Advanced Lattice Sieving on GPUs, with Tensor Cores

Léo Ducas, Marc Stevens, **Wessel van Woerden** (CWI).

CWI

Centrum Wiskunde & Informatica

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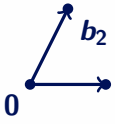
- First optimized implementation of the asymptotic best known sieve [BDGL].

Lattice

$\mathcal{L} := B\mathbb{Z}^d$



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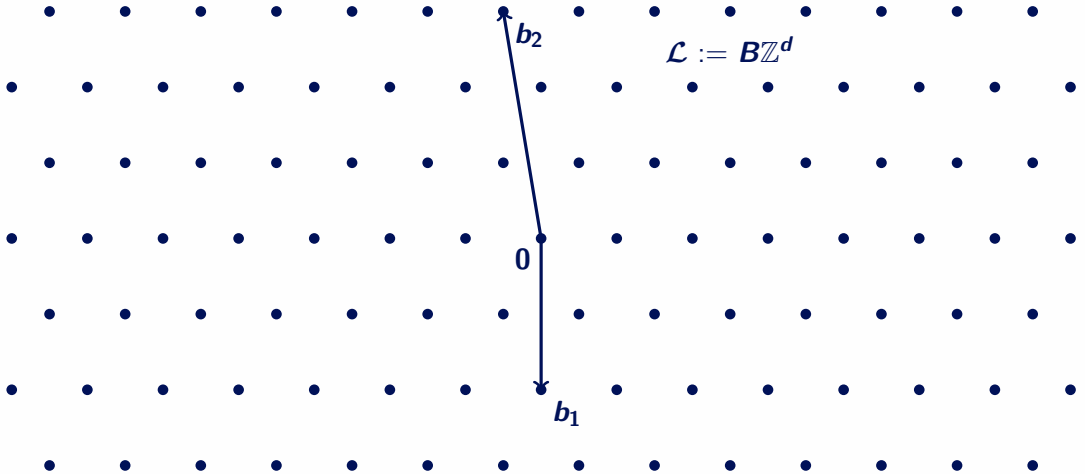
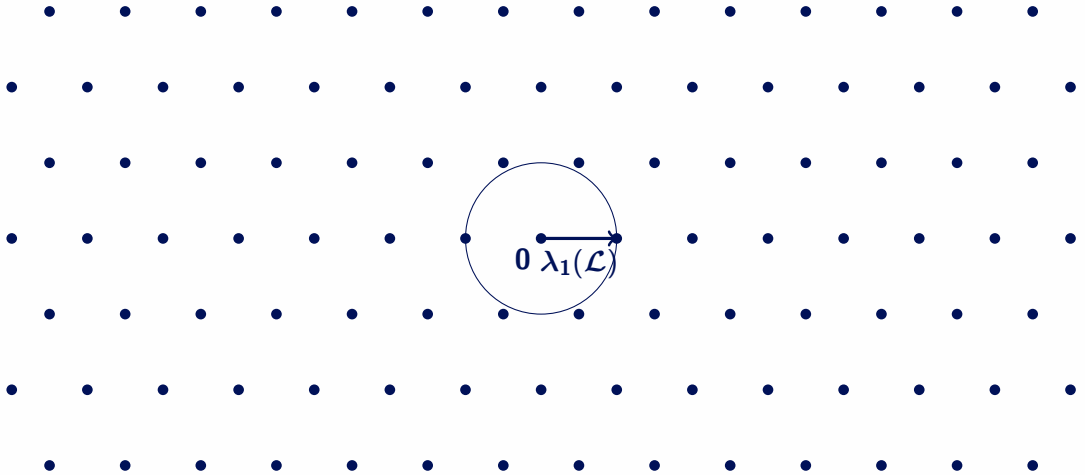
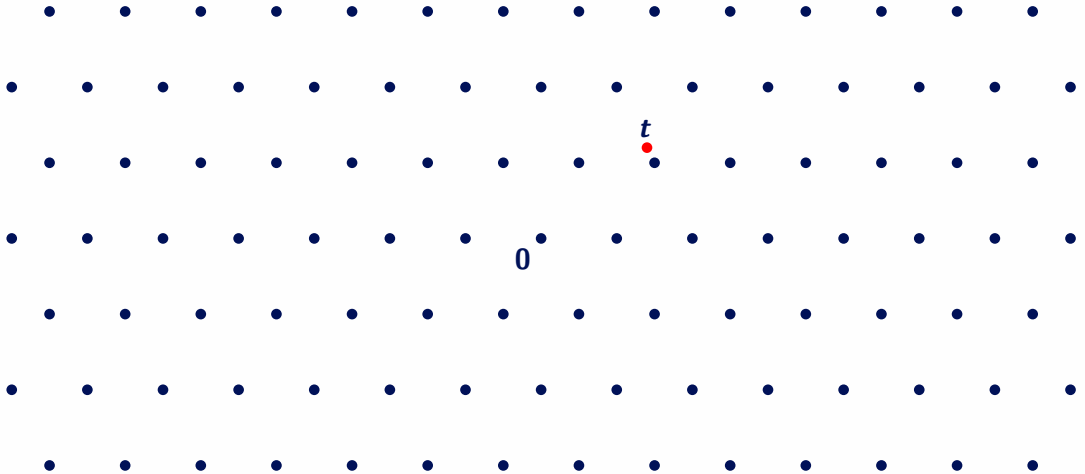


Shortest Vector Problem



Bounded Distance Decoding



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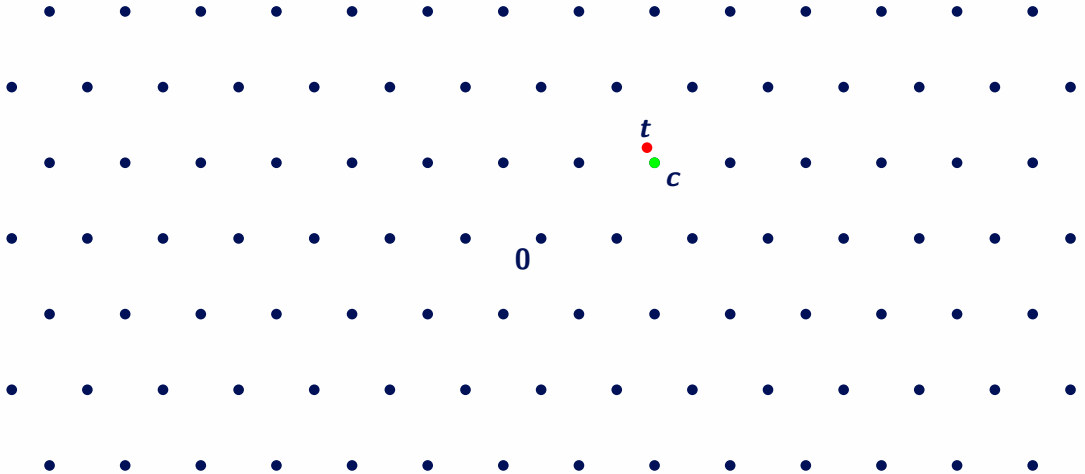

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- Given: 'Random' $\boldsymbol{d}$-dimensional lattice $\boldsymbol{\mathcal{L}}$ (Goldstein and Mayer)
- Goal: Find a $\boldsymbol{v} \in \boldsymbol{\mathcal{L}}$ s.t.

$$\|\boldsymbol{v}\| \leq \mathbf{1.05} \cdot \mathrm{GH}(\boldsymbol{\mathcal{L}}) \approx \mathbf{1.05} \cdot \boldsymbol{\lambda_1}(\boldsymbol{\mathcal{L}}).$$

Lattice Sieving

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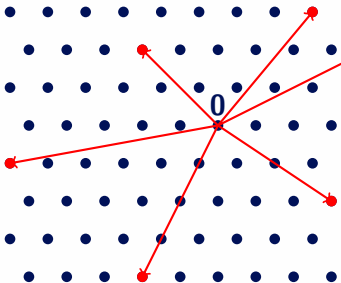

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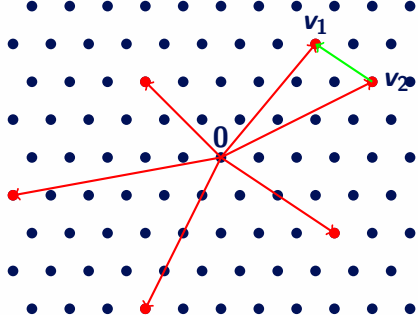

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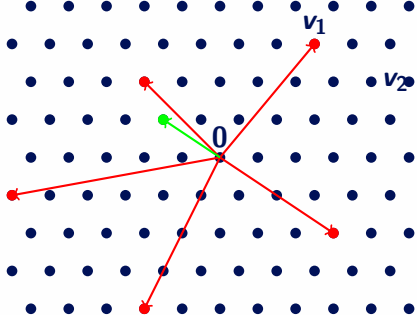

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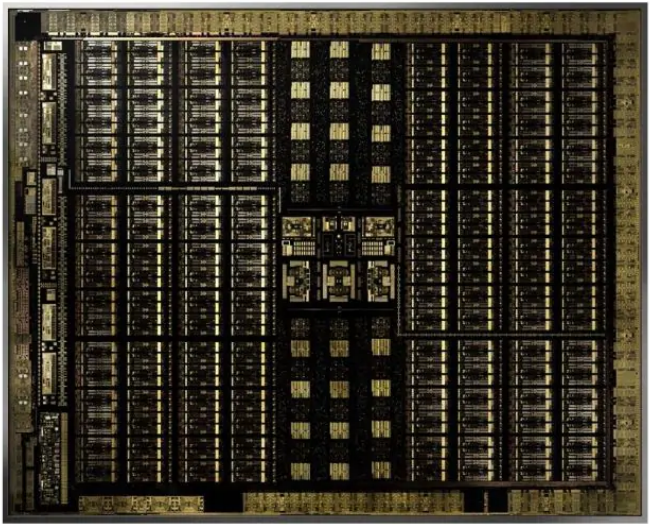

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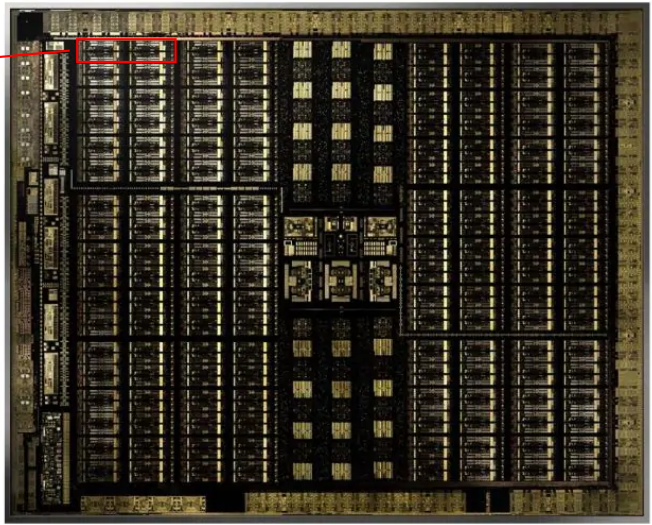
64 FP32 cores
64 INT32 cores
8 Tensor cores.

Thousands of cores.



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Per **32** cores:
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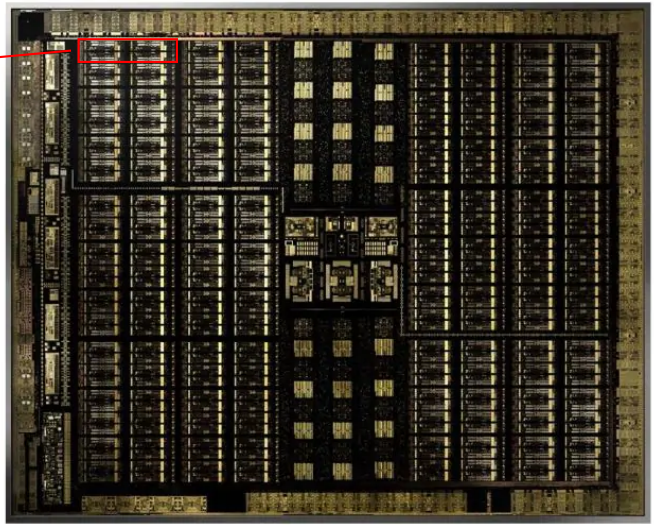

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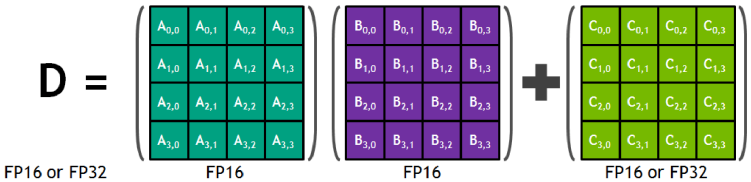

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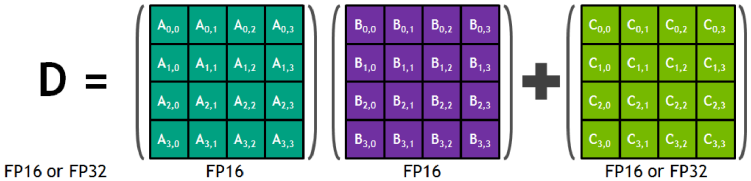

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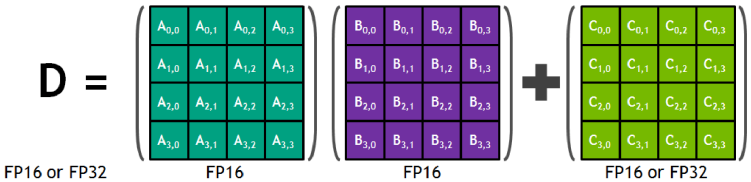

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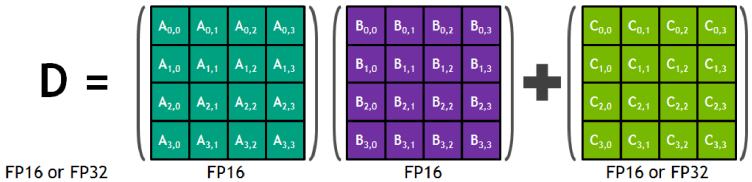

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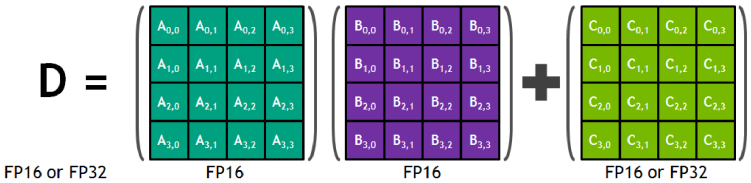

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- Newest model $>$ **300** 16-bit **Tflops**.
- The current best CPU would reach at most $\approx$ **5** 16-bit **Tflops**.

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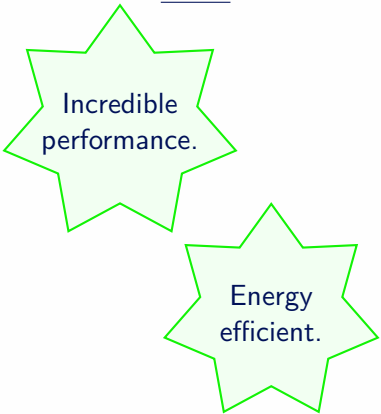

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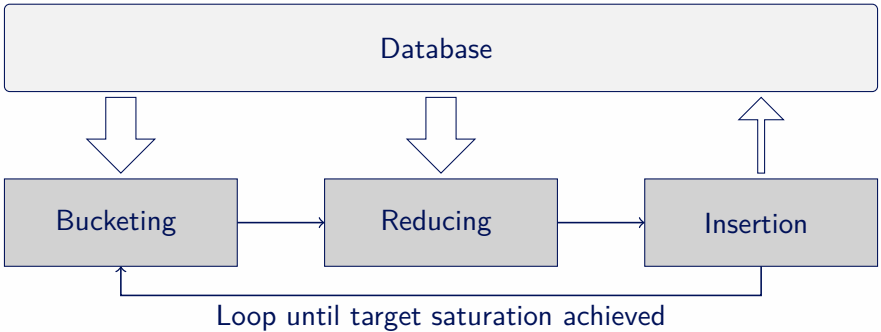
Advanced Sieving on GPUs

Sieving Process



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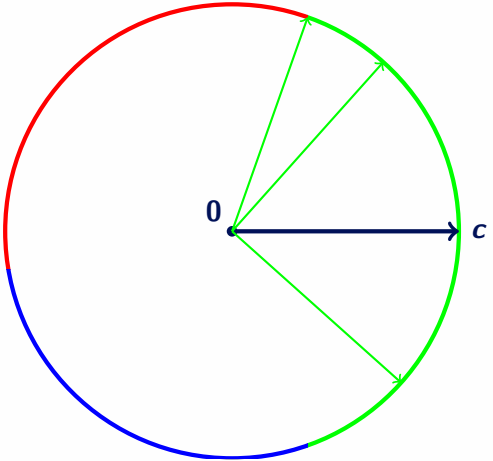

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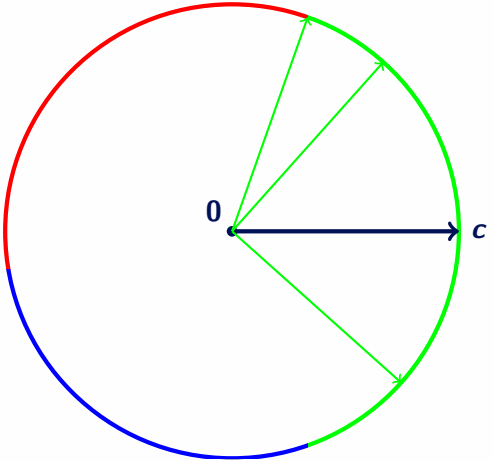

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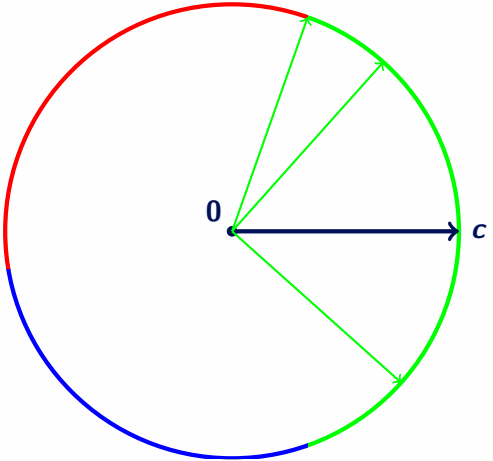

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Increases reduction probability per pair.

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  - AVX2 CPU implementation merged into G6K, fastest CPU sieve.

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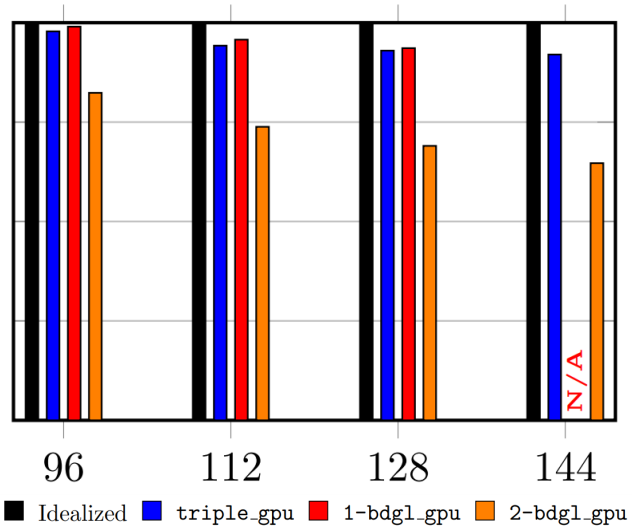

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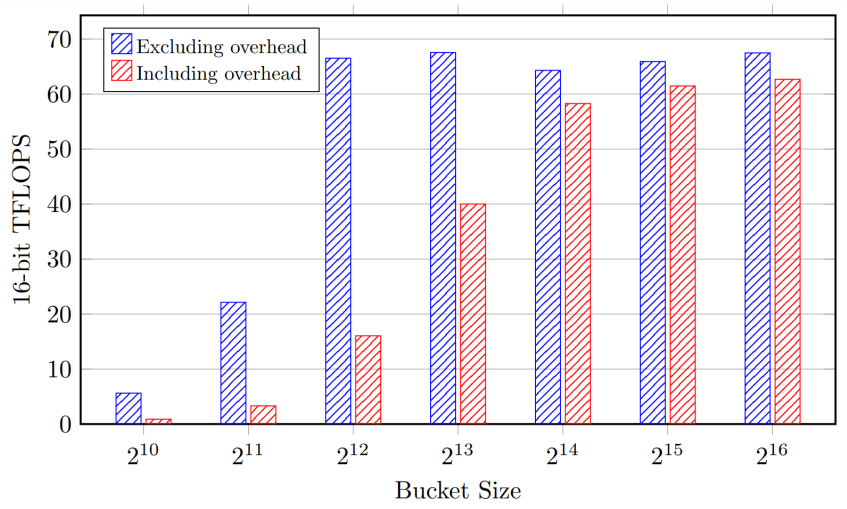
- FLOP: $\boldsymbol{O}(\boldsymbol{d} \cdot \boldsymbol{B}^2)$, data: $\boldsymbol{O}(\boldsymbol{d} \cdot \boldsymbol{B})$, ratio improves for larger bucket size $\boldsymbol{B}$.

Amortizing data throughput



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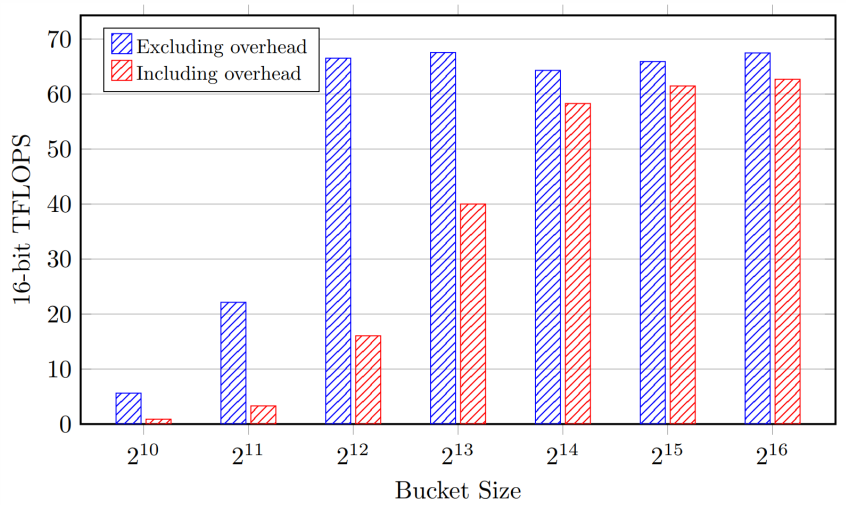

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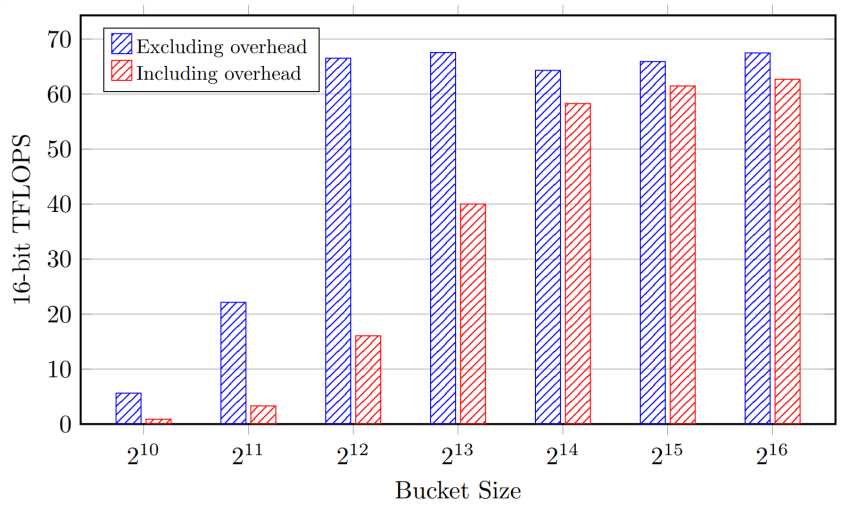

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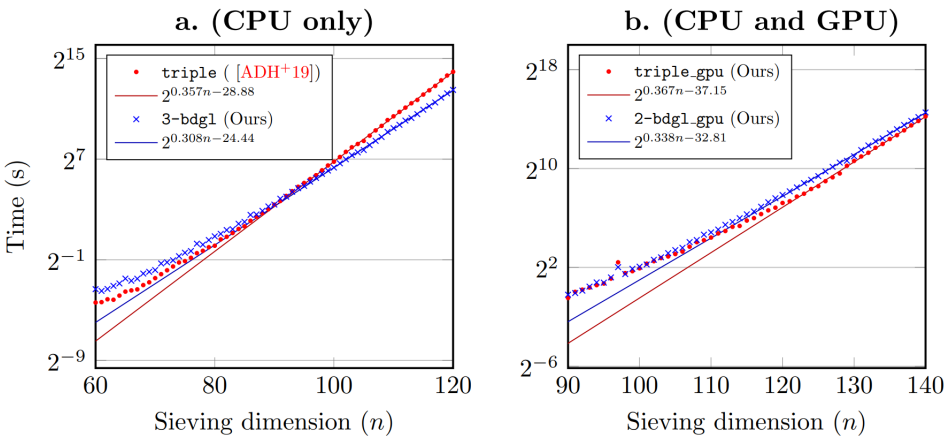
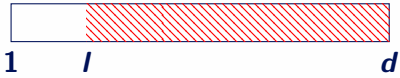
- Large buckets to reach optimal performance.

BGJ1 vs BDGL


a. (CPU only)
b. (CPU and GPU)
Time (s)
Sieving dimension (n)
Sieving dimension (n)
triple ([ADH+19])
$2^{0.357n-28.88}$
3-bdgl (Ours)
$2^{0.308n-24.44}$
triple_gpu (Ours)
$2^{0.367n-37.15}$
2-bdgl_gpu (Ours)
$2^{0.338n-32.81}$

Dimensions for Free [Duc18]

- Sieve in a projected sublattice $\pi_l(\mathcal{L})$ (projected away from first l basis vectors).



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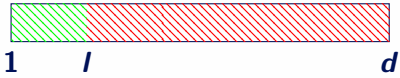

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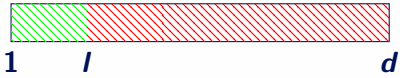

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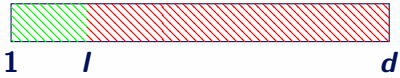

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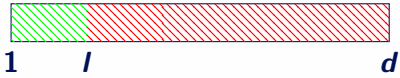

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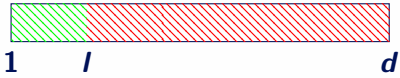

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- Progressive sieving: decrease l step-by-step.
- On the fly lifting: lift any shortish vector we encounter.
- Can we efficiently detect if $v_i - v_j$ might lift to a short vector [BDD problem]?

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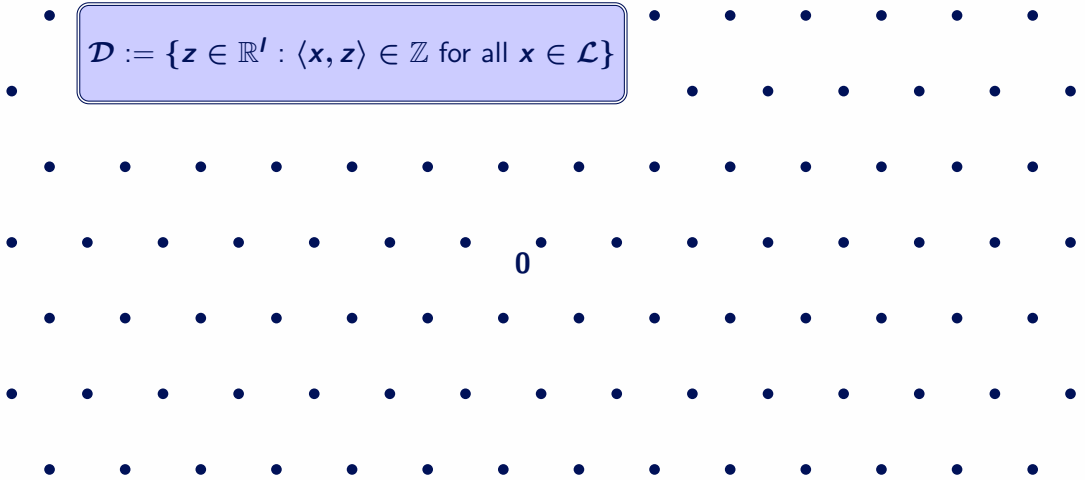

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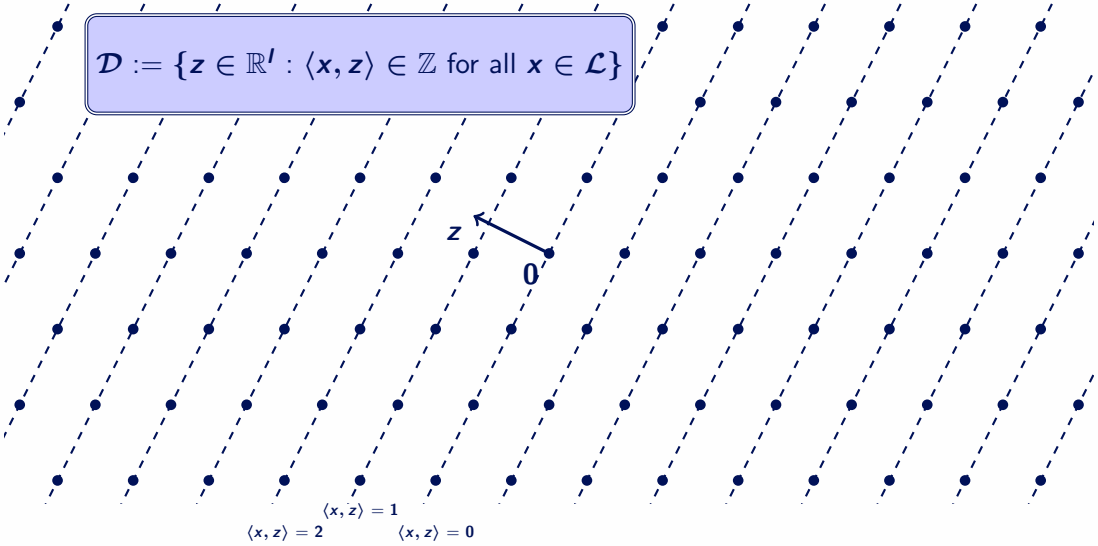

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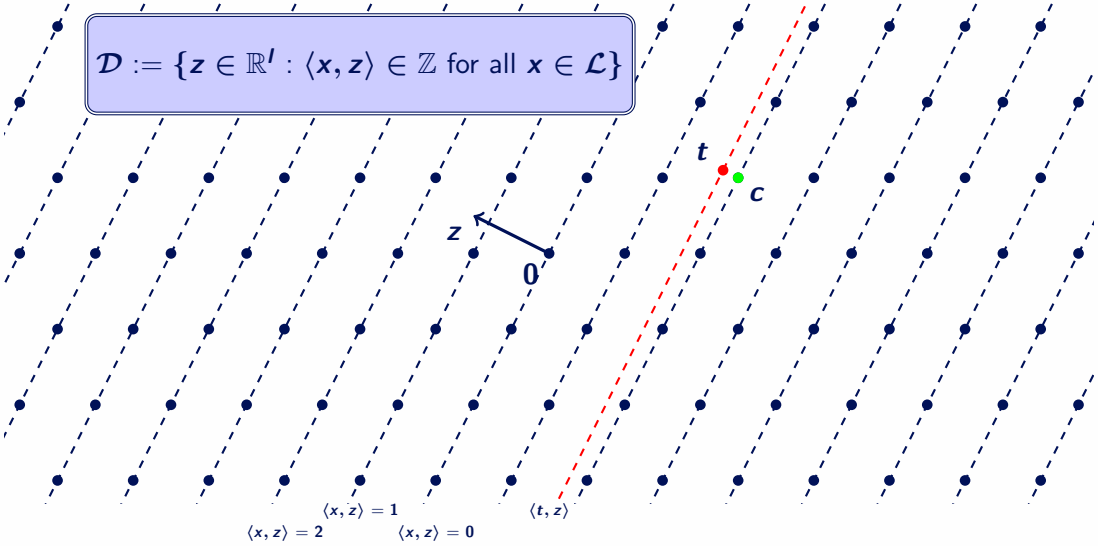

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- Suitable for GPUs.

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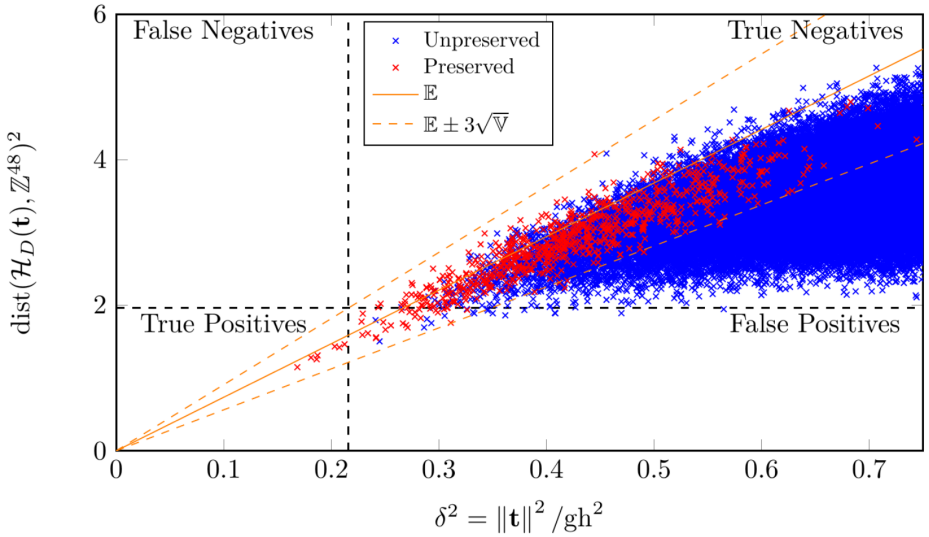
False Negatives
True Negatives
True Positives
False Positives
Unpreserved
Preserved
$\mathbb{E}$
$\mathbb{E} \pm 3\sqrt{\mathbb{V}}$
$\mathrm{dist}(\mathcal{H}_D(\mathbf{t}), \mathbb{Z}^{48})^2$
$\delta^2 = \|\mathbf{t}\|^2 / \mathrm{gh}^2$
0
2
4
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

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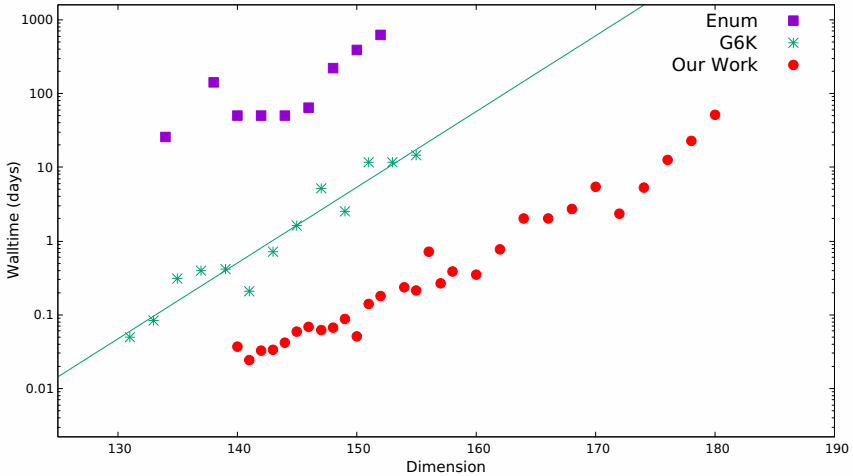
- Compute everything on the GPU, overhead of $\boldsymbol{O(B \cdot d^2)}$ for a bucket size $\boldsymbol{B}$.

New SVP records

Dimension 180!



maximum RAM size of 1.5TB reached for **180**.

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- Extra benefit of saving memory with negligible overhead.
- BDGL beats BGJ1 in practice on CPUs, but the cross-over for GPUs lies much higher.

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