Chair of Applied Cryptography

ChaAC

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

FACULTY OF ENGINEERING

Subtractive Sets over Cyclotomic Rings

Limits of Schnorr-like Arguments over Lattices

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Perspective

This work

Subtractive sets $\leftrightarrow$ lattice-based Schnorr-like arguments

Perspective

This work

Subtractive sets $\leftrightarrow$ lattice-based Schnorr-like arguments

Concurrent works on Lattice-based Schnorr-like arguments

- [BCS21] Jonathan Bootle, Alessandro Chiesa, Katerina Sotiraki: Sumcheck Arguments and their Applications, CRYPTO'21
- [ACK21] Thomas Attema, Ronald Cramer, Lisa Kohl: A Compressed Sigma-Protocol Theory for Lattices, CRYPTO'21

Every other lattice talk needs this slide!

Short Integer Solution (SIS) over $\mathcal{R}$

Fix q, β. Given $(\mathbf{A}, \mathbf{y})$, find $\mathbf{x}$ such that

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{y} \bmod q \\ \|\mathbf{x}\| \leq \beta. \end{cases}$$

- $h, k \in \mathbb{N}$: dimensions
- $q \in \mathbb{N}$: modulus
- $\beta \in \mathbb{N}$: norm bound
- $\mathcal{R}$: ring ($+$, $-$ and $\times$ but not always $\div$, e.g. $\mathbb{Z}$)
- $\mathcal{R}_q := \mathcal{R}/q\mathcal{R}$
- $\mathbf{A} \in \mathcal{R}_q^{h \times k}$: matrix
- $\mathbf{x} \in \mathcal{R}^k$: vector
- $\mathbf{y} \in \mathcal{R}_q^h$: vector
- $\|\cdot\|$: infinity norm

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Motivating Problem

Proving knowledge of SIS witness $\mathbf{x}$.

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- $\mathbf{y} \in \mathcal{R}_q^h$: vector
- $\|\cdot\|$: infinity norm

Proving Knowledge of SIS

$$R_{s,\beta}((\mathbf{A},\mathbf{y}),\mathbf{x}) := (\mathbf{Ax} = s \cdot \mathbf{y} \bmod q \wedge \|\mathbf{x}\| \leq \beta)$$

where $s \in \mathcal{R}$ is called the "slack" ($s = 1 \implies$ no slack)

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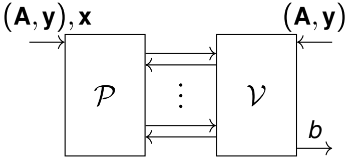

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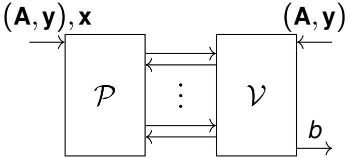

- Completeness for $R_{1,\beta}$: If $R_{1,\beta}\left((\mathbf{A},\mathbf{y}),\mathbf{x}\right) = 1$ then $\mathcal{V}$ accepts $(\mathbf{A},\mathbf{y})$, i.e. $b = 1$.

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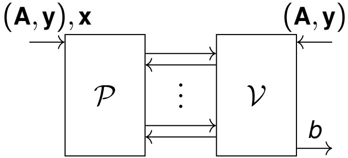

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- κ-Knowledge Soundness for $R_{s,\beta'}$: There exists efficient *knowledge extactor* $\mathcal{E}$ such that

if	$\mathcal{P}$ convinces $\mathcal{V}$ to accept $(\mathbf{A},\mathbf{y})$	with probability $\rho > \kappa$,
then	$\mathcal{E}^{\mathcal{P}}$ extracts $\tilde{\mathbf{x}}$ such that $R_{s,\beta'}\left((\mathbf{A},\mathbf{y}),\tilde{\mathbf{x}}\right) = 1$	with probability $\rho - \kappa$.

Proving Knowledge of SIS

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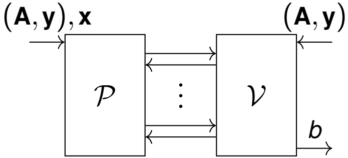

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  - knowledge error κ
  - "slack" s

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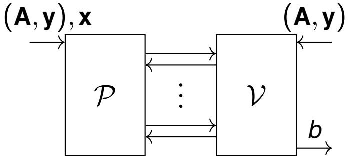

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- Challenge: Design $\langle \mathcal{P}, \mathcal{V} \rangle$ to minimise
  - knowledge error κ
  - "slack" s
  - "stretch" $\frac{\beta'}{\beta}$

Landscape

Pre-2019

- PCP-based: *// Probabilistically-checkable proofs*
  - ℹ PCP (e.g. for R1CS) + commitments
  - ⊞ logarithmic-size proof, no slack ($s = 1$), no stretch $\beta' = \beta$
  - ⊟ Super-polynomial modulus q
- Stern-like:
- Schnorr-like:

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  - ℹ combinatorial (cut-and-choose)
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- Schnorr-like:
  - ℹ algebraic
  - ⊞ $1/\text{poly}(\lambda)$ knowledge error ($O(\lambda/\log\lambda)$ repetition)
  - ⊞ linearity $\implies$ recursive composition (“Bulletproof folding”) $\implies$ logarithmic-size proof
  - ⊟ slack $s \neq 1$, stretch $\beta'/\beta > 1$ (amplified by recursive composition)

Landscape

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- Stern+Schnorr:
  - ℹ Schnorr but with extra non-linear constraints
  - ⊞ $1/\text{poly}(\lambda)$ knowledge error, no slack ($s = 1$), no stretch $\beta' = \beta$
  - ⊟ non-linearity $\implies$ not "Bulletproof" compatible

Landscape

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  - ⊞ $1/\mathrm{poly}(\lambda)$ knowledge error, no slack ($s = 1$), no stretch $\beta' = \beta$
  - ⊟ non-linearity $\Longrightarrow$ not “Bulletproof” compatible

Question

Keep linearity and $1/\mathrm{poly}(\lambda)$ knowledge error of Schnorr, but reduce slack and stretch?

Schnorr-like Protocol 1

Parameters: $\mathcal{C} \subseteq \mathcal{R}$: challenge set, $\gamma \in \mathbb{N}$: norm bound, $\kappa = \frac{1}{|\mathcal{C}|}$: knowledge error

$\mathcal{P}((\mathbf{A}, \mathbf{y}), \mathbf{x})$		$\mathcal{V}(\mathbf{A}, \mathbf{y})$
$\mathbf{u} \leftarrow\$ \mathcal{R}^k$		
$\mathbf{v} := \mathbf{A}\mathbf{u}$	$\xrightarrow{\mathbf{v}}$	
	$\xleftarrow{c}$	$c \leftarrow\$ \mathcal{C}$
$\hat{\mathbf{x}} := \mathbf{u} + c \cdot \mathbf{x}$	$\xrightarrow{\hat{\mathbf{x}}}$	return $\begin{cases} \mathbf{A}\hat{\mathbf{x}} = \mathbf{v} + c \cdot \mathbf{y} \bmod q \\ \|\hat{\mathbf{x}}\| \leq \gamma \end{cases}$

Knowledge Extractor

- Recall verification equation

$$\mathbf{A}\hat{\mathbf{x}} \stackrel{?}{=} \mathbf{v} + c \cdot \mathbf{y} \bmod q$$

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- Run $\mathcal{P}$ twice on c_0, c_1 to get $\mathbf{v}, \hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1$ such that

$$\mathbf{A}\begin{pmatrix}\hat{\mathbf{x}}_0 & \hat{\mathbf{x}}_1\end{pmatrix} = \begin{pmatrix}\mathbf{v} & \mathbf{y}\end{pmatrix}\begin{pmatrix}1 & 1 \\ c_0 & c_1\end{pmatrix} \bmod q$$

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- **Try** to solve the following dual Vandermonde system for $\mathbf{z}$ over $\mathcal{R}$:

$$\begin{pmatrix} 1 & 1 \\ c_0 & c_1 \end{pmatrix} \mathbf{z} = s \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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- Output $\tilde{\mathbf{x}} := \begin{pmatrix}\hat{\mathbf{x}}_0 & \hat{\mathbf{x}}_1\end{pmatrix}\mathbf{z}$ such that

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Schnorr-like Protocol 2: Lattice Bulletproof [Bootle et al. @ Crypto'20]

Parameters: $\mathcal{C} \subseteq \mathcal{R}$: challenge set, $\gamma \in \mathbb{N}$: norm bound, $\kappa = \frac{2}{|\mathcal{C}|}$: knowledge error

Structural Assumptions: $\mathbf{A} = (\mathbf{A}_0 \quad \mathbf{A}_1)$, $\mathbf{x} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{pmatrix}$, $\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{A}_0\mathbf{x}_0 + \mathbf{A}_1\mathbf{x}_1 \mod q$

$\mathcal{P}((\mathbf{A},\mathbf{y}),\mathbf{x})$		$\mathcal{V}(\mathbf{A},\mathbf{y})$
$\mathbf{y}_{01} := \mathbf{A}_0\mathbf{x}_1,\ \mathbf{y}_{10} := \mathbf{A}_1\mathbf{x}_0$	$\xrightarrow{\mathbf{y}_{01},\mathbf{y}_{10}}$	
	$\xleftarrow{c}$	$c \leftarrow\$ \mathcal{C}$
$\hat{\mathbf{x}} := \mathbf{x}_0 + c \cdot \mathbf{x}_1$	$\xrightarrow{\hat{\mathbf{x}}}$	**return** $\begin{cases} (c \cdot \mathbf{A}_0 + \mathbf{A}_1)\hat{\mathbf{x}} = \mathbf{y}_{10} + c \cdot \mathbf{y} + c^2 \cdot \mathbf{y}_{01} \mod q \\ \|\hat{\mathbf{x}}\| \leq \gamma \end{cases}$

Lattice Bulletproof Knowledge Extractor

- Recall verification equation $(c \cdot \mathbf{A}_0 + \mathbf{A}_1)\hat{\mathbf{x}} \stackrel{?}{=} \mathbf{y}_{10} + c \cdot \mathbf{y} + c^2 \cdot \mathbf{y}_{01} \bmod q$

Lattice Bulletproof Knowledge Extractor

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- Run $\mathcal{P}$ 3 times on c_0, c_1, c_2 to get $\mathbf{y}_{01}, \mathbf{y}_{10}, \hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2$ such that

$$\mathbf{A} \begin{pmatrix} c_0\hat{\mathbf{x}}_0 & c_1\hat{\mathbf{x}}_1 & c_2\hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_0 & \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{y}_{10} & \mathbf{y} & \mathbf{y}_{01} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ c_0 & c_1 & c_2 \\ c_0^2 & c_1^2 & c_2^2 \end{pmatrix} \mod q$$

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(s,t)-Subtractive Sets over $\mathcal{R}$

For what challenges $c_0, \ldots, c_{t-1}$ and slack s is the following dual Vandermonde system solvable over $\mathcal{R}$?

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ c_0 & c_1 & \ldots & c_{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_0^{t-1} & c_1^{t-1} & \ldots & c_{t-1}^{t-1} \end{pmatrix} \mathbf{z} = s \cdot \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{t-1} \end{pmatrix} \quad (\star)$$

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Observation. If $\prod_{j \in \mathbb{Z}_t \setminus \{i\}} (c_i - c_j) \mid s$ for all $i \in \mathbb{Z}_t$, then Equation $(\star)$ is solvable over $\mathcal{R}$.

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Definition. A set $\mathcal{C} \subseteq_n \mathcal{R}$ is (s,t)*-subtractive* if for any t-subset $T = \{c_0, \ldots, c_{t-1}\} \subseteq_t \mathcal{C}$ it holds that $\prod_{j \in \mathbb{Z}_t \setminus \{i\}} (c_i - c_j) \mid s$ for all $i \in \mathbb{Z}_t$. If $s = 1$ we say $\mathcal{C}$ is subtractive.

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A Note about Secret Sharing over $\mathcal{R}$. If $\mathcal{C} \subseteq_n \mathcal{R}$ is (s,t)-subtractive, then for any $T = \{c_0, \ldots, c_{t-1}\} \subseteq_t \mathcal{C}$, the following Vandermonde system is solvable over $\mathcal{R}$:

$$\begin{pmatrix} 1 & c_0 & \cdots & c_0^{t-1} \\ 1 & c_1 & \cdots & c_1^{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_{t-1}^{t-1} & \cdots & c_{t-1}^{t-1} \end{pmatrix} \mathbf{z} = s \cdot \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{t-1} \end{pmatrix}$$

$\Longrightarrow$ t-out-of-n secret sharing over $\mathcal{R}$.

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Sample Implications.

- $(s,3)$-subtractive set of size $n \implies$ Lattice Bulletproof with slack s and knowledge error $2/n$
- (s,t)-subtractive set of size $n \implies$ Lattice-based t-out-of-n threshold primitives

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Sample Implications.

- $(s,3)$-subtractive set of size $n \implies$ Lattice Bulletproof with slack s and knowledge error $2/n$
- (s,t)-subtractive set of size $n \implies$ Lattice-based t-out-of-n threshold primitives

Challenge. Find large (poly-size) (s,t)-subtractive sets with small slack s over interesting $\mathcal{R}$, e.g. cyclotomic rings $\mathcal{R} = \mathbb{Z}[\zeta_m]$ where ζ_m is a primitive m-th root of unity, $m = \mathrm{poly}(\lambda)$.

Our Results over $\mathcal{R} = \mathbb{Z}[\zeta_m]$

- Power-of-2 cyclotomic rings $m = 2^\ell$:
  - ⊞ Construct family of (s, t)-subtractive sets of size n for a wide range of s, t, n,
    e.g. $(2, 3)$-subtractive set of size $n = m/2 + 1$ ($\implies$ Bulletproof with slack 2)
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- Proof system for SIS over $\mathcal{R}$:
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- Application to threshold secret sharing over $\mathcal{R}$, e.g. distributed PRF

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Prime-Power Cyclotomic Rings $\mathbb{Z}[\zeta_{p^\ell}]$

Theorem. The set $\mathcal{C}$ is subtractive and $|\mathcal{C}| = p$.

$$\mathcal{C} = \{\mu_0, \mu_1, \ldots, \mu_{p-1}\} \qquad \mu_k = \frac{1-\zeta^k}{1-\zeta}$$

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- For $i < j < p$ we have

$$\mu_j - \mu_i = \frac{1-\zeta^j}{1-\zeta} - \frac{1-\zeta^i}{1-\zeta} = \frac{\zeta^i - \zeta^j}{1-\zeta} = \zeta^i \cdot \frac{1-\zeta^{j-i}}{1-\zeta} = \zeta^i \cdot \mu_{j-i}$$

which is invertible over $\mathcal{R}$.

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- $\mathcal{C}$ is subtractive $\implies$ $c_0 - c_1$ is invertible $\implies$ $1 \in \mathcal{I}$, a contradiction.

Power-of-2 Cyclotomic Rings $\mathbb{Z}[\zeta_{2^\ell}]$

Theorem. For $0 \leq i \leq \ell$, $s \in \langle 1-\zeta \rangle^{\lceil \log t \rceil 2^{i-1}}$, the set $\mathcal{C}$ is (s,t)-subtractive and $|\mathcal{C}| = 2^i + 1$.

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- $\mathcal{C}$ is $(2,t)$-subtractive $\implies$ $(c_0 - c_1) \mid 2 \implies 2 \in \mathcal{I}$, a contradiction.

Conclusion

- Formalisation of (s, t)-subtractive sets
- Applications to Schnorr-like arguments and threshold secret sharing
- Construction of poly(λ)-size (s, t)-subtractive sets with (almost) matching impossibility results
- Improved lattice Bulletproof instantiation
- Impossibility of better knowledge error assuming algebraic extractors

Paper	`ia.cr/2021/202`
Blog Post	`russell-lai.hk/2021/07/15/subtractive-sets-over-cyclotomic-rings/`

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