On the Success Probability of Solving Unique SVP via BKZ

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Preliminaries

Search-LWE

Let $\boldsymbol{s} \leftarrow \chi_s^n$, $\boldsymbol{e} \leftarrow \chi_e^m$ and $\boldsymbol{A} \leftarrow U(\mathbb{Z}_q^{m \times n})$. Given $(\boldsymbol{A}, \boldsymbol{A}\boldsymbol{s} + \boldsymbol{e} \bmod q)$, recover $\boldsymbol{s}$.

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Primal attack strategy

Construct a lattice containing $\boldsymbol{t} := (\boldsymbol{s}, \boldsymbol{e}, 1)$ as the unique shortest vector. Use lattice reduction to recover $\boldsymbol{t}$. This reduces LWE to the unique Shortest Vector Problem (uSVP).

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Immediate questions:

- How do we cost the attack?
- How does the cost vary if we change χ_e, χ_s?

Preliminaries

Lattice basis

Given d linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d \in \mathbb{R}^d$, we say they form a basis for the (full-rank) lattice $\Lambda = \{\sum x_i \boldsymbol{b}_i : x_i \in \mathbb{Z}\}$.

Basis profile

Given a basis $\boldsymbol{B}$ of a lattice, we define the profile of $\boldsymbol{B}$ as the set $\{\|\boldsymbol{b}_i^*\|^2\}_{i=1}^d$ of its orthogonal Gram–Schmidt vectors.

Orthogonal projections

Given a basis $\boldsymbol{B}$ of $\mathbb{R}^d$ we denote by $\pi_{\boldsymbol{B},k} : \mathbb{R}^d \to \mathbb{R}^d$ the linear operator projecting vectors orthogonally to the subspace $\operatorname{span}_{\mathbb{R}}(\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k-1}\})$.

Preliminaries

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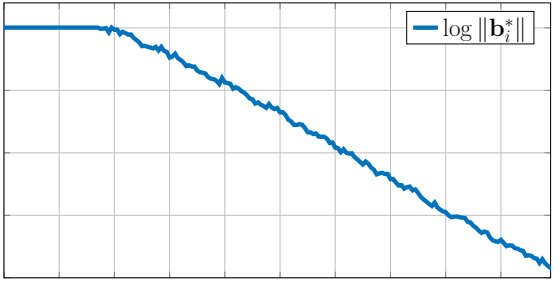
We now look at the main step in the lattice reduction algorithms we will use.

The BKZ-β tour

BKZ-β tour

for $i \leftarrow 1$ **to** d **do** // $i = 1$
  $j \leftarrow \min(i + \beta - 1, d)$
  $\boldsymbol{v} = x_i\pi_i(\boldsymbol{b}_i) + \cdots + x_j\pi_i(\boldsymbol{b}_j) \leftarrow O_{\mathrm{SVP}}(\pi_i(\boldsymbol{B}[i:j]))$
  if $\|\boldsymbol{v}\| < \|\boldsymbol{b}_i^*\|$ **then**
    $\boldsymbol{v'} \leftarrow x_i\boldsymbol{b}_i + \cdots + x_j\boldsymbol{b}_j$
    extend $\boldsymbol{B}$ by inserting $\boldsymbol{v'}$ into $\boldsymbol{B}$ at index i
    LLL on $\boldsymbol{B}$ to remove linear dependencies
  return *whether an insertion was made*

Lattice basis profile



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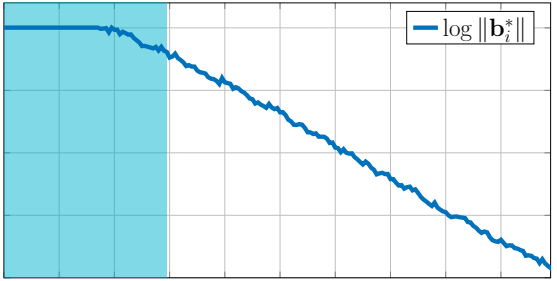

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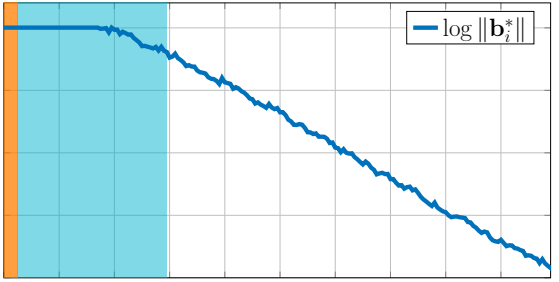

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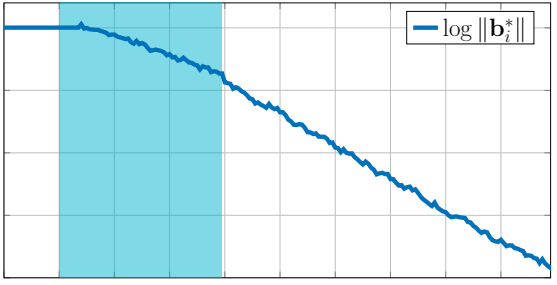

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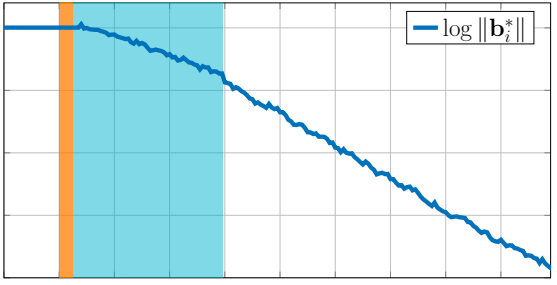

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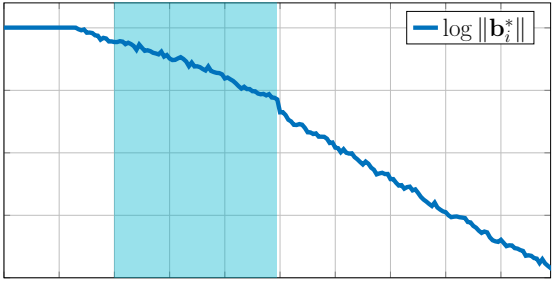

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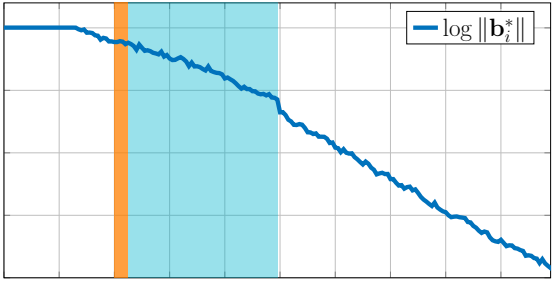

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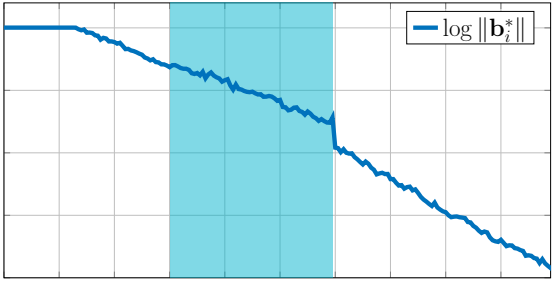

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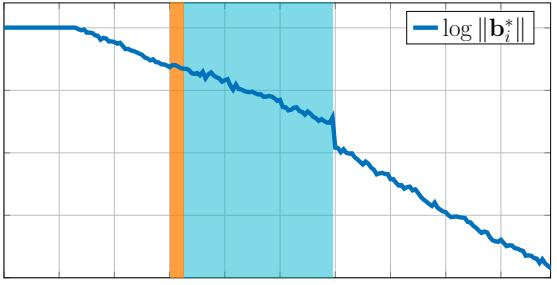

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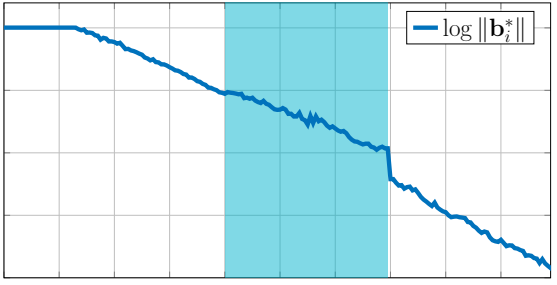

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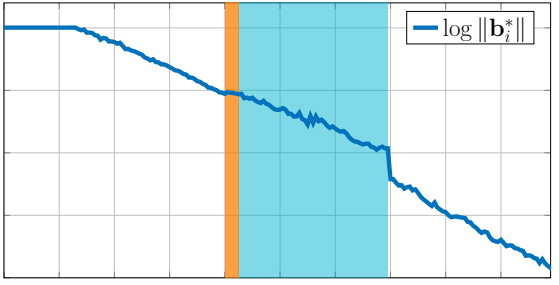

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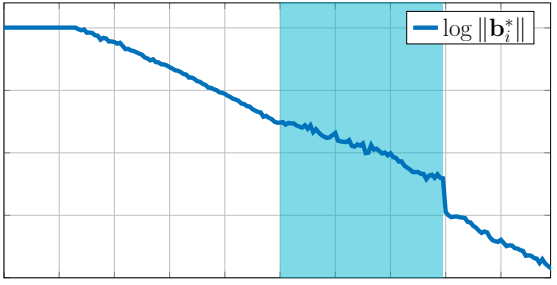

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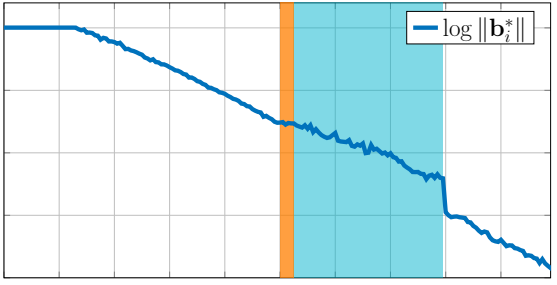

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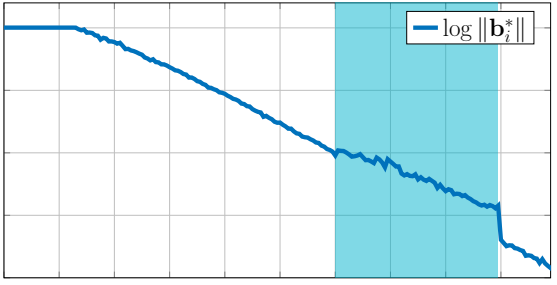

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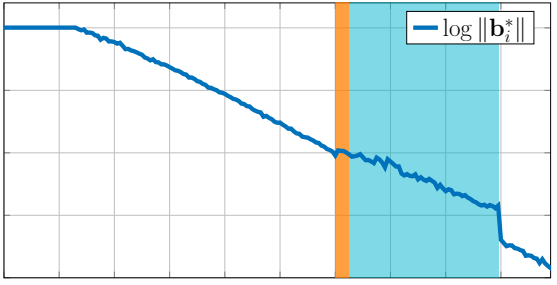

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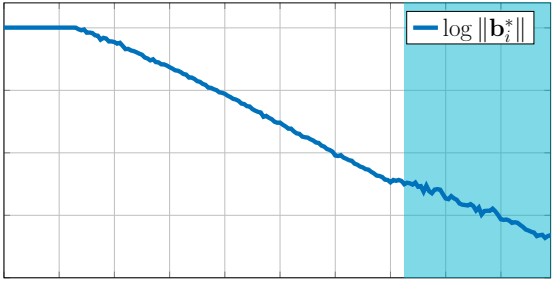

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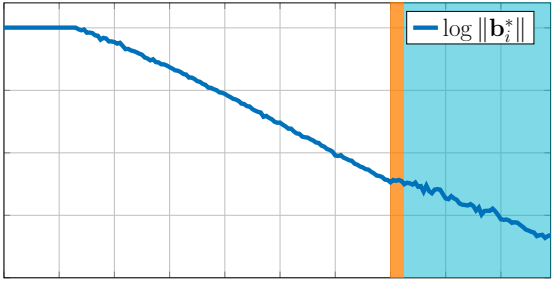

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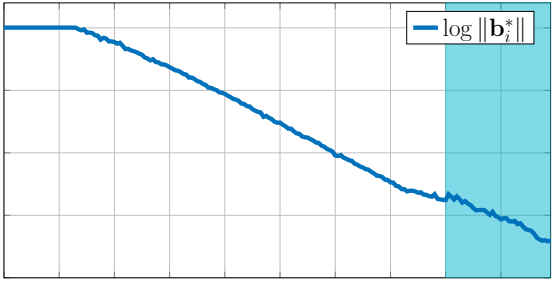

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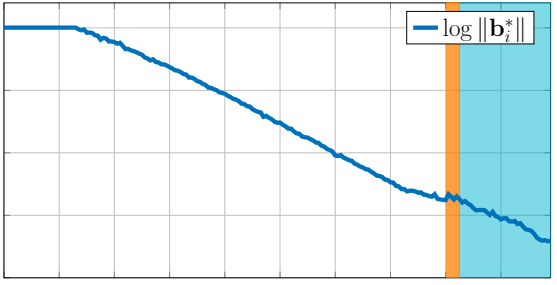
for $i \leftarrow 1$ **to** d **do** // 81
  $j \leftarrow \min(i + \beta - 1, d)$
  $\boldsymbol{v} = x_i \pi_i(\boldsymbol{b}_i) + \cdots + x_j \pi_i(\boldsymbol{b}_j) \leftarrow O_{\mathrm{SVP}}(\pi_i(\boldsymbol{B}[i:j]))$
  if $\|\boldsymbol{v}\| < \|\boldsymbol{b}_i^*\|$ **then**
    $\boldsymbol{v'} \leftarrow x_i \boldsymbol{b}_i + \cdots + x_j \boldsymbol{b}_j$
    extend $\boldsymbol{B}$ by inserting $\boldsymbol{v'}$ into $\boldsymbol{B}$ at index i
    LLL on $\boldsymbol{B}$ to remove linear dependencies
  return *whether an insertion was made*

Lattice basis profile



The BKZ-β tour

BKZ-β tour

for $i \leftarrow 1$ **to** d **do** // 91

- $j \leftarrow \min(i + \beta - 1, d)$
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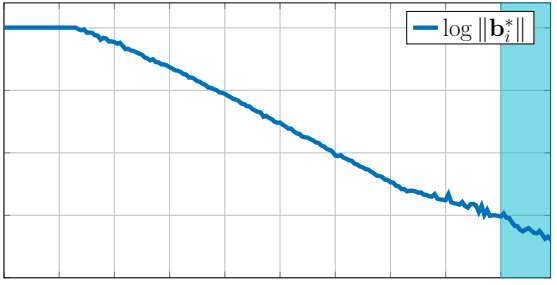

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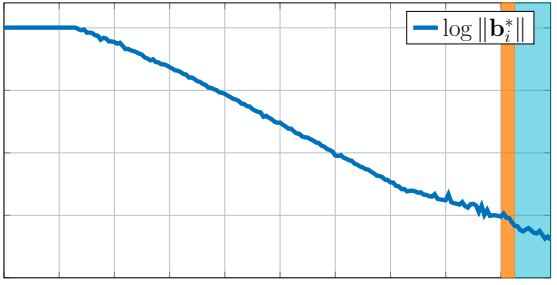

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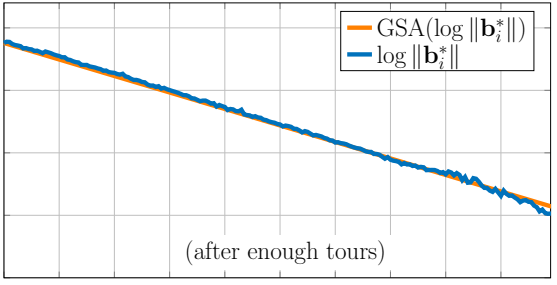

Block-wise lattice reduction

From the BKZ-β tour, we can define two algorithms.

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- BKZ-β [SE91] runs τ tours with block size β.

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Input: $\tau \in \mathbb{Z}_+$
repeat τ *times*
  run a BKZ-β tour
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Block-wise lattice reduction

From the BKZ-β tour, we can define two algorithms.

- BKZ-β [SE91] runs τ tours with block size β.
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Input: $\tau \in \mathbb{Z}_+$
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At the end of each tour we can check if a solution to LWE was found.

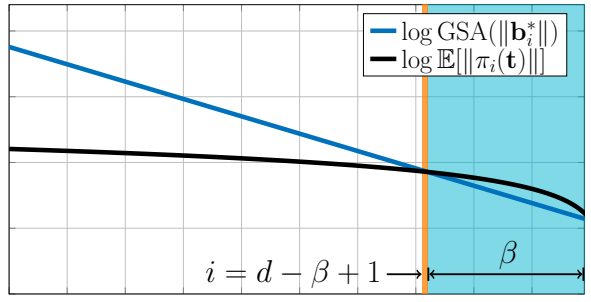
The cost of LWE

- Objective: recover $\boldsymbol{t} = (\boldsymbol{s}, \boldsymbol{e}, 1) \in \Lambda$.
- In practice, recovery of $\boldsymbol{t}$ follows from recovery of $\pi_{d-\beta+1}(\boldsymbol{t})$.

The cost of LWE

- Objective: recover $\boldsymbol{t} = (\boldsymbol{s}, \boldsymbol{e}, 1) \in \Lambda$.
- In practice, recovery of $\boldsymbol{t}$ follows from recovery of $\pi_{d-\beta+1}(\boldsymbol{t})$.
- [ADPS16]: choose smallest β^* s.t. $\|\pi_{d-\beta^*+1}(\boldsymbol{t})\| \leq \mathrm{GSA}(\|\boldsymbol{b}^*_{d-\beta^*+1}\|)$.

[ADPS16] success condition



- [ADPS16]'s approach was originally experimentally verified in [AGVW17].
- Success probabilities near 1 using BKZ-β^*, but stay high also for smaller block sizes $\beta^* - h$.
- This could be an issue if h grows with n.

n	65	80	100
β^*	93.3%	94.2%	88.8%
β^*-5	52.8%	60.6%	39.6%
β^*-10	4.8%	8.9%	5.8%
β^*-15	—	0.2%	0.2%

Table: Measured success probability for solving Search-LWE with secret dimension n, using block size β, as reported in [AGVW17].

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Our main contribution

We extend [ADPS16, DDGR20]'s approach to predict successes for $\beta < \beta^*$.

We extend [DDGR20] to simulate the probability of recovering $\pi_{d-\beta+1}(\boldsymbol{t})$.

Estimating succcess probability of BKZ-β

Input: (n, q, χ, m), β, τ
$p_{\text{tot}} \leftarrow 0, \sigma^2 \leftarrow \mathbb{V}(\chi)$
$d \leftarrow n + m + 1$
for *tour* $\leftarrow 1$ **to** τ **do**
  $\texttt{prof} \leftarrow \text{BKZSim}((n, q, \chi, m), \beta, \text{tour})$
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- Use a basis profile simulator 'BKZSim' (such as [CN11]) to keep track of the basis profile as BKZ tours happen.

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- Accumulate success probabilities as tours progress.[1]

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For PBKZ: same idea, but increase β after τ tours until success probability ≈ 1.

[1] We model the win events as independent, since tours re-randomise the basis (until it is reduced).

Experimental verification

To test our success probability estimations we chose three sets of parameters estimated to need $\beta^* \approx 60$.

n	q	σ	β^*
72	97	1	61
93	257	1	61
100	257	$\sqrt{2/3}$	60

Table: LWE parameters, σ is the standard deviation for a discr. Gaussian distribution.

Experimental verification

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Two batches

- Discrete Gaussian error and secret.
- Binary (resp. ternary) error and secret for $n < 100$ (resp. $n = 100$).

The simulator *does not* receive the distribution as input, just its variance.

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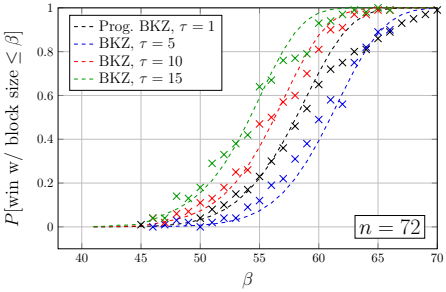
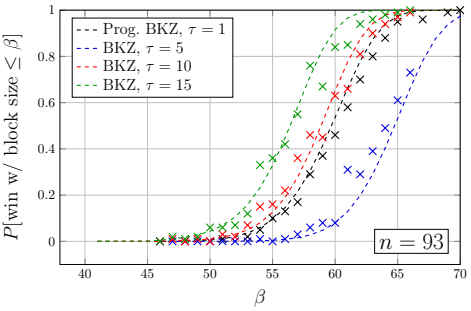
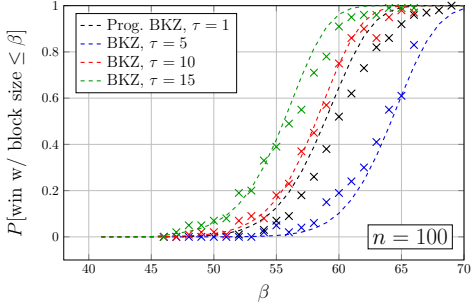
We run many variants of these experiments. We now look at some results, more can be found in the paper.

Gaussian $\boldsymbol{e}$ and $\boldsymbol{s}$, BKZ and PBKZ



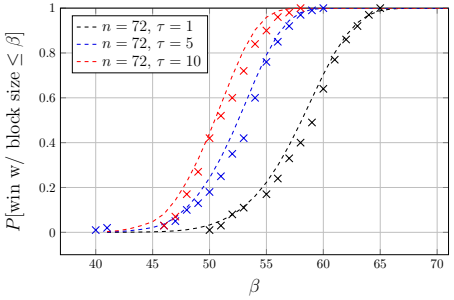


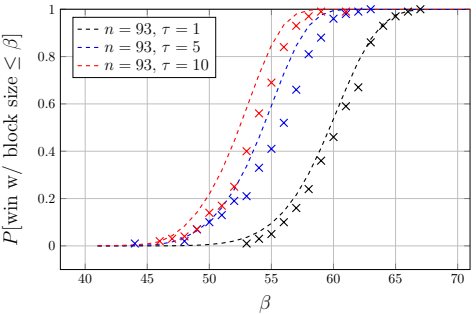


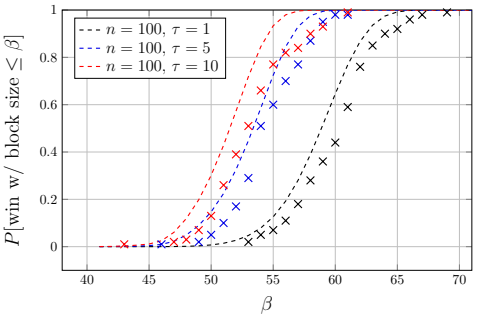
Discrete Gaussian experiments, PBKZ and various tours of BKZ. All discrete Gaussian errors.

Dashed lines are simulations, crosses experiments.

Binary/Ternary $\boldsymbol{e}$ and $\boldsymbol{s}$, PBKZ







Progressive BKZ experiments and simulations for binary and ternary secrets.

No changes are made the simulator.

The effect of sample variance

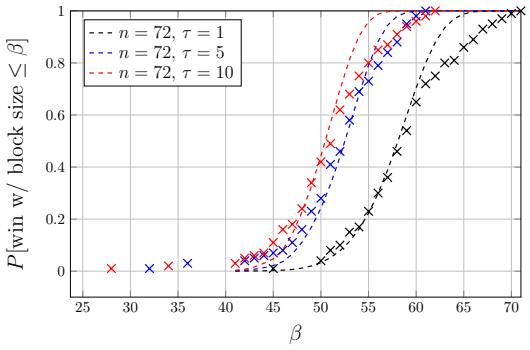
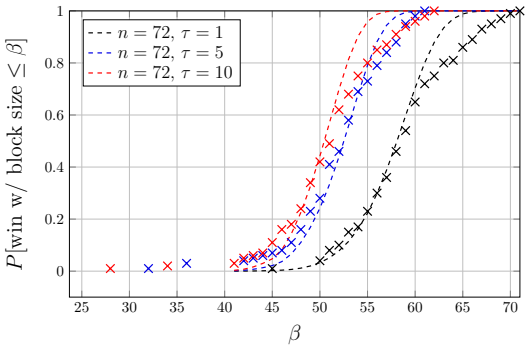


Figure: Gaussian $\boldsymbol{s}$ and $\boldsymbol{e}$ instance, reduced using PBKZ.

The effect of sample variance

- Let $\boldsymbol{t} = (t_1, \ldots, t_d)$ with each $t_i \xleftarrow{iid} D$.
- Let $\bar{t} = \frac{1}{d}\sum_{i=1}^{d} t_i$ be the sample mean.
- Then $s^2 = \frac{1}{d}\sum_{i=1}^{d} (t_i - \bar{t})^2$ is the sample variance.



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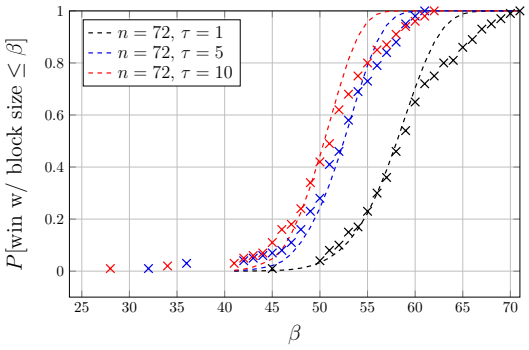
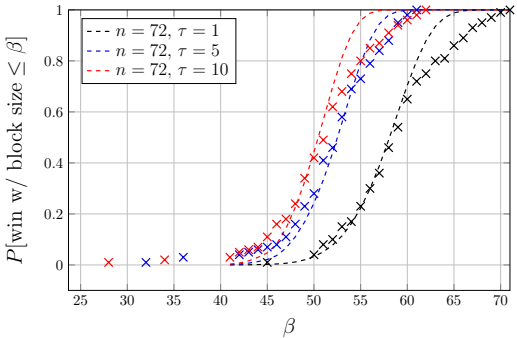
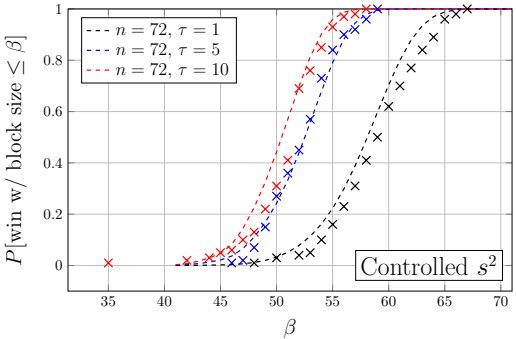


Figure: Gaussian $\boldsymbol{s}$ and $\boldsymbol{e}$ instance, reduced using PBKZ.

Given $\boldsymbol{t}$, s^2 affects the norms $\|\pi_{d-\beta+1}(\boldsymbol{t})\|^2$, not σ^2. Simulations are off when $s^2 \not\approx \sigma^2$.

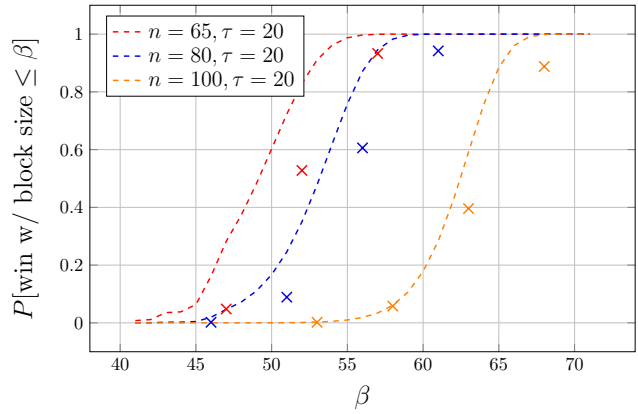
Controlling s^2, Gaussian $\boldsymbol{e}$ and $\boldsymbol{s}$, PBKZ





- On the right we control the sample variance to be within 2% of σ^2.
- Should not affect crypto instances: $\mathbb{V}(s^2) \in O(1/d)$.

Finally, we use the uSVP simulator to explain the [AGVW17] success probabilities.



Our predictions for [AGVW17]'s parameters seem to match their reported probabilities.

- The uSVP simulators allow us to predict success probabilities for $\beta < \beta^*$.
- What is the impact on cryptographic parameters? Consider β a random variable.

	BKZ 2.0, $\tau = 15$			PBKZ, $\tau = 1$		PBKZ, $\tau = 5$	
scheme	β^*	$\mathbb{E}(\beta)$	$\sqrt{\mathbb{V}}(\beta)$	$\mathbb{E}(\beta)$	$\sqrt{\mathbb{V}}(\beta)$	$\mathbb{E}(\beta)$	$\sqrt{\mathbb{V}}(\beta)$
Kyber 512	381	386.06	2.56	389.53	2.88	385.70	2.32
Kyber 768	623	634.41	2.96	638.23	3.30	634.00	2.66
Kyber 1024	873	891.13	3.31	895.24	3.66	890.63	2.96
LightSaber	404	408.81	2.65	412.24	2.96	408.35	2.39
Saber	648	659.36	3.00	663.10	3.32	658.85	2.68
FireSaber	890	907.76	3.34	911.78	3.68	907.16	2.97
ntruhps2048509	374	375.93	2.58	379.56	2.92	375.71	2.36
ntruhps2048677	521	522.78	2.82	526.77	3.18	522.67	2.57
ntruhps4096821	621	628.78	2.83	632.54	3.17	628.43	2.55
ntruhrss701	471	477.20	2.48	480.51	2.77	476.72	2.23

Table: Cryptographic parameters for NIST PQC finalists, using 2nd round parameters.

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LightSaber	404	408.81	2.65	412.24	2.96	408.35	2.39
Saber	648	659.36	3.00	663.10	3.32	658.85	2.68
FireSaber	890	907.76	3.34	911.78	3.68	907.16	2.97
ntruhps2048509	374	375.93	2.58	379.56	2.92	375.71	2.36
ntruhps2048677	521	522.78	2.82	526.77	3.18	522.67	2.57
ntruhps4096821	621	628.78	2.83	632.54	3.17	628.43	2.55
ntruhrss701	471	477.20	2.48	480.51	2.77	476.72	2.23

Table: Cryptographic parameters for NIST PQC finalists, using 2nd round parameters.

- The uSVP simulators allow us to predict success probabilities for $\beta < \beta^*$.
- What is the impact on cryptographic parameters? Consider β a random variable.

	BKZ 2.0, $\tau = 15$			PBKZ, $\tau = 1$		PBKZ, $\tau = 5$	
scheme	β^*	$\mathbb{E}(\beta)$	$\sqrt{\mathbb{V}}(\beta)$	$\mathbb{E}(\beta)$	$\sqrt{\mathbb{V}}(\beta)$	$\mathbb{E}(\beta)$	$\sqrt{\mathbb{V}}(\beta)$
Kyber 512	381	386.06	2.56	389.53	2.88	385.70	2.32
Kyber 768	623	634.41	2.96	638.23	3.30	634.00	2.66
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Observations: $\mathbb{E}(\beta) > \beta^*$, $\mathbb{V}(\beta)$ stays small.

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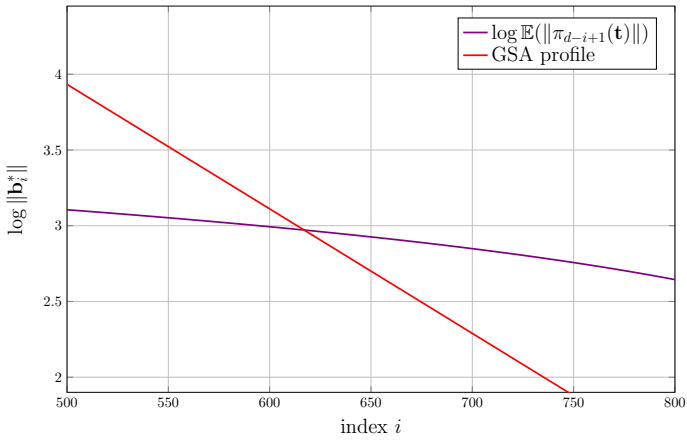
- $\mathbb{V}(\beta)$ stays small: successful β can not be much smaller than $\mathbb{E}(\beta)$. Low probability attacks by just picking smaller β should not be significantly cheaper.
- $\mathbb{E}(\beta) > \beta^*$: it seems [ADPS16] underestimates the hardness of LWE. This is surprising, by accounting for $\beta < \beta^*$ we could expect the opposite effect.

$\mathbb{E}(\beta) > \beta^*$ is caused by internally using the [CN11] simulator in place of the GSA.

- We modified the LWE estimator to use [CN11] in place of the GSA.
- Let's look at Kyber 512.



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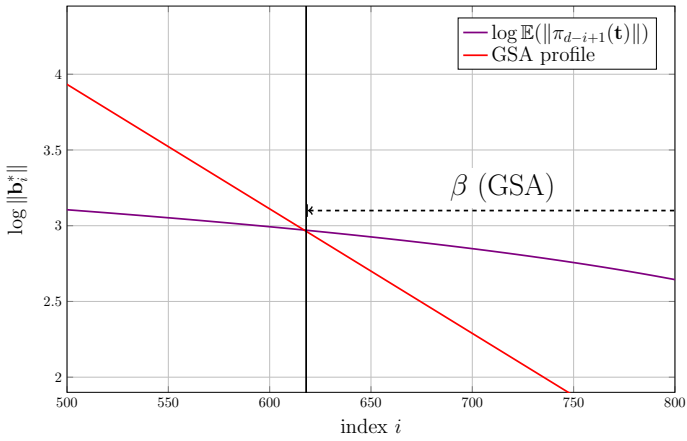

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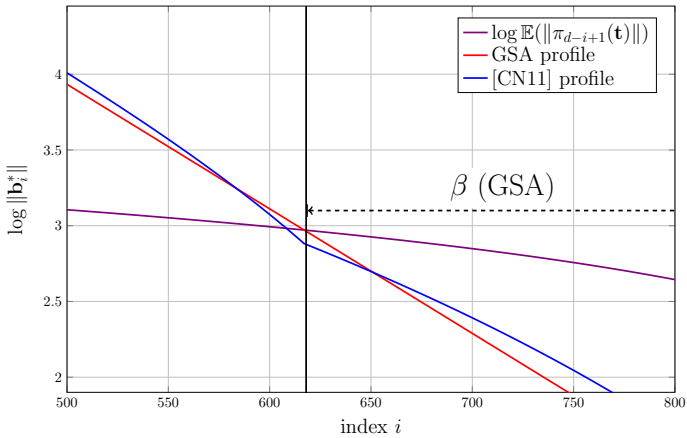

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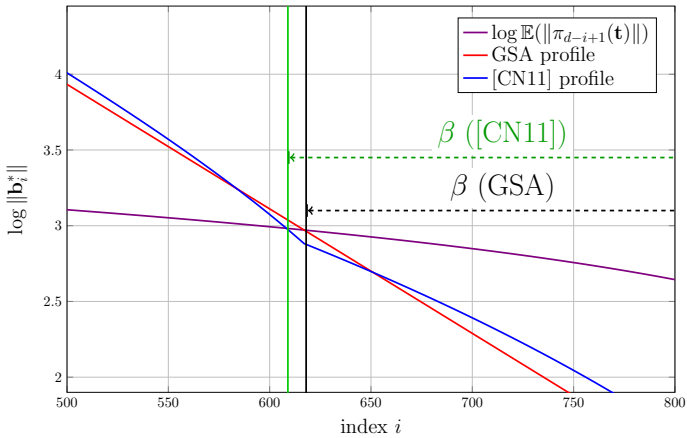

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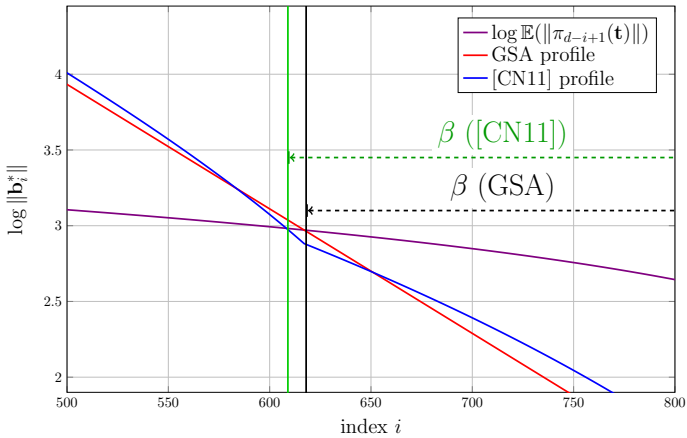
- This effect carries to our uSVP simulators, and similarly to [DDGR20]'s code.



Conclusions

- We capture the success probabilities of smaller-than-expected-successful block sizes.
- Effect seem consistent across secret and error distributions.
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Thank you


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