Synthesizing Quantum Circuits of AES with Lower T-depth and Less Qubits

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Outline

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Motivation

- Quantum Cryptanlysis for Symmetric Ciphers
  - Grover's algorithm: the attacker needs to construct a Grover oracle to search the key.
  - Simon's algorithm (Kuwakado and Mori, ISIT 2010; Kaplan et al. Crypto 2016): The attacker needs to access an online quantum encryption oracle.
  - Offline Simon's algorithm (Bonnetain et al. Asiacrypt 2019): the attacker needs to construct different quantum encryption oracles for different keys.
  - **The quantum circuit for the encryption process** is a part of the Grover oracle or the quantum encryption oracle.
- NIST's call for proposals for PQC
  - **The complexity of quantum key search circuit for AES** is used as a baseline to categorize the post-quantum public-key schemes

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From classical circuits to quantum circuits:

- Classicial gates: XOR, NOT, AND
  $\Rightarrow$ CNT gate set: CNOT, NOT(Pauli-X), Toffoli
  $\Rightarrow$ Clifford+T gates: $\{$Pauli gates, CNOT, S, H$\}$ + T

Optimization Goals:

- Width: the number of qubits
- Gate count
- Depth: The number of layers of the circuit (gates acting on disjoint sets of qubits can be applied in parallel)
- In fault-tolerant quantum computation (Surface code), the cost of the T **gate is greatly higher** than that of a Clifford gate, and the running time of a circuit is **dominated by the T-depth**.

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The Pipeline Structure

Round Transformation:

- $Round_i : (key_i, x) \rightarrow (key_i, O(R_i))$
- $O(R_i)$: the output of the round function
- $\mathcal{R}_i : |key_i\rangle |x\rangle |0\rangle \rightarrow |key_i\rangle |x\rangle |O(R_i)\rangle$, **an out-of-place implementation**

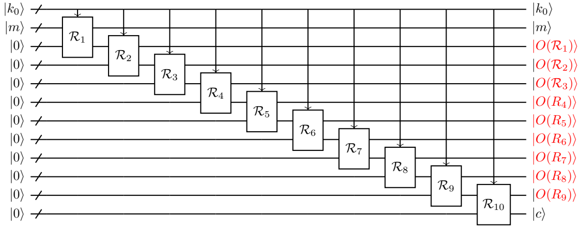

Figure: The pipeline structrue for AES-128

- Generates redundant output $O(\mathcal{R}_i)$ after each round

The Zig-zag Structure

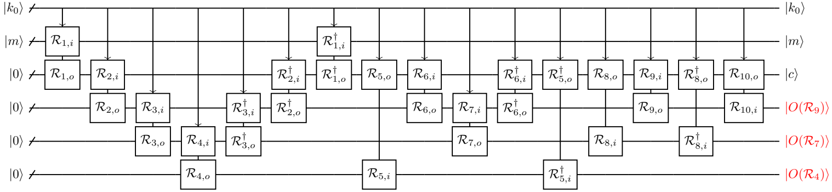

Figure: The zig-zag structure for AES-128

- The reverse circuit $R^\dagger$ is used to clean some redundant outputs.

The Out-of-Place Based Round-in-Place Structure

- For symmetric ciphers, each round is invertible, so theoretically there is an in-place quantum circuit for each round.
- However, directly obtain a such in-place circuit is very hard.
- We can construct it by combing two out-of-place sub-circuits.
  - Round transformation $R : (k, x) \to (k, T(x, k))$
  - T': the inverse function of T, $T'(k, T(x, k)) = x$

Figure: The op-based in-place circuit

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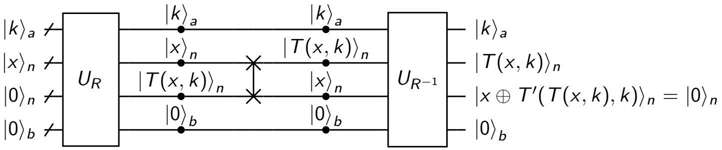

Figure: The op-based in-place circuit

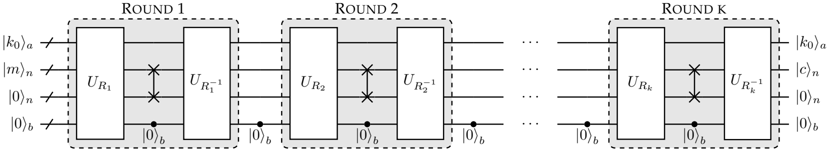

Figure: The OP-based round-in-place structure

- The width does not increase after each round

Comparison of Different Structures

- n qubits for input, n qubits for output, and r rounds.
- Same out-of-place round circuit using αn ancilla qubits .

Table: The widths (number of qubits) of different structures, where t is the minimal number such that $\sum_{i=1}^{t} i > r$.

Pipeline	Zig-zag	Round-in-place
$(r+\alpha+1)n$	$(t+1+\alpha)n \approx (\sqrt{2r}+\alpha)n$	$(2+\alpha)n$

Table: The depths and DW-costs of the oracles based on different structures

Metric	Type	Pipeline	Zig-zag	Round-in-place
Depth	Grover	$2r \cdot d$	$\approx 4r \cdot d$	$4r \cdot d$
	Encrypt	$2r \cdot d$	$\approx 4r \cdot d$	$2r \cdot d$
DW-cost	Grover	$2r(r+1+\alpha)nd$	$2r(\sqrt{2r}+\alpha)nd$	$4r(2+\alpha)nd$
	Encrypt	$2r(r+1+\alpha)nd$	$2r(\sqrt{2r}+\alpha)nd$	$2r(2+\alpha)nd$

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Synthesizing Optimal CNOT Circuits

- Invertible linear transformation :
  $|x_1, x_2, \ldots, x_n\rangle \to |L_1(x_1, \ldots, x_n), \ldots, L_n(x_1, \ldots, x_n)\rangle$
- **Invertible linear transformation $\Rightarrow$ In-place CNOT circuit**
  - CNOT gate: $|x_1, x_2\rangle \to |x_1, x_1 \oplus x_2\rangle$, seen as a row addition elementary matrix
  - PLU decomposition: number of gates is large
  - Heuristic algorithm (Xiang et al. FSE 2020): greatly reduce the number of gates, but is not optimal.
- A new SAT-based method for implementing linear transformations with minimal number of CNOT gates
  - Encode the problem of finding a circuit with k gates into a SAT problem
  - $k-1$ (UNSAT), k (SAT) $\Rightarrow$ using k gates is optimal

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The Way of Encoding

- Variable sets: $B = (b_{ij})_{k\times n}$, $C = (c_{ij})_{k\times n}$, $F = (f_{ij})_{n\times n}$, $\Psi = \{\psi_{i,j,s}\}_{k\times n\times n}$.
- B, C: $b_{ij_1} = c_{ij_2} = 1 \Rightarrow$ CNOT$_i$: Adds Wire$_{j_1}$ to Wire$_{j_2}$.
- F: $f_{ij} = 1 \Rightarrow L_i$ is the output of Wire$_j$.
- Ψ: $\psi_{i,j,k} = 1 \Rightarrow$ After CNOT$_i$, in the boolean expression (ANF) of Wire$_j$, coeff(x_k) is 1.

Boolean Equations for the CNOT circuit problem

$$EQN_b = \left\{ \begin{array}{c} b_{ij_1} b_{ij_2} = 0, \\ b_{i1} + b_{i2} + \cdots + b_{in} + 1 = 0, \\ \text{for } 1 \leq i \leq k, 1 \leq j_1 \neq j_2 \leq n \end{array} \right\} \quad EQN_c = \left\{ \begin{array}{c} c_{ij_1} c_{ij_2} = 0, \\ c_{i1} + c_{i2} + \cdots + c_{in} + 1 = 0, \\ \text{for } 1 \leq i \leq k, 1 \leq j_1 \neq j_2 \leq n \end{array} \right\}$$

$$EQN_a = \left\{ \begin{array}{c} f_{i,j}(\psi_{k,j,s} + a_{is}) = 0 \\ \text{for } 1 \leq i,j \leq n, 1 \leq s \leq m \end{array} \right\} \quad EQN_f = \left\{ \begin{array}{c} f_{ij_1} f_{ij_2} = 0, \\ f_{i1} + f_{i2} + \cdots + f_{in} + 1 = 0, \\ \text{for } 1 \leq i \leq n, 1 \leq j_1 \neq j_2 \leq n \end{array} \right\}$$

$$EQN_\psi = \left\{ \begin{array}{c} \psi_{i,j,s} + \sum_{t=1}^{n} c_{ij} b_{it} \psi_{i-1,t,s} + \psi_{i-1,j,s} = 0, \\ \text{for } 1 \leq i \leq k, 1 \leq j \leq n, 1 \leq s \leq m \end{array} \right\}$$

- Problems with size < 9 bits can be solved in a reasonable time.
  - 8-bit: 56 threads, SAT:200-300 sec, UNSAT: 1 day

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$\mathfrak{C}^0$-circuit and $\mathfrak{C}^*$-circuit

$\mathfrak{C}^0$-circuit of f: $|x\rangle_a |0\rangle_b |0\rangle_c \to |x\rangle_a |f(x)\rangle_b |0\rangle_c$.

$\mathfrak{C}^*$-circuit of f : $|x\rangle_a |y\rangle_b |0\rangle \to |x\rangle_a |y \oplus f(x)\rangle_b |0\rangle_c$

- A $\mathfrak{C}^*$-circuit is always a $\mathfrak{C}^0$-circuit.
- Building a $\mathfrak{C}^0$-circuit is much easier than building a $\mathfrak{C}^*$-circuit.
- Some circuits using the output wires as temporary storage space to save the cost of qubits, are $\mathfrak{C}^0$-circuits but not $\mathfrak{C}^*$-circuits.
  - AES S-box circuits proposed in [GLRS16,ASAM18,LPS19]

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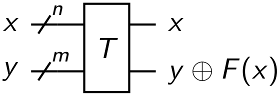
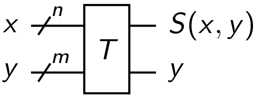
Implementing nonlinear transformations in-place (I)



1) Feistel-like



2) Substitution-like

Figure: Two kinds of classical invertible nonlinear transformations

- **Feistel-like** (Fesitel cipher, NFSR, Key schedule).

$$T : (x, y) \to (x, y \oplus F(x))$$

To implement T in-place, we only need a $\mathfrak{C}^*$-circuit of F:
$|x\rangle\,|y\rangle \to |x\rangle\,|y \oplus F(x)\rangle$;

Implementing nonlinear transformations in-place (II)

- **Substitution-like** (S-box):

$$T : (x, y) \to (S(x, y), y)$$

can be implemented by the OP-based in-place circuit

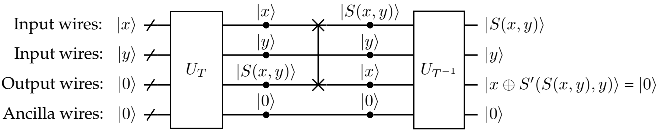


Figure: An OP-based in-place circuit for a substitution-like transformation. S': a function satisfying $S'(S(x, y), y) = x$.

1. $|x\rangle |y\rangle |0\rangle \to |x\rangle |y\rangle |S(x, y)\rangle$: $\mathfrak{C}^0$-circuit of S
2. $|S(x, y)\rangle |y\rangle |x\rangle \to |S(x, y)\rangle |y\rangle |x \oplus S'(S(x, y), y)\rangle$: we don't need a $\mathfrak{C}^*$-circuit of S'.

   $z = S(x, y)$, $|z\rangle |y\rangle |S'(z, y)\rangle \to |z\rangle |y\rangle |0\rangle$. Only need to design a $\mathfrak{C}^0$-circuit of S', and use its reverse circuit.

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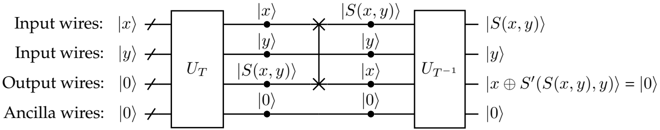

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Constructing a $\mathfrak{C}^*$-circuit from a $\mathfrak{C}^0$-circuit

- Some criteria for efficiently designing $\mathfrak{C}^*$-circuits are proposed
- Under these criteria, a $\mathfrak{C}^*$-circuit of f can be constructed from a special $\mathfrak{C}^0$-circuit called **Simplex $\mathfrak{C}^0$-circuit**:

$$|x\rangle_a |y\rangle_b |0\rangle_c \to |x\rangle_a |A(y) \oplus f(x)\rangle_b |0\rangle_c\,, \quad A: \text{ a linear function}$$

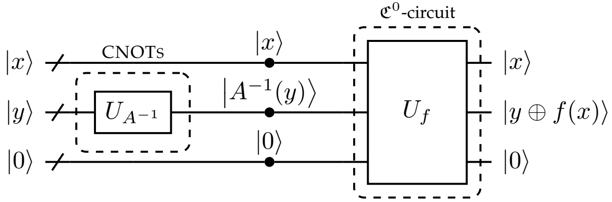


Figure: A $\mathfrak{C}^*$-circuit based on a simplex $\mathfrak{C}^0$-circuit

- $U_{A^{-1}}$: a CNOT sub-circuit; in most times uses ≤ 8 qubits.

Application in AES Key Schedule

- We can construct a $\mathfrak{C}^*$-circuit of AES S-box (used in the key schedule) from the $\mathfrak{C}^0$-circuit proposed in previous works without increasing #qubit and #Toffoli

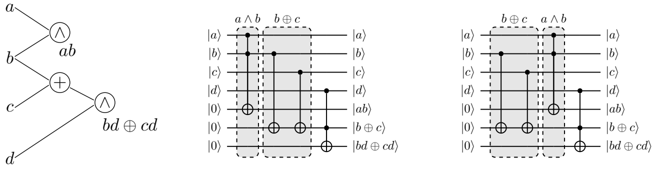
Table: Quantum resources for implementing the S-box of AES

	#ancilla	Toffoli-depth	#Toffoli	#CNOT	#NOT	source
$\mathfrak{C}^0$-S-box	6	41	52	326	4	Asiacrypt2020
$\mathfrak{C}^*$-S-box	7	60	68	352	4	Asiacrypt2020
	6	41	52	336	4	This paper

Outline

Classical AND-depth v.s. Quantum T-depth

- T gates only appears in the Toffoli gates, quantum AND gates (Eurocrypt 2020, a $\mathfrak{C}^0$ circuit of AND), and their adjoint (all have **T-depth-1 implementations**).
- Classical AND-depth = Quantum T-depth ?



(1) classical circuit (2) T-depth-2 (3) T-depth-1

Figure: Quantum implementations of a classical circuit with AND-depth 1

- AES S-box circuit in Eurocrypt 2020: AND-depth 4, but T-depth 6.

The Lowest-T-depth Circuit

Theorem

Given a classical circuit with **AND-depth s**, the T-depth of the quantum circuit implementing all the nodes of the classical circuit **is not smaller than s**. Moreover, with sufficiently many ancillae, we can construct a quantum circuit implementing all the nodes of the classical circuit with **T-depth s** .

- Based on Boyar and Peralta's classical circuit for AES S-box (AND-depth-4), we construct a T-depth-4 quantum circuit for AES S-box.
- We construct a new improved classical circuit for AES S-box (AND-depth-3), and induce a T-depth-3 quantum circuit for AES S-box.
- AES S-box has algebraic degree 7. Needs at least 3 multiplication layers, hence T-depth-3 is optimal.

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Low-width Circuits for AES

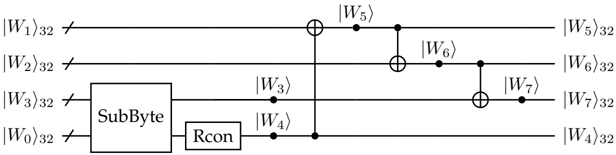

Figure: An in-place circuit for generating the first round key

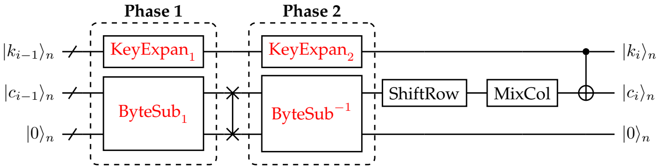

Figure: The in-place implementation of the i-th round of AES-128

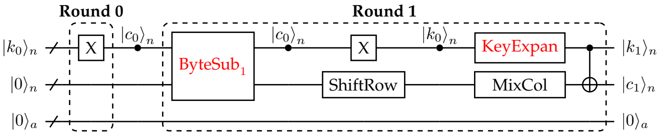

Figure: The implementation of the round 0 and round 1 of AES

Table: Quantum resources for implementing AES-128

Width	Toffoli-Depth	#Toffoli	#CNOT	#Pauli-X	source
512	2016	19788	128517	4528	Asiacrypt2020
492	820	17888	126016	2528	$p = 18$
374	1558	17888	126016	2528	$p = 9$

- p: number of S-boxes applied in parallel

Low-depth circuits for AES

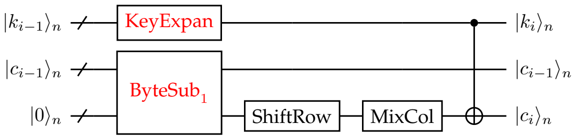

Figure: The out-of-place implementation of the i-th round of AES-128

Table: Quantum resources for implementing AES and AES†.

#CNOT	#1qClifford	$\#T$	#M	T-depth	Full depth	width	source
291150	83116	54400	13600	120 (60)	2827	3936	Eurocrypt 2020
298720	83295	54400	13600	80 (40)	2198	3936	with S-box$_3$
570785	189026	124800	31200	60 (30)	2312	5576	with S-box$_4$

- S-box$_4$: T-depth-4 S-box, S-box$_3$: T-depth-3 S-box
- ($*$) : for only implementing the forward circuit of AES

Tradeoff between Width and T-depth

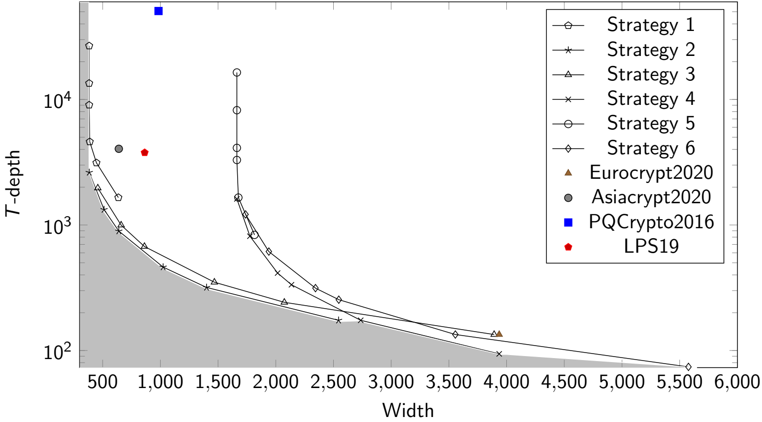

Figure: The width and T-depth for implementing the Grover oracle of AES-128

Thank you for your attention!

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