SIDH Proof of Knowledge

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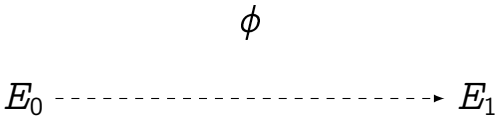
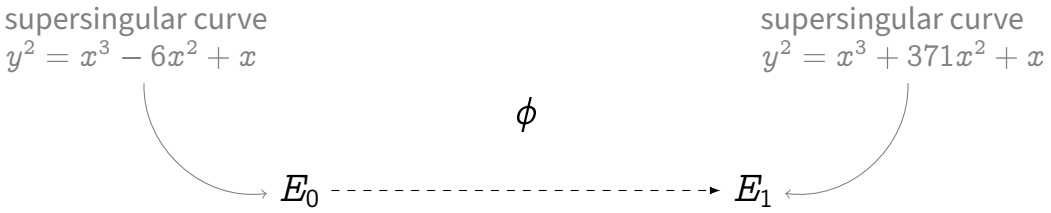
Proofs of Isogeny Knowledge…Why?

$$\phi : E_0 \longrightarrow E_1$$

- Signatures,
- Verifiable *<insert your favorite primitive>*,
- Non-interactive SIDH (†) key exchange,
- …Why not?


ϕ
E_0
E_1


supersingular curve
$y^2 = x^3 - 6x^2 + x$
supersingular curve
$y^2 = x^3 + 371x^2 + x$
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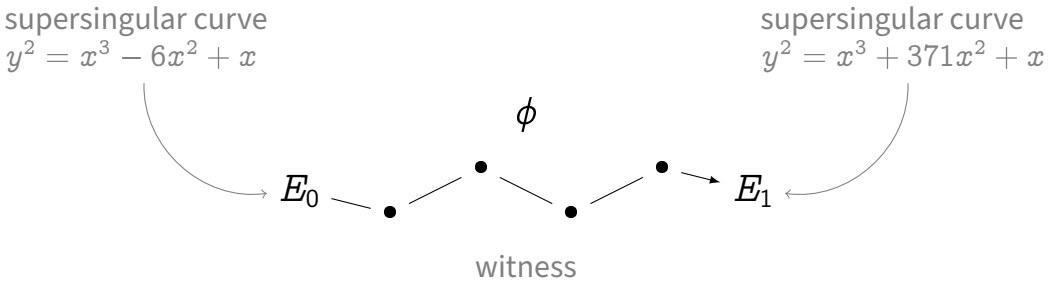
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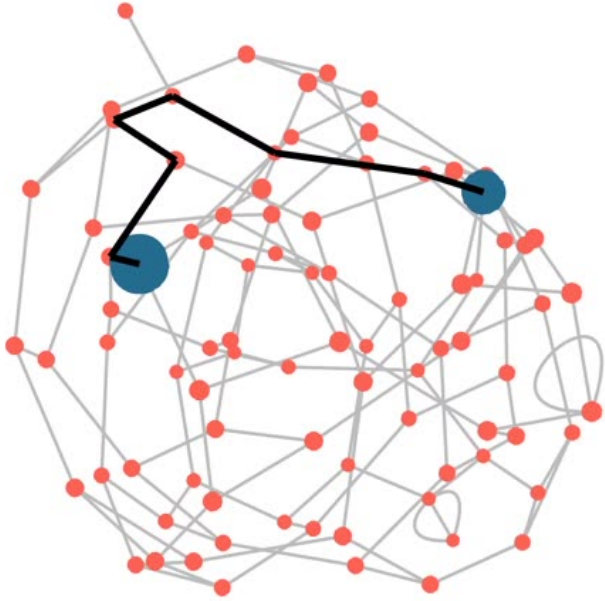
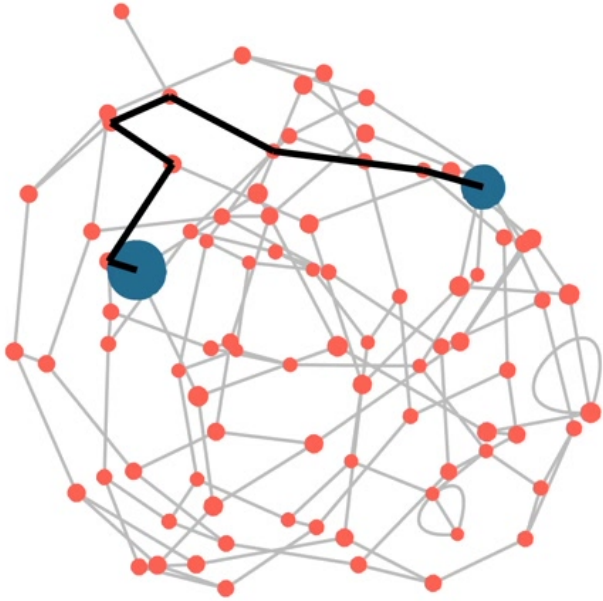
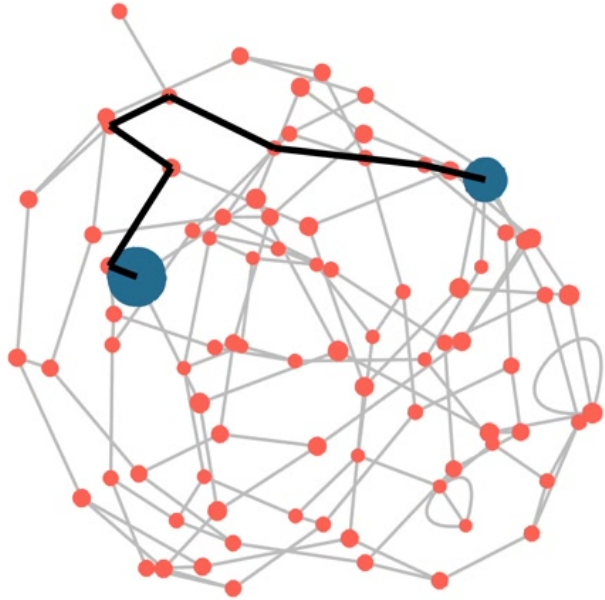
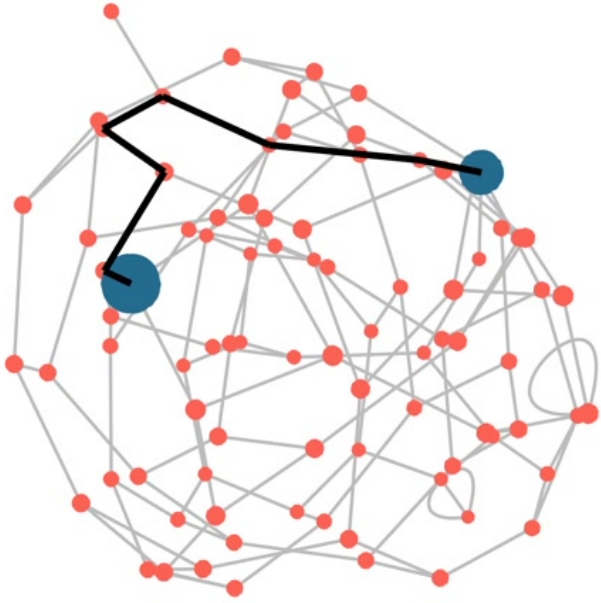
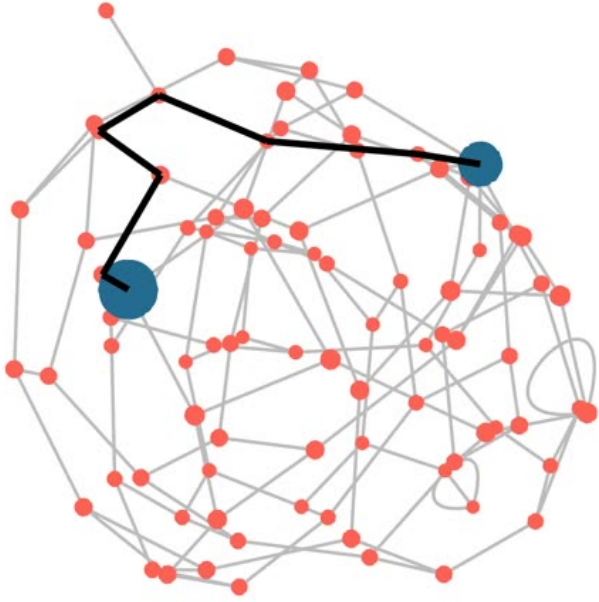
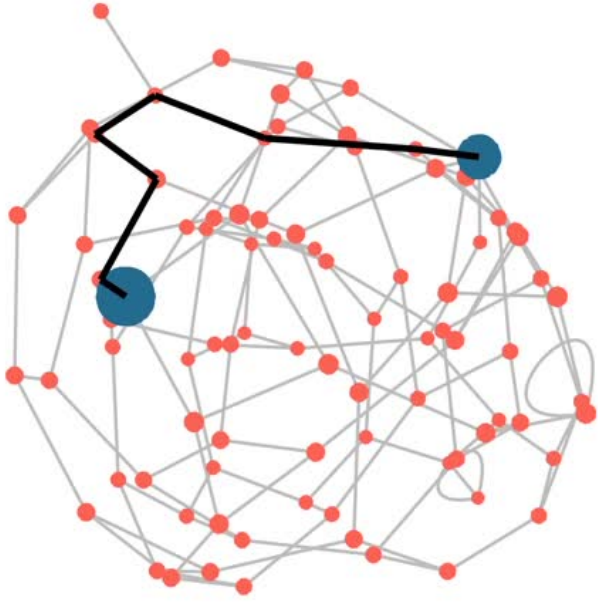


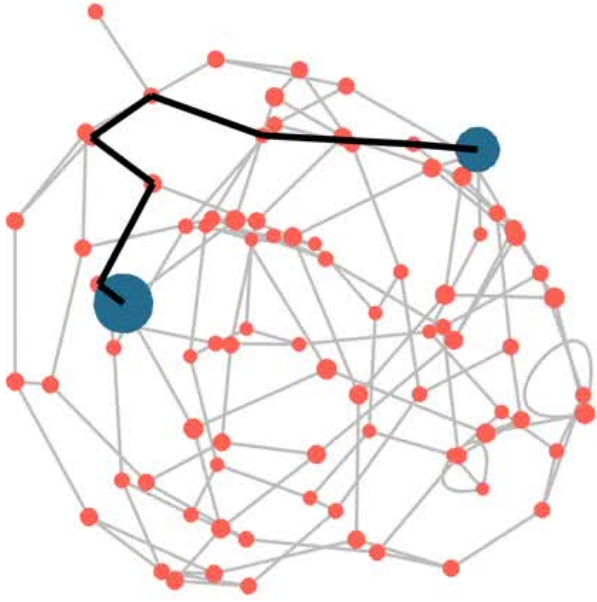


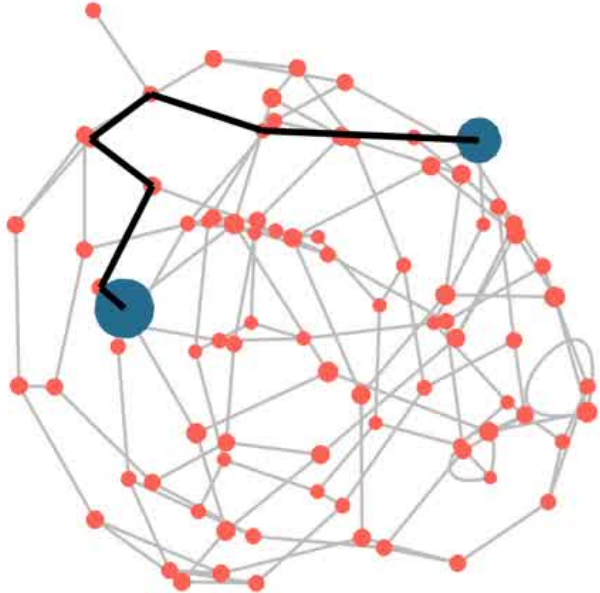


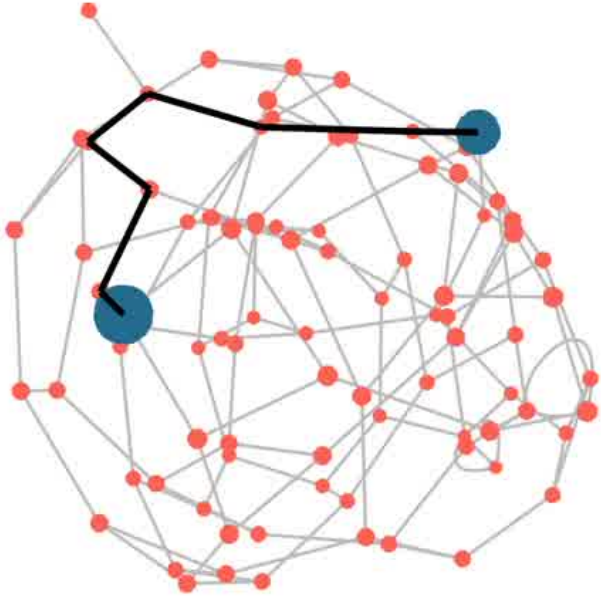


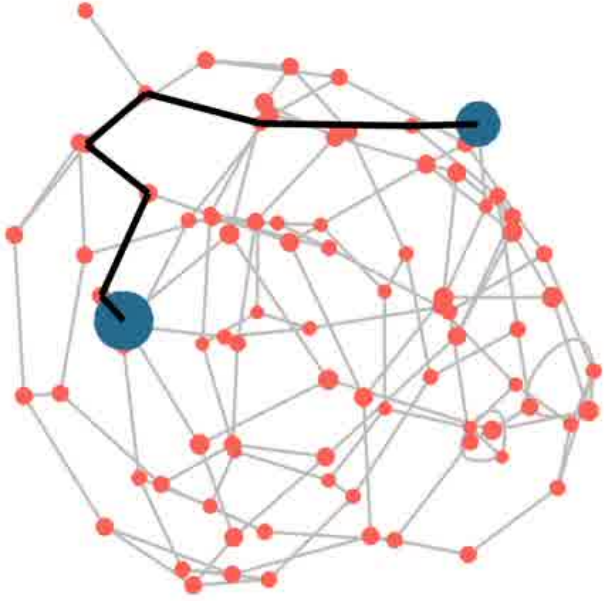




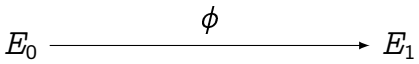








Something that doesn't work



- Ok for CSIDH / ordinary curves / group actions (see SeaSign, CSI-FiSh).
- Galbraith-Petit-Silva: ok if E_0 is supersingular and has known endomorphism ring.
- Not known how to make ZK for general supersingular curves.

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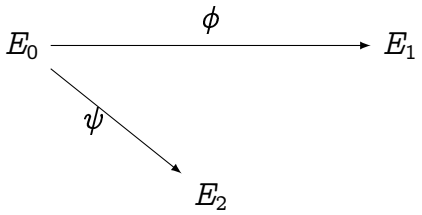

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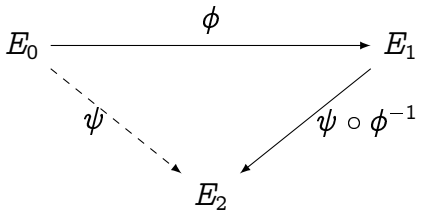

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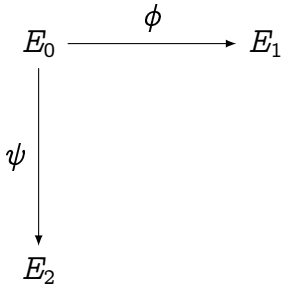
SIDH squares (pushouts)



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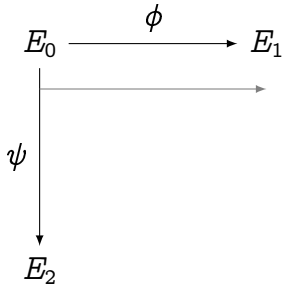

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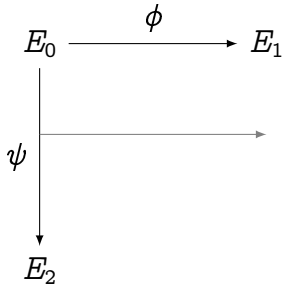

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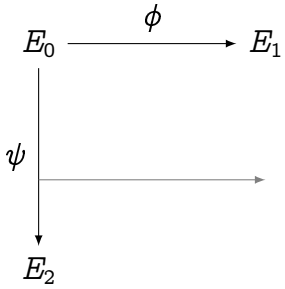

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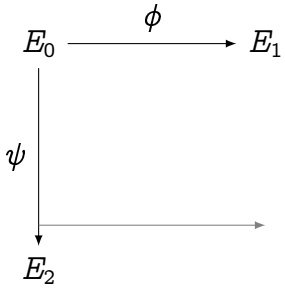

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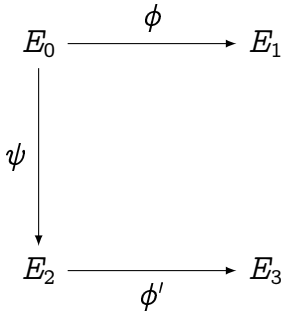

- $\ker \phi' = \psi(\ker \phi)$,

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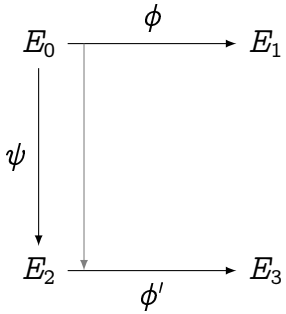

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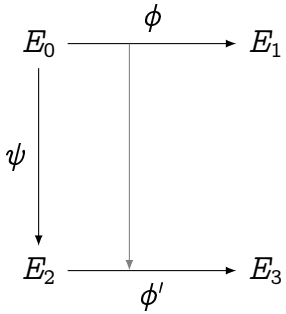

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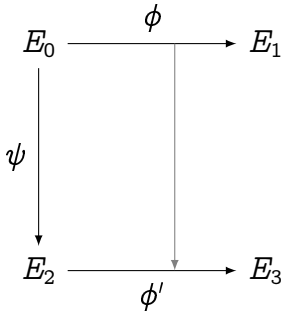

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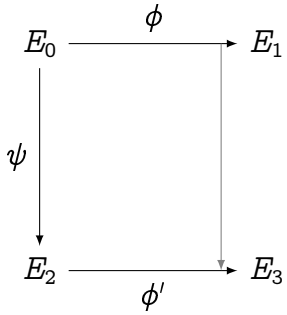

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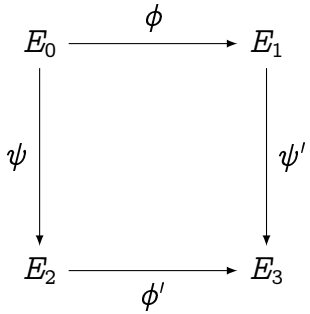

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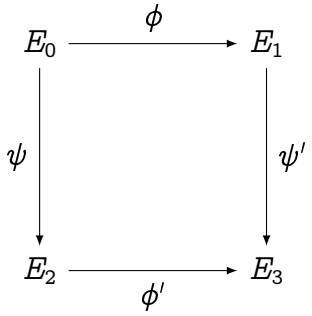
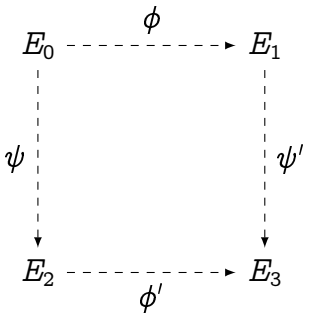


- $\ker \phi' = \psi(\ker \phi)$,
- $\ker \psi' = \phi(\ker \psi)$,
- $\deg \phi = \deg \phi' = A = 2^n$,
- $\deg \psi = \deg \psi' = B = 3^n$,
- $\gcd(A, B) = 1$,

Something that should work: De Feo–Jao–Plût, 2012 (roughly)



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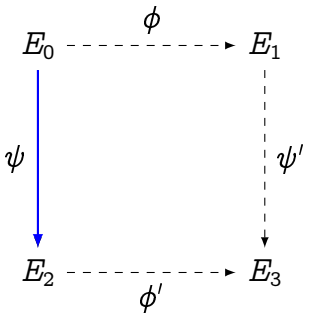

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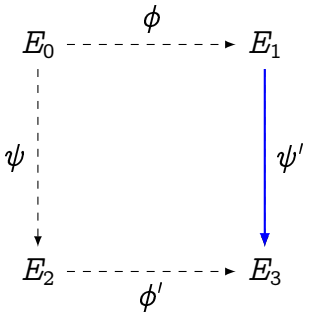

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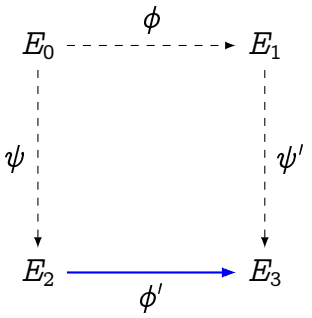
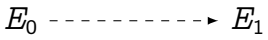


Special soundness (as claimed in DFJP11)



E_2 E_3

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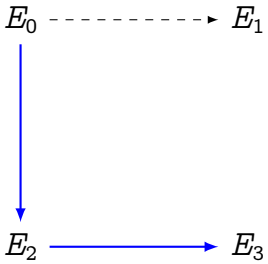

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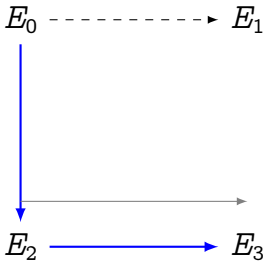

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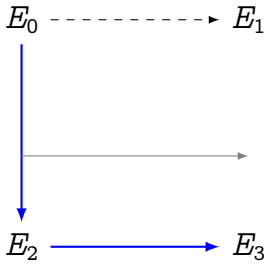

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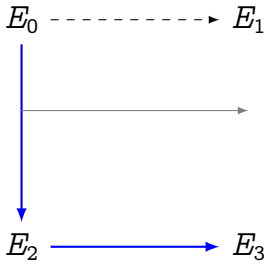

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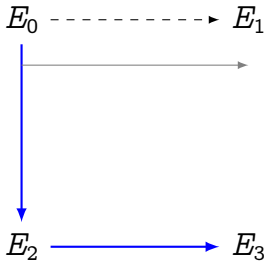

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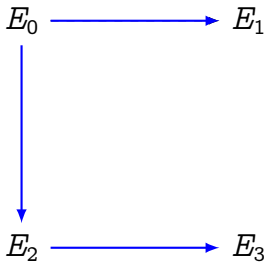
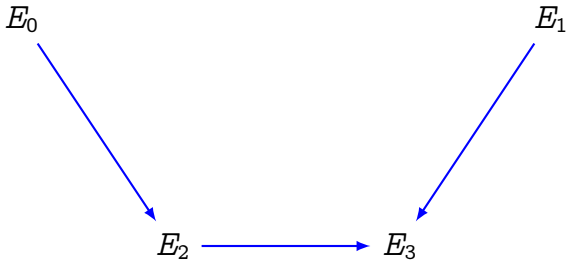


Epic fail!



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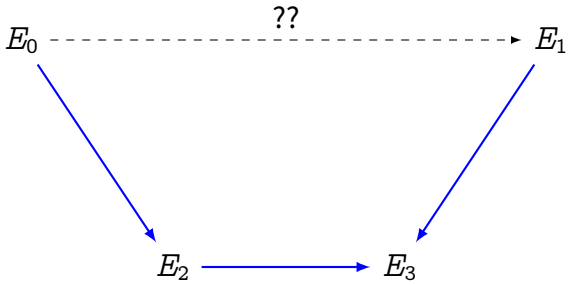

Our (not) elegant fix!

E_0 E_1

$E_2[B] = \langle P_2, Q_2 \rangle$ $E_3[B] = \langle P_3, Q_3 \rangle$

Our (not) elegant fix!

E_0 E_1

a, b

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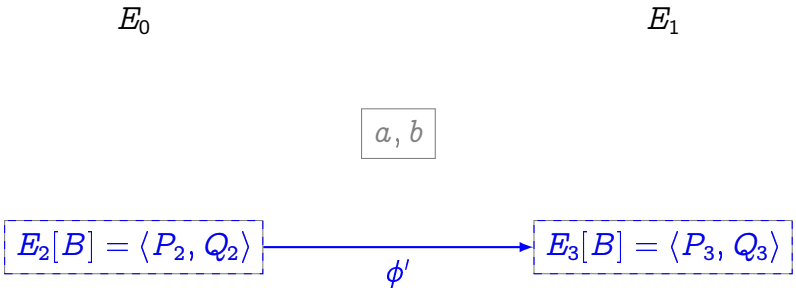

$$P_3 = \phi'(P_2), \qquad Q_3 = \phi'(Q_2)$$

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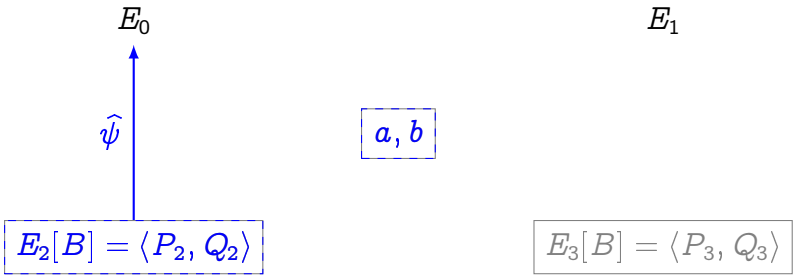

$$P_3 = \phi'(P_2), \qquad Q_3 = \phi'(Q_2)$$

$$\ker \widehat{\psi} = aP_2 + bQ_2$$

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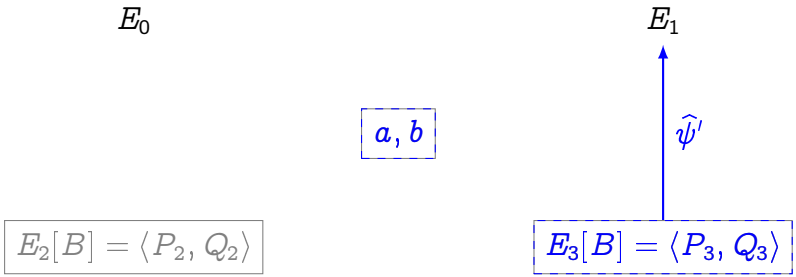

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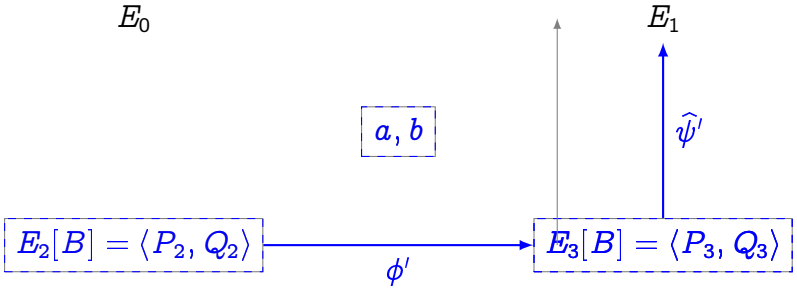

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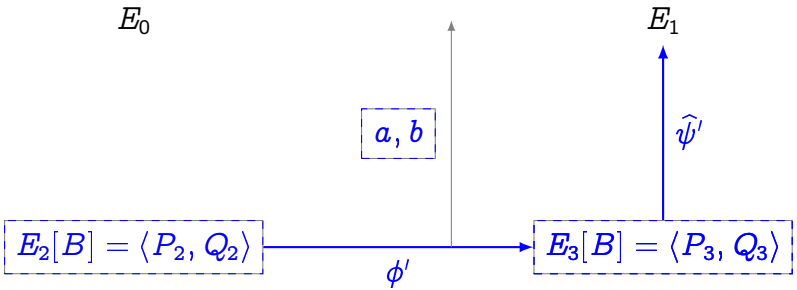

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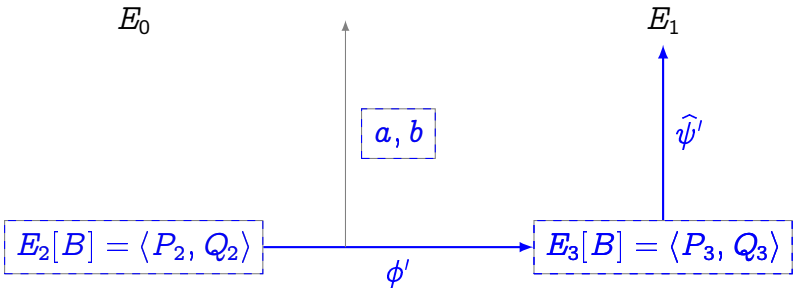

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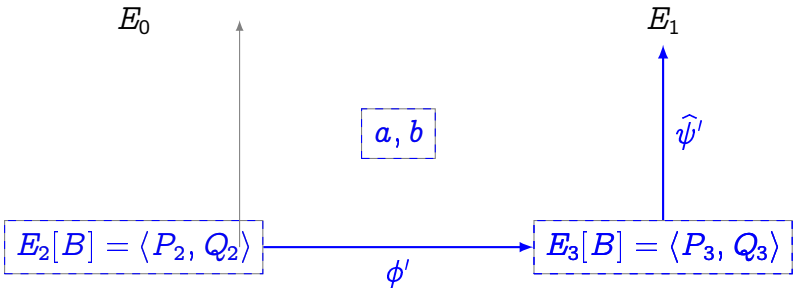

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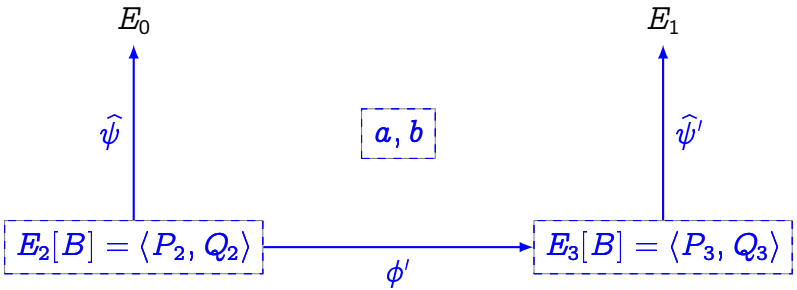

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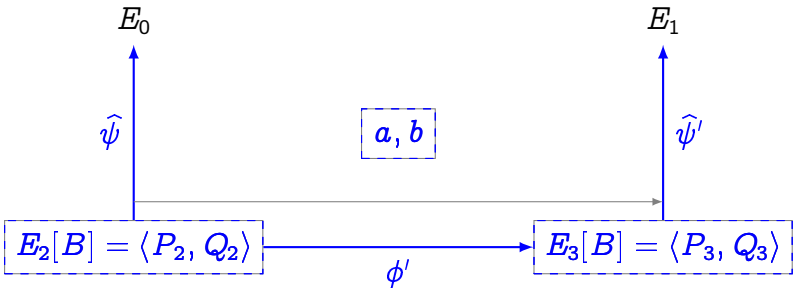
$$\ker \widehat{\psi} = aP_2 + bQ_2$$
$$\ker \widehat{\psi}' = aP_3 + bQ_3$$

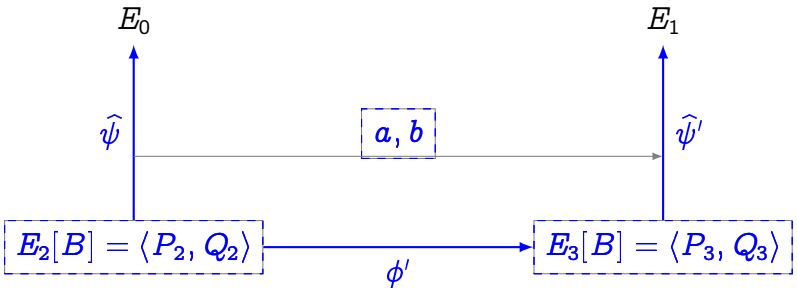
$$\Rightarrow \quad \ker \widehat{\psi}' = \phi'(\ker \widehat{\psi})$$

Our (not) elegant fix!



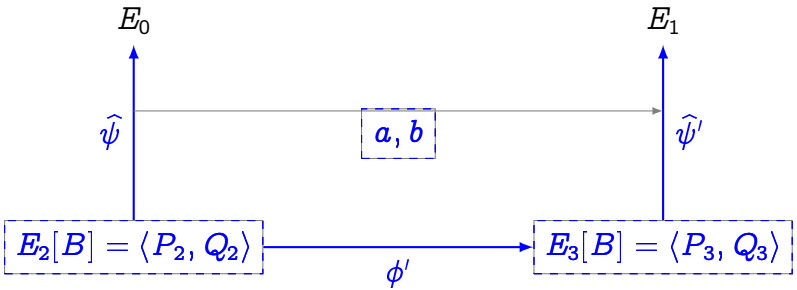
$$\left.\begin{aligned} P_3 = \phi'(P_2), \qquad Q_3 = \phi'(Q_2) \\ \ker \widehat{\psi} = aP_2 + bQ_2 \\ \ker \widehat{\psi}' = aP_3 + bQ_3 \end{aligned}\right\} \Rightarrow \quad \ker \widehat{\psi}' = \phi'(\ker \widehat{\psi})$$

Our (not) elegant fix!



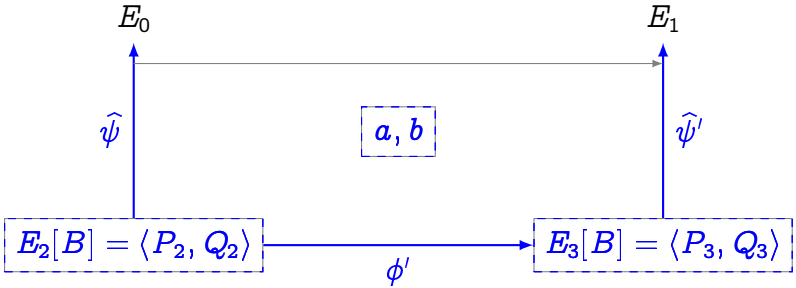
$$\left.\begin{aligned} P_3 = \phi'(P_2), \qquad Q_3 = \phi'(Q_2) \\ \ker \widehat{\psi} = aP_2 + bQ_2 \\ \ker \widehat{\psi}' = aP_3 + bQ_3 \end{aligned}\right\} \Rightarrow \quad \ker \widehat{\psi}' = \phi'(\ker \widehat{\psi})$$

Our (not) elegant fix!



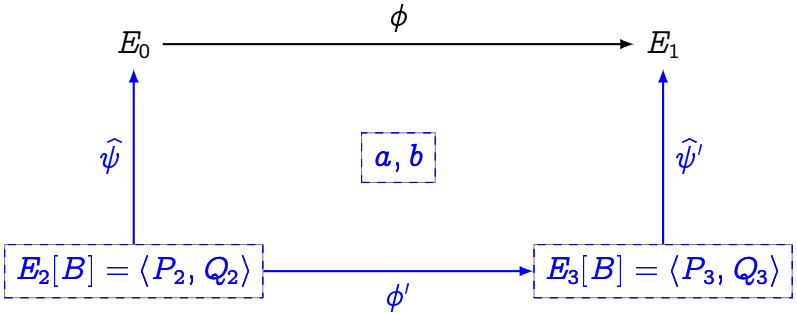
$$\left.\begin{aligned} P_3 = \phi'(P_2), \qquad Q_3 = \phi'(Q_2) \\ \ker \widehat{\psi} = aP_2 + bQ_2 \\ \ker \widehat{\psi}' = aP_3 + bQ_3 \end{aligned}\right\} \Rightarrow \quad \ker \widehat{\psi}' = \phi'(\ker \widehat{\psi})$$

Our (not) elegant fix!



$$\left.\begin{aligned} P_3 = \phi'(P_2), \qquad Q_3 = \phi'(Q_2) \\ \ker \widehat{\psi} = aP_2 + bQ_2 \\ \ker \widehat{\psi}' = aP_3 + bQ_3 \end{aligned}\right\} \Rightarrow \quad \ker \widehat{\psi}' = \phi'(\ker \widehat{\psi})$$

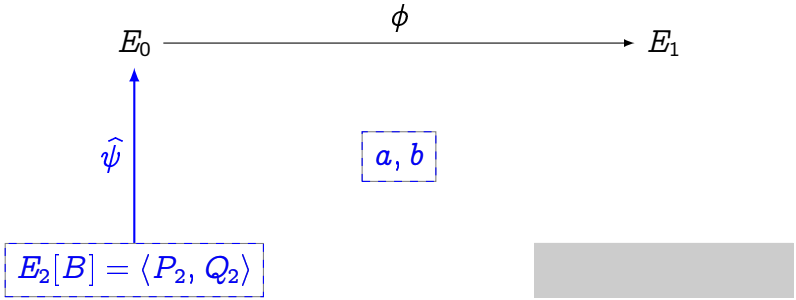
Our (not) elegant fix!



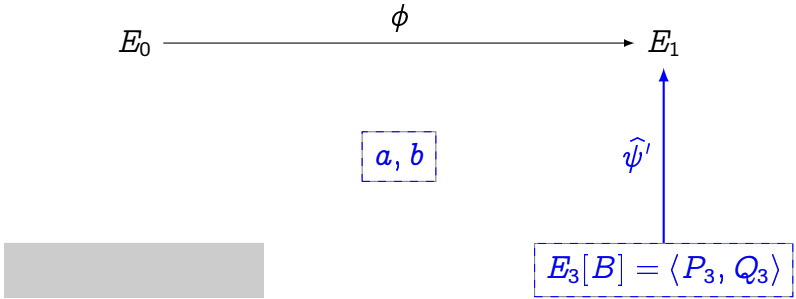
$$P_3 = \phi'(P_2), \qquad Q_3 = \phi'(Q_2)$$
$$\ker \widehat{\psi} = aP_2 + bQ_2$$
$$\ker \widehat{\psi}' = aP_3 + bQ_3$$

$$\Rightarrow \quad \ker \widehat{\psi}' = \phi'(\ker \widehat{\psi})$$
$$\ker \phi = \widehat{\psi}(\ker \phi')$$

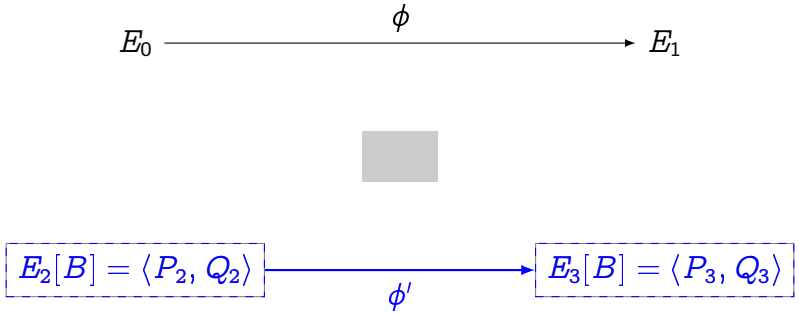
Zero-knowledge



Zero-knowledge

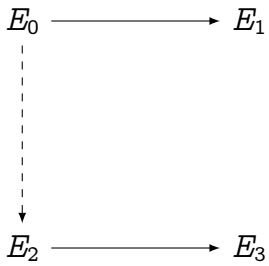


Zero-knowledge



Assumption: Decisional Supersingular Product (DSSP)

Statistical ZK (BCCDFLMPPW ’22)



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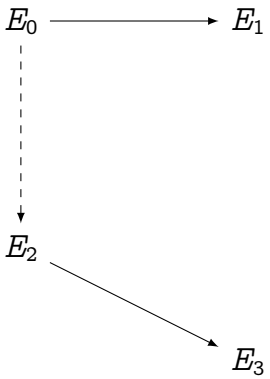
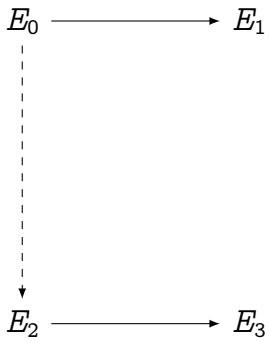


Statistical ZK (BCCDFLMPPW ’22)



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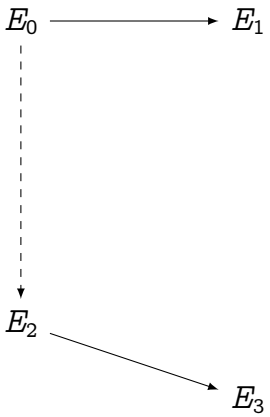

Statistical ZK (BCCDFLMPPW ’22)

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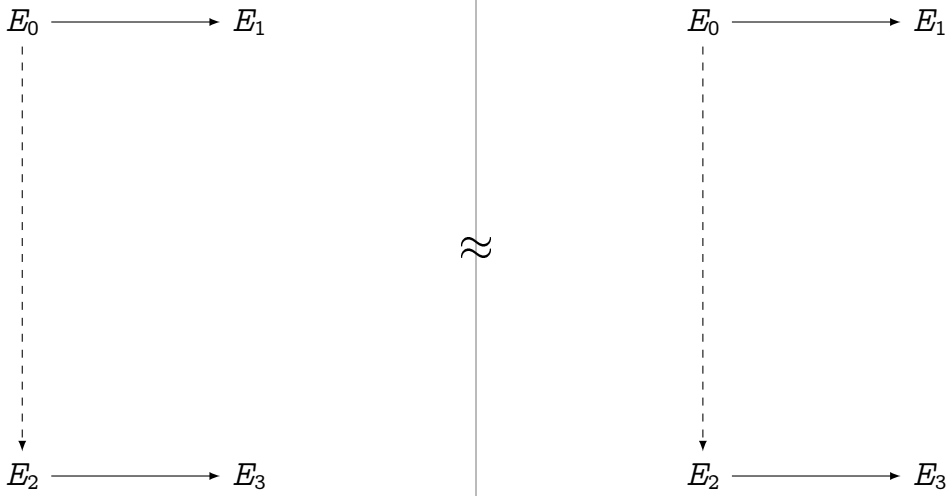
Statistical ZK (BCCDFLMPPW '22)



Summary

This work:

- A knowledge sound, computational ZK protocol to prove knowledge of an isogeny of fixed degree… Finally!
- A variant to prove knowledge of an SIDH key… Because, why not?

Recent development:

- A statistically ZK protocol for the same… but knowledge soundness not as good.

Open problems:

- Statistical ZK + strong Knowledge soundness.
- Efficiency.
- An even remotely efficient protocol for the CSIDH setting:
  (SeaSign sucks and CSI-FiSh doesn't scale).