Rate-1 Quantum Fully Homomorphic Encryption

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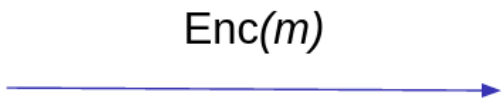
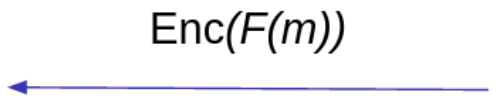
Fully Homomorphic Encryption Scheme

A FHE scheme allows Alice to encrypt some message m such that later Bob (holding any function F) can compute $\mathrm{Enc}(F(m))$.



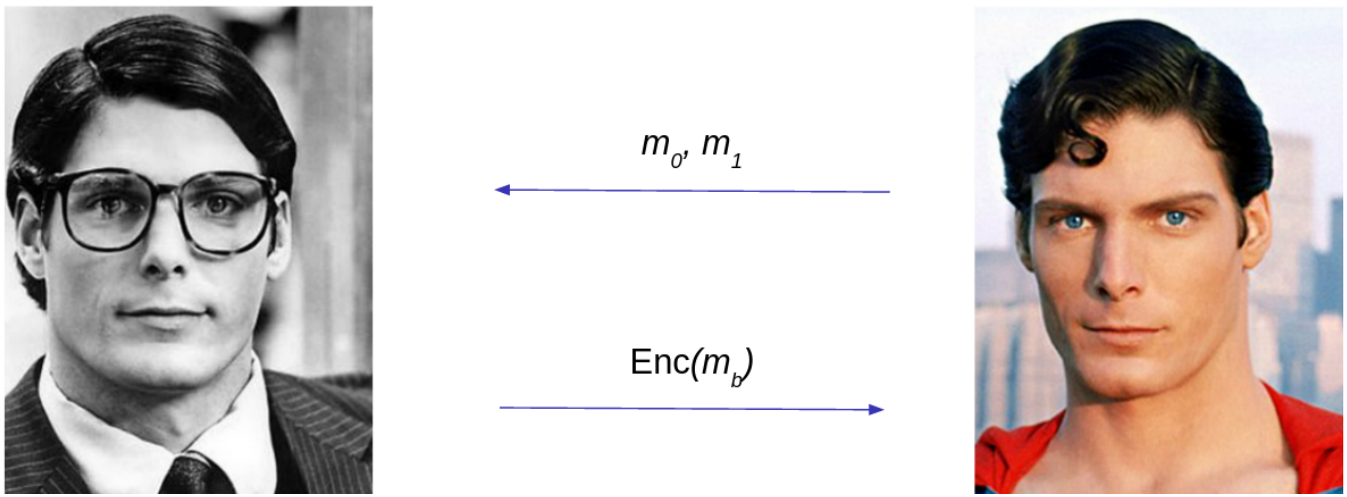


Fully Homomorphic Encryption Scheme

Semantic Security

- Semantic Security: Alice's input m must be hidden in an indistinguishability sense.

$$\mathrm{FHE.Enc}(pk, m_0) \approx_c \mathrm{FHE.Enc}(pk, m_1)$$



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- **Semi-Honest Circuit Privacy**: Statistical circuit privacy is required to hold only for well-formed messages from Alice (pk and ciphertext).

vs

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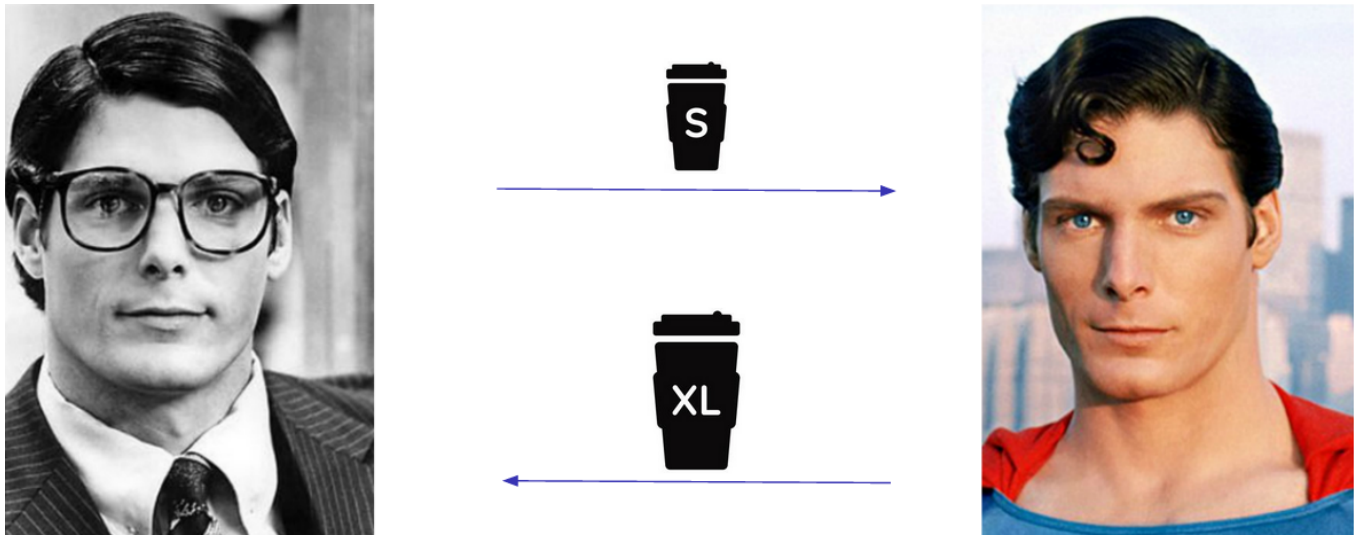
Malicious circuit privacy in classical FHE

Any classical FHE scheme can be converted into one with malicious circuit privacy [OPP14].

Fully Homomorphic Encryption Scheme

Communication Complexity



Ensure that communication overhead introduced for security doesn't nullify the efficiency of outsourcing computations.

Fully Homomorphic Encryption Scheme

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- Useful in applications such as:
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  - private information retrieval [CGKS95]
- Best communication complexity approaches that of the insecure protocol (where Alice sends her input m in plain), assuming the hardness of LWE [BDGM19].

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- Rate ρ: $\frac{\text{size of } F(m)}{\text{size of FHE.Eval}(F,m)} \geq \rho$.
- Rate-1 schemes asymptotically approach rate $\rho = 1$.

Quantum FHE Scheme

In a Quantum FHE scheme Alice can encrypt a quantum state $|\psi\rangle$ whereas Bob (holding any unitary U) can compute

$\text{Enc}(|\psi\rangle)$

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Communication complexity with a quantum output?

Results

We construct quantum FHE in a malicious setting with communication complexity:

$$(|\,|\psi\rangle\,| + |C(|\psi\rangle)|) \cdot (1 + o(1))$$

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Assuming the quantum hardness of LWE, there exists a (levelled) QFHE scheme with rate-1.

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Classical Techniques for Quantum FHE

Interference

Elements of a superposition representing the same bit string but with opposite amplitudes must cancel out.

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Example

- Hadamard Transformation:

$$H(|0\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \; H(|1\rangle) = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

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- $\frac{1}{2}(|Enc(0)\rangle + |Enc(1)\rangle + |Enc(0)\rangle - |Enc(1)\rangle)$

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- For a superposition $|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$ we sample $\text{otk} = (x, z) \in \{0,1\}^2$ and implement the QOTP as

$$\text{QOTP}(\text{otk}, |\phi\rangle) \equiv X^x Z^z |\phi\rangle = \alpha |x\rangle + (-1)^z \beta |1 \oplus x\rangle$$

Quantum FHE [Mah18]

$X^x Z^z |\psi\rangle$, $Enc(x,z)$

$X^{x'} Z^{z'} U |\psi\rangle$, $Enc(x',z')$

Quantum FHE

Mahadev Protocol

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  - Clifford Gates: $U\, X^x Z^z |\psi\rangle = X^{x'} Z^{z'} U |\psi\rangle$.
  - Toffoli Gate: quantum operation dependent on the classically encrypted keys.
- Quantum Capable FHE Scheme: A classical FHE that can be used to evaluate quantum circuits.

Quantum FHE

Properties of Interest

- The scheme has hybrid ciphertexts consisting of:
  - a Quantum OTP state.
  - a classical encryption of the completely classical *otk*.
- The classical component of the ciphertext satisfies (semi-honest) circuit privacy.

Quantum FHE

Our Contribution

Maliciously Circuit Private QFHE

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Rate-1 QFHE

We construct a QFHE scheme with nearly optimal ciphertext expansion.

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Existing QFHE Rate

- Rate ρ (Mahadev's scheme):

$$\frac{|C(|\psi\rangle)|}{|\mathsf{QFHE.QEval}\,(pk, C, |\phi\rangle)|} = \frac{\ell}{\ell + \text{size of HE keys}} \geq \rho$$

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- Classical FHE is not rate-1.

- Overall Inverse Polynomial Rate!

Failed Approach

- Shrink classical information.

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- Stuck with FHE.Enc(pk, otk), where two classical bits are necessary to encrypt a qubit.

Spooky Interactions [BDGM19]

- Some FHE schemes pack k classical bits $(m_1, \ldots, m_k)$ in ciphertexts of the form $c = (\boldsymbol{c}_0, c_1, \ldots, c_k) \in \mathbb{Z}_q^{n+1} \times \{0,1\}^k$.

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- The last k-bits of the ciphertexts are *non-locally* correlated with the secret key sk.
- Spooky Decryption:

$$\mathsf{Dec}(sk, c) = F(sk, \boldsymbol{c}_0) \oplus (c_1, \ldots, c_k)$$

Construction

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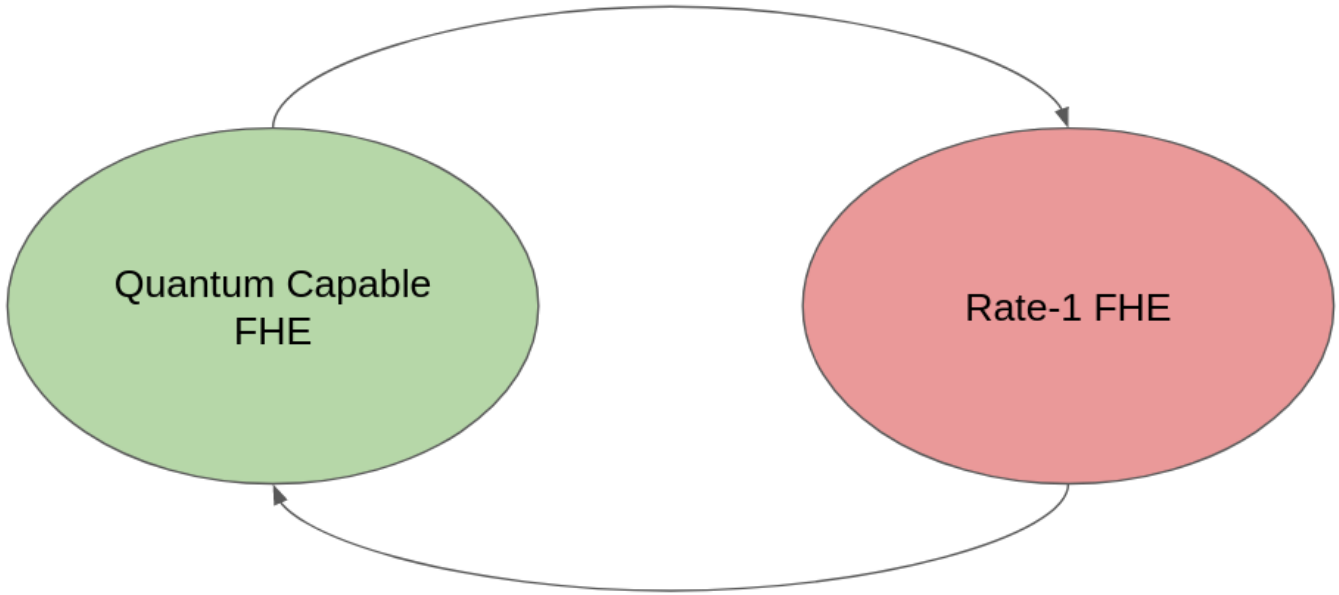
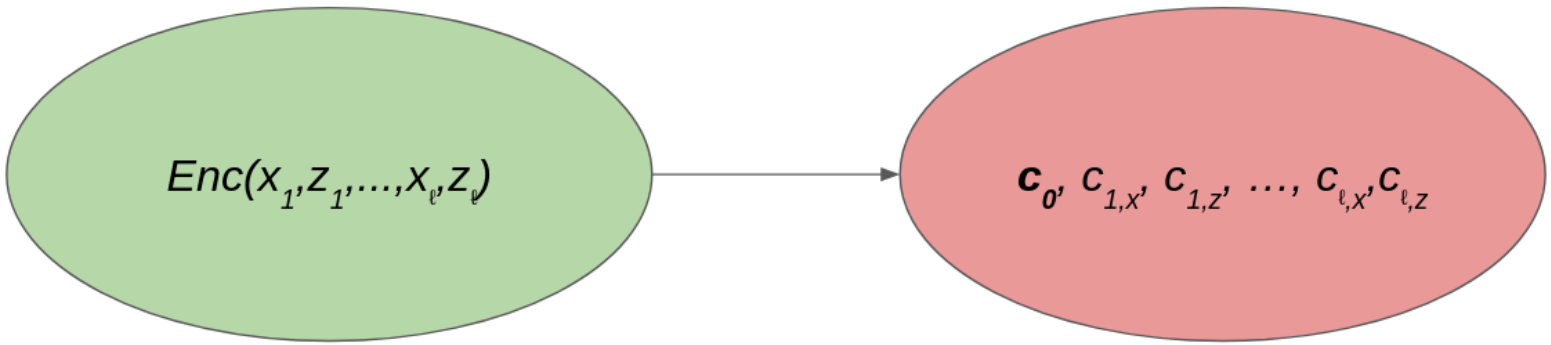


- Use the quantum capable FHE to perform homomorphic evaluations and the rate-1 to store the classical information.

QFHE Construction

- Convert Enc(pk, otk) into an FHE ciphertext with spooky decryption via bootstrapping:



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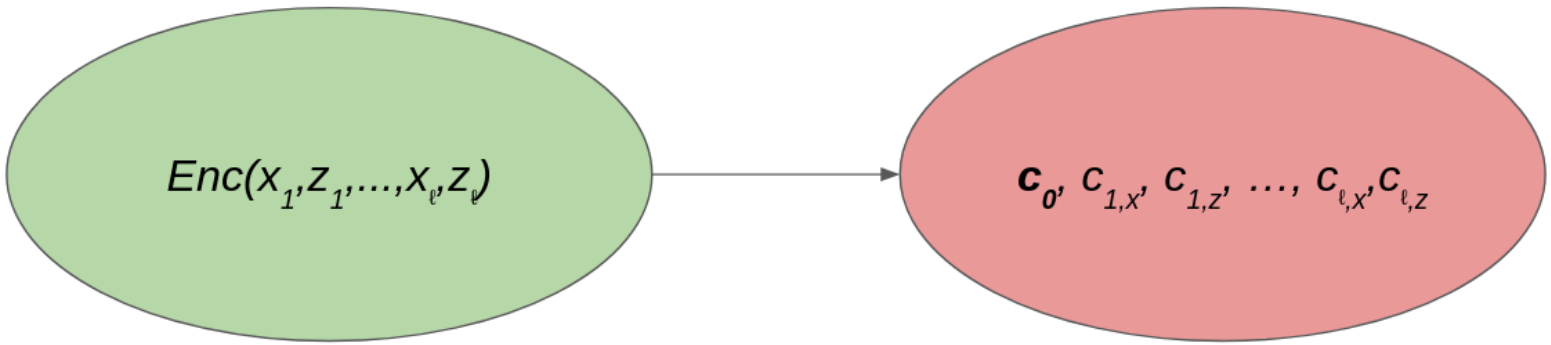

- Return $\boldsymbol{c}_0$ and $\bigotimes_{i\in[l]} (X^{c_{i,x}} Z^{c_{i,z}}) \cdot \mathrm{QOTP}(\mathrm{otk}, |\psi\rangle)$.

QFHE Construction

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- Alter the one time key: $x_i, z_i \rightarrow x_i \oplus c_{i,x}, z_i \oplus c_{i,z}$
- The result of the function F is:

$$\begin{aligned} F(sk, \boldsymbol{c}_0) &= \mathrm{Dec}(sk, c) \oplus (c_{1,x}, c_{1,z}, \ldots, c_{\ell,x}, c_{\ell,z}) \\ &= (x_1, z_1, \ldots, x_\ell, z_\ell) \oplus (c_{1,x}, c_{1,z}, \ldots, c_{\ell,x}, c_{\ell,z}) \\ &= \text{the updated one time key} \end{aligned}$$

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- By setting the parameters accordingly, the rate asymptotically approaches 1, assuming standard LWE.

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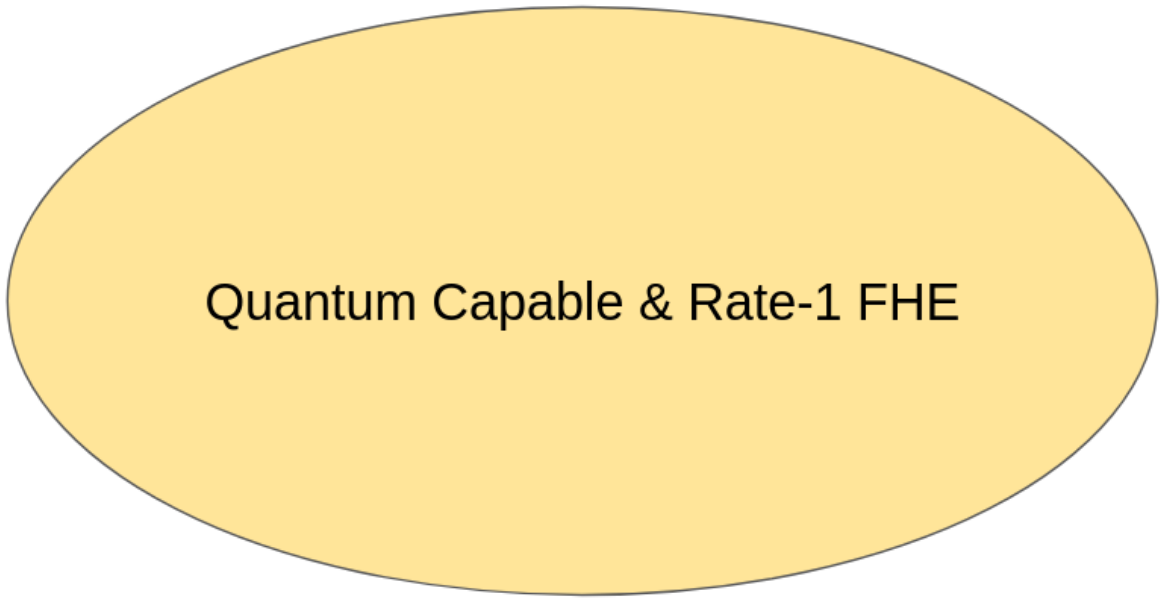

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- https://eprint.iacr.org/2020/1454
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