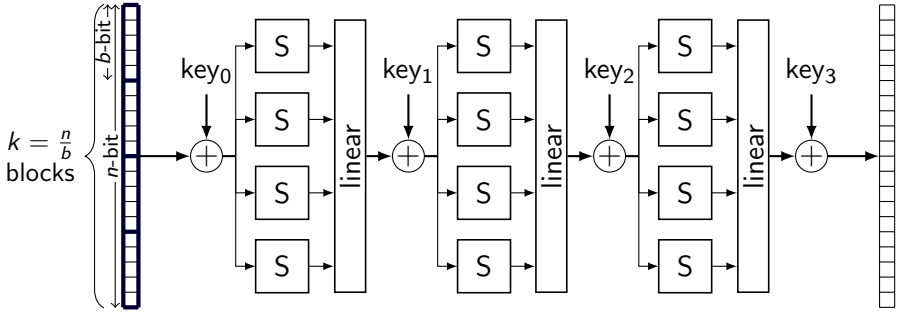
Proof Overview (KAC)



Proof Overview (KAC)



Proof Overview (SPN & AES)



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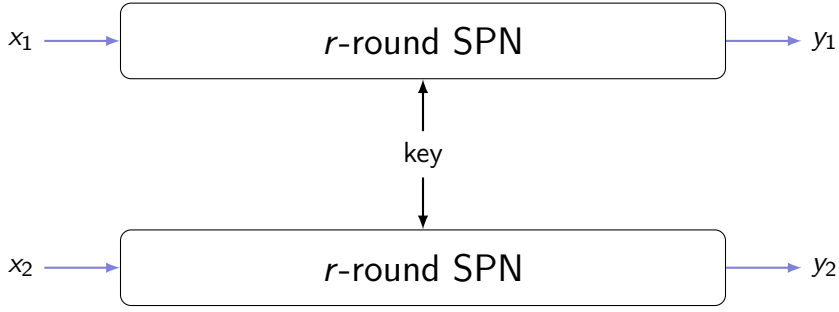
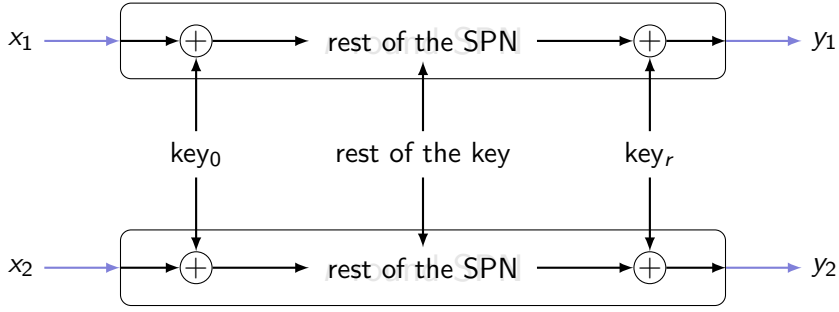
Proof Overview (SPN & AES)

Only the difference matters



Proof Overview (SPN & AES)

Only the difference matters



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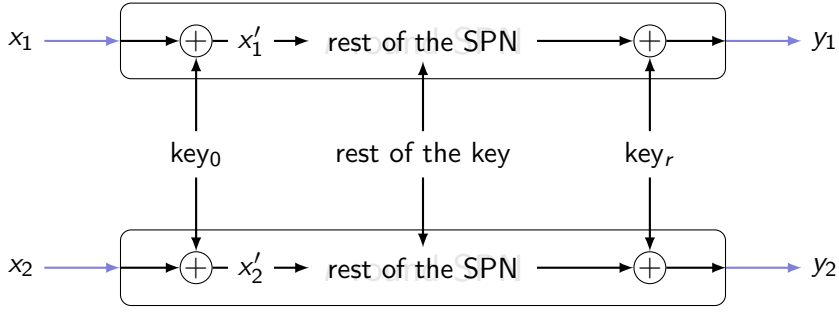

(x_1', x_2') is random conditioning on $x_1' - x_2' = x_1 - x_2$

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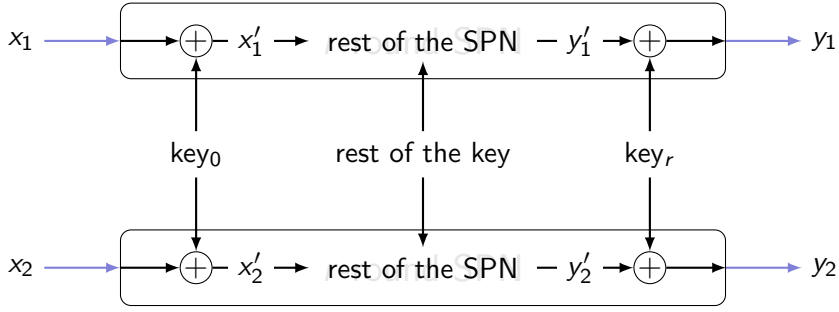
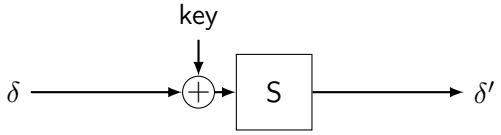


(x_1', x_2') is random conditioning on $x_1' - x_2' = x_1 - x_2$

$$\text{SD}((y_1, y_2), \text{uniform}) \;=\; \text{SD}(y_1' - y_2', \text{uniform})$$

Proof Overview (SPN & AES)

S-box: input difference $\delta \quad \mapsto \quad$ output difference δ'



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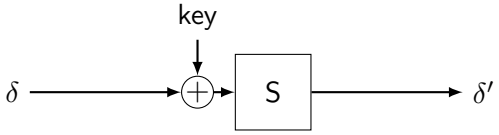

given inputs x_1, x_2 s.t. $x_1 \oplus x_2 = \delta$,
what is the distribution of $\delta' = S(x_1 \oplus \text{key}) \oplus S(x_2 \oplus \text{key})$?

Proof Overview (SPN & AES)

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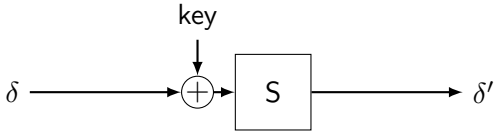

$$S(x) = x^{-1} \quad or \quad S(x) = x^3$$

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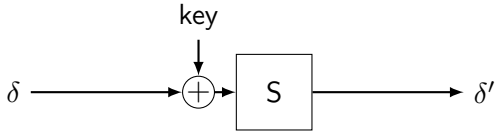

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Subspace Sampling Lemma

View δ, δ' as dimension-n vectors in $\mathbb{F}_2^n$

δ' is a random vector orthogonal to δ!

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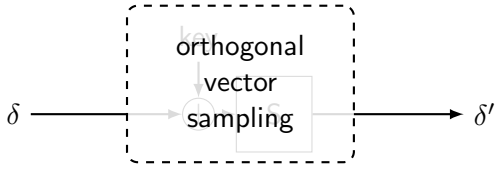

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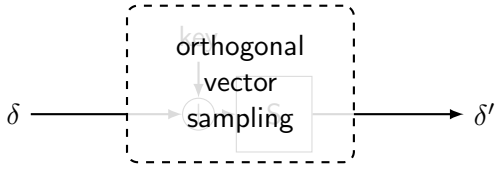

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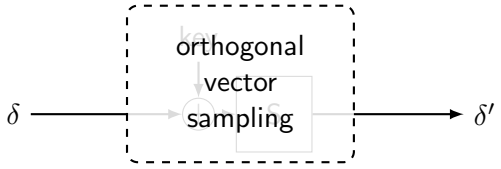

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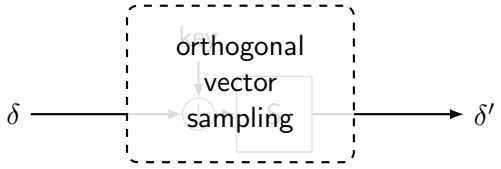

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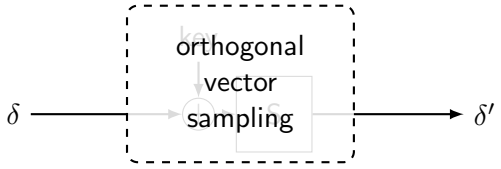

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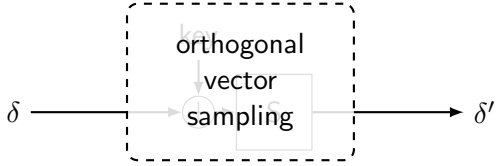

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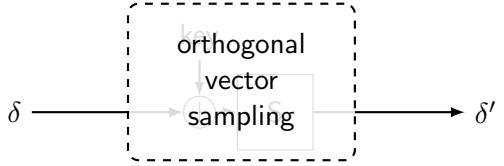

fixed $\delta \neq 0$ $\Longrightarrow$ $\mathrm{H}_\infty(\delta') = n - 1$

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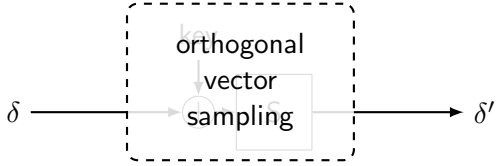

$\mathsf{H}_\infty(\delta) \geq n - 1 \quad \Longrightarrow \quad ???$

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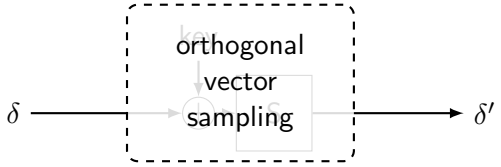

Extraction Lemma

$$\mathrm{H}_\infty(\delta) \geq n - 1 \quad \Longrightarrow \quad \delta' \text{ close to uniform}$$

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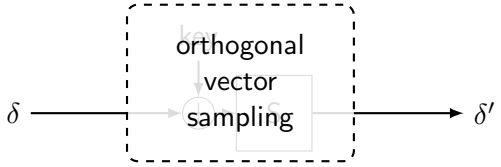

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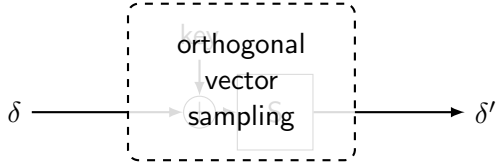

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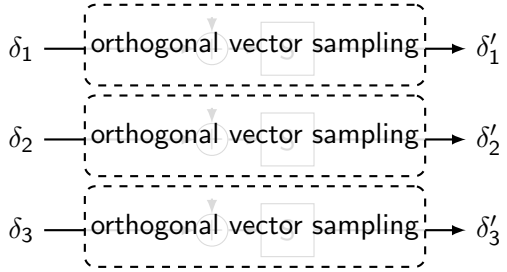

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$$\forall i \ \mathrm{H}_\infty(\delta_i) \geq n-1 \qquad \Longrightarrow \qquad (\delta'_1, \ldots, \delta'_k) \text{ close to uniform}$$

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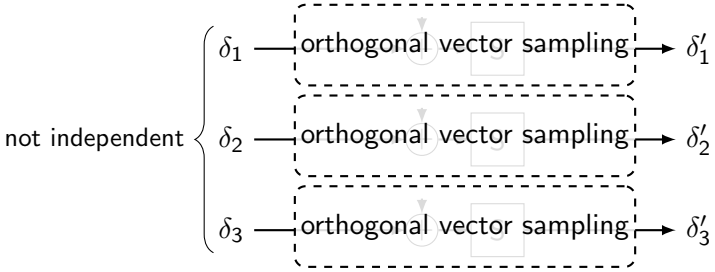

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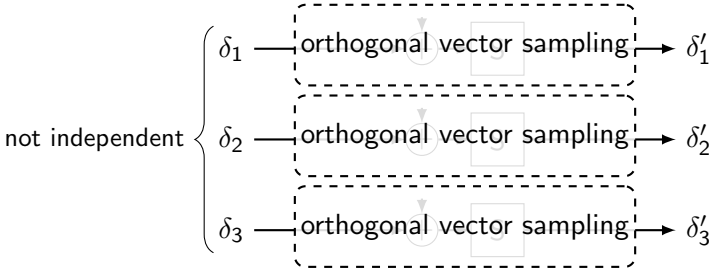

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$\forall i\ \mathrm{H}_\infty(\delta_i) \geq n-1 \quad \Longrightarrow \quad (\delta'_1, \ldots, \delta'_k)$ close to uniform

$\mathrm{H}_\infty(\{\delta_i\}_{i\in S}) \geq (n-1)\cdot|S|$ for any subset $S \subseteq [k]$ $\quad \Longrightarrow \quad (\delta'_1, \ldots, \delta'_k)$ very close to uniform

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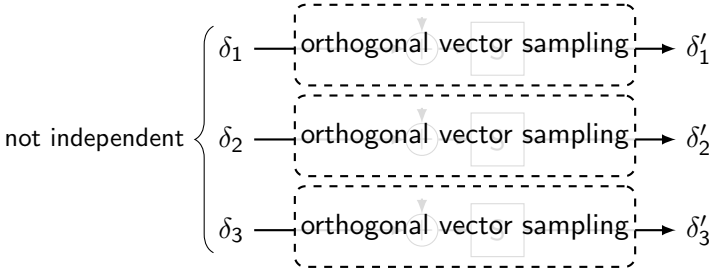

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$$\forall i\ \mathrm{H}_\infty(\delta_i) \geq n-1 \implies \mathrm{SD}((\delta'_1,\ldots,\delta'_k),\text{uniform}) \leq \sqrt{\frac{2^k-1}{2^b}}$$

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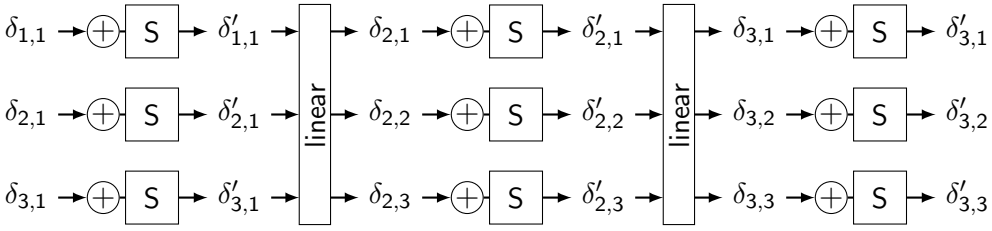

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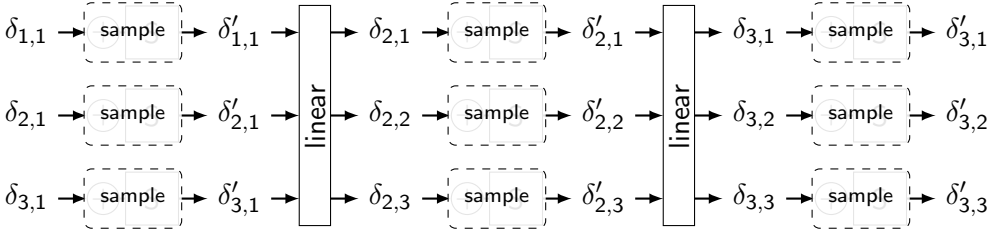

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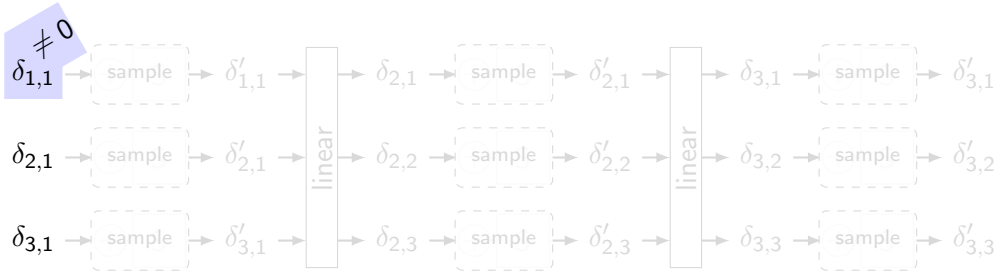

$\xrightarrow{\text{w.l.o.g.}}$ $\delta_{1,1} \neq 0$

Proof Overview (SPN & AES)

$$\xRightarrow{\text{w.l.o.g.}} \delta_{1,1} \neq 0 \Longrightarrow \mathsf{H}_\infty(\delta'_{1,1}) = n - 1$$

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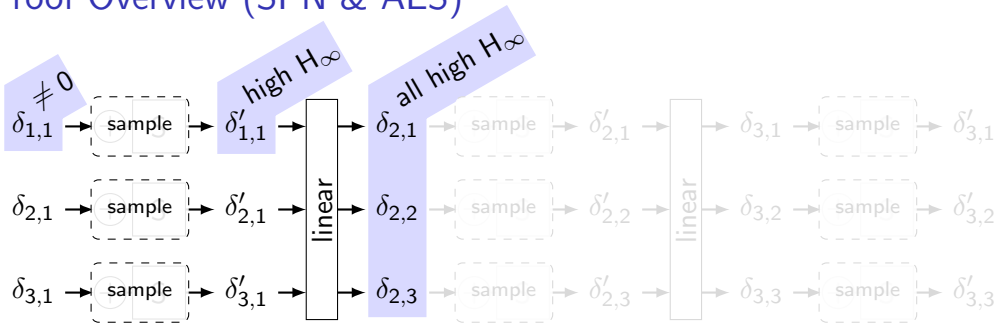

$$\overset{\text{w.l.o.g.}}{\Longrightarrow} \delta_{1,1} \neq 0 \Longrightarrow \mathrm{H}_\infty(\delta'_{1,1}) = n-1 \overset{(\star)}{\Longrightarrow} \begin{array}{c} \forall i \\ \mathrm{H}_\infty(\delta_{2,i}) \geq n-1 \end{array}$$

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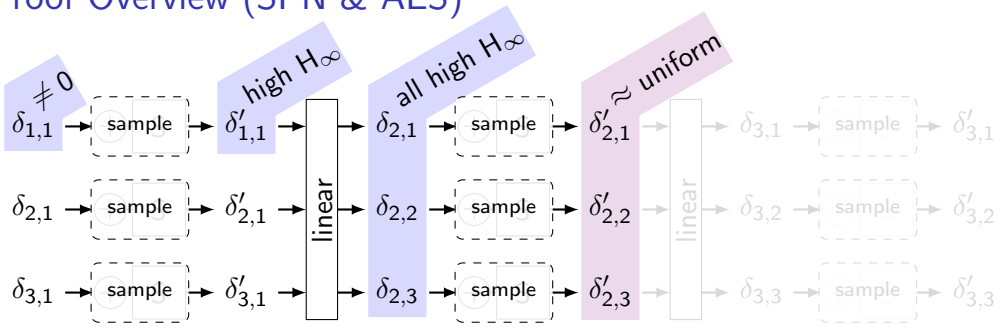
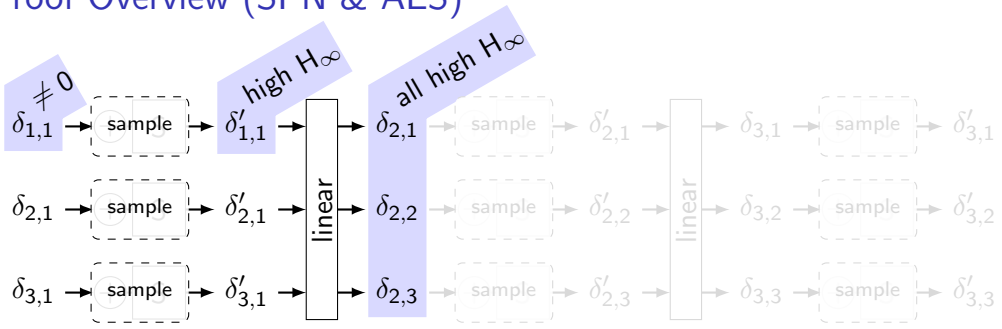


w.l.o.g. $\Longrightarrow$ $\delta_{1,1} \neq 0$ $\Longrightarrow$ $H_\infty(\delta'_{1,1}) = n-1$ $\overset{(\star)}{\Longrightarrow}$ $\forall i$ $H_\infty(\delta_{2,i}) \geq n-1$ $\overset{\text{extraction lemma}}{\Longrightarrow}$ $(\delta'_{2,1}, \ldots, \delta'_{2,k})$ close to unifrom

$SD = \sqrt{\frac{2^k}{2^d}} + O(\frac{k}{2^d})$

Proof Overview (SPN & AES)



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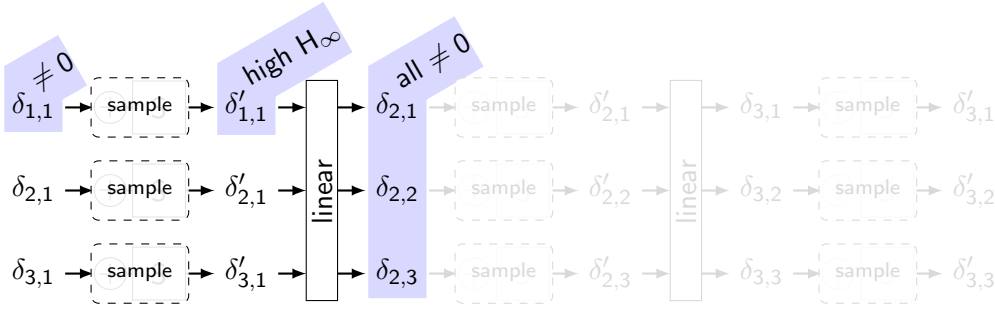

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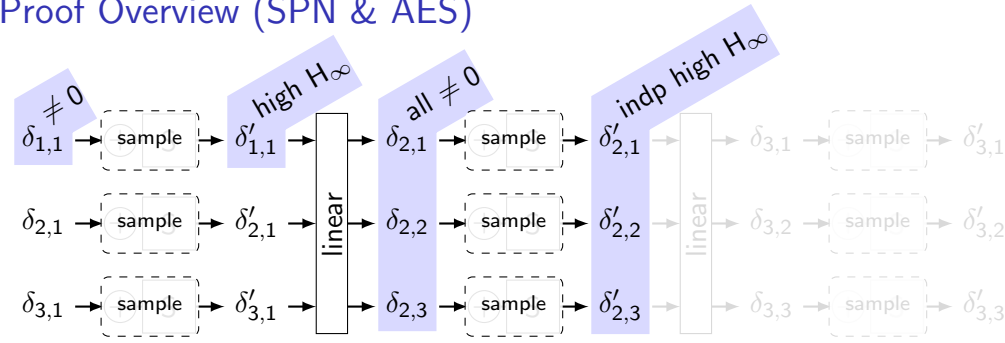

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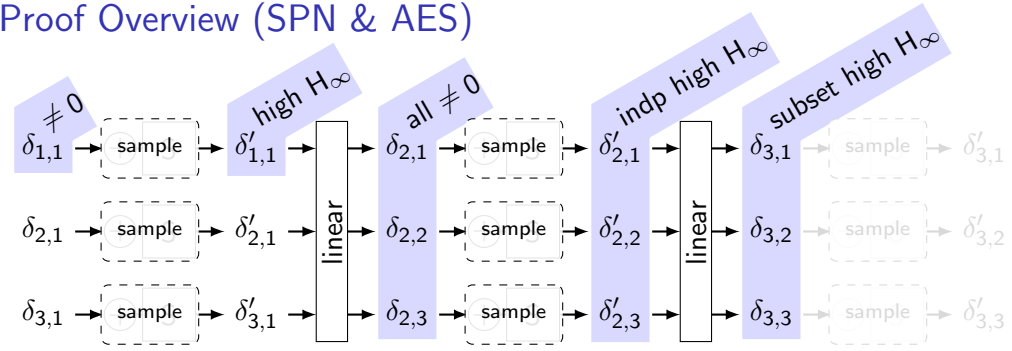

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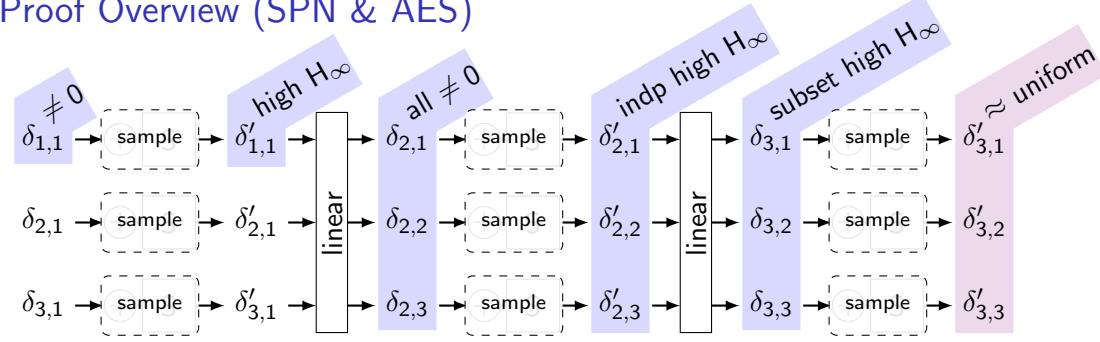

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Our Results (SPN & AES)

2-round SPN is $(\frac{4k}{2^b} + \sqrt{\frac{2^k}{2^b}})$-close to 2-wise independent.

3-round SPN is $(\frac{8k}{2^b} + \sqrt{\frac{k}{2^b}})$-close to 2-wise independent.

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6-round AES is 0.472-close to 2-wise independent.

Our Results (KAC)

r-round KAC$(\pi_1, \ldots, \pi_r)$ is close to $(r - o(r))$-wise indp for most $\pi_1, \ldots, \pi_r$

Our Results (SPN & AES)

2-round SPN is $(\frac{4k}{2^b} + \sqrt{\frac{2^k}{2^b}})$-close to 2-wise independent.

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t-**wise independence** has a really rich body of problems . . .

- Amplify independence like what we did in KAC
  - 3-wise independence of a concrete cipher
- The role of key scheduling
- Analysis of other concrete cipher design
  - e.g. add–rotate–xor (ARX) cipher
- The relationship between *t*-wise independent and other class(es) of attack