FINAL: Faster FHE instantiated with NTRU and LWE.

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RLWE vs NTRU

Hardness assumptions for FHE

Let $R := \mathbb{Z}[X]/\langle X^N + 1\rangle$

RLWE problem

- Secret: $s \in R$
- $a_i \leftarrow \mathcal{U}(R_q)$
- $e_i \leftarrow \chi$
- $b_i := a_i \cdot s + e_i \bmod q$

Then $(a_i, b_i) \approx_C \mathcal{U}(R_q^2)$.

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NTRU problem

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Ideally, we should halve the memory consumption and running time.

Past NTRU constructions

- Indeed, past NTRU-based FHE schemes, like YASHE, were more efficient than the corresponding RLWE-based schemes [LN14].

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- While first ciphers based on NTRU used very small q, like $q \in O(N)$.
- It turns out that the NTRU problem is insecure when q is too big in comparison with N.

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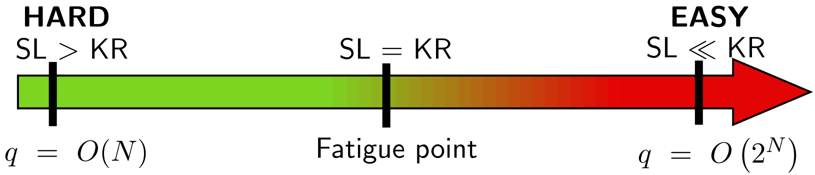
Overstretched regime

- Key recovery attacks (KR): exponential time in N.
- New sublattice attacks (SL): hardness varies depending on q.



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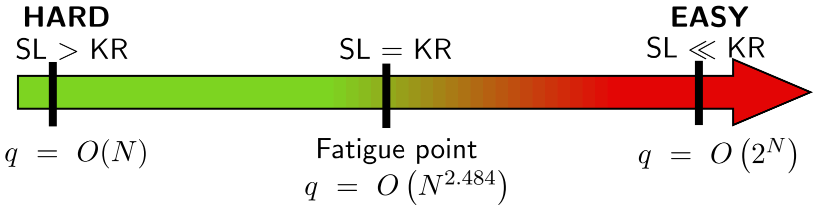

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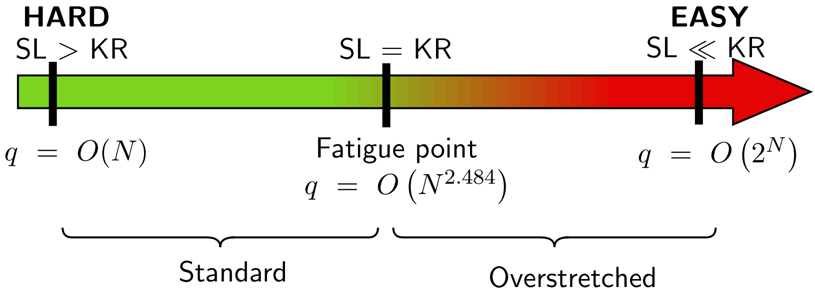

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- Our LWE-NTRU scheme is 28% faster than TFHE.
- Bootstrapping keys 45% smaller than in TFHE.

NTRU-based GSW-like Scheme

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- Secret: sk is a small $f \in R$.
- Scalar ciphertext: $c = g/f + \Delta m \in R_q$, for a random g.
- Vector ciphertext:

$$\mathbf{c} = \mathbf{g}/f + (B^0, B^1, ..., B^{\ell-1}) \cdot m \in R_q^\ell$$

where $\ell = \lceil \log_B(q) \rceil$.

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$$\mathbf{G} = \begin{bmatrix} B^0 & 0 \\ B^1 & 0 \\ \vdots & \vdots \\ B^{\ell-1} & 0 \\ 0 & B^0 \\ 0 & B^1 \\ \vdots & \vdots \\ 0 & B^{\ell-1} \end{bmatrix}$$

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- External product:
  - Let $\mathbf{y} = \text{Decomp}_B(c) \in R_q^\ell$
  - Compute $c_{mult} = \mathbf{y} \cdot \mathbf{c} = \sum_{i=0}^{\ell-1} y_i \cdot c_i \in R_q$
  - Cost: ℓ multiplications on R_q

NGS vs ring-GSW

	NGS	GSW
Scalar ciphertext	1 poly	2 polys
"Full" ciphertext	ℓ polys	$4 \cdot \ell$ polys
External prod.	ℓ mults	$4 \cdot \ell$ mults

FHE with fast bootstrapping

Overview of bootstrapping in FHEW and TFHE

LWE encryption of m with noise large e

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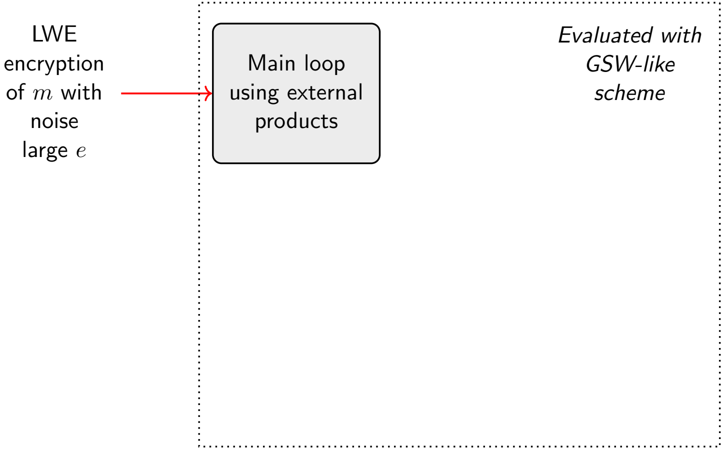

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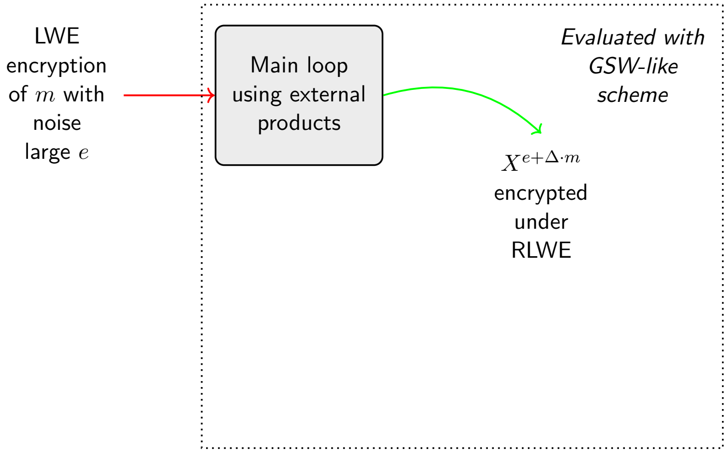

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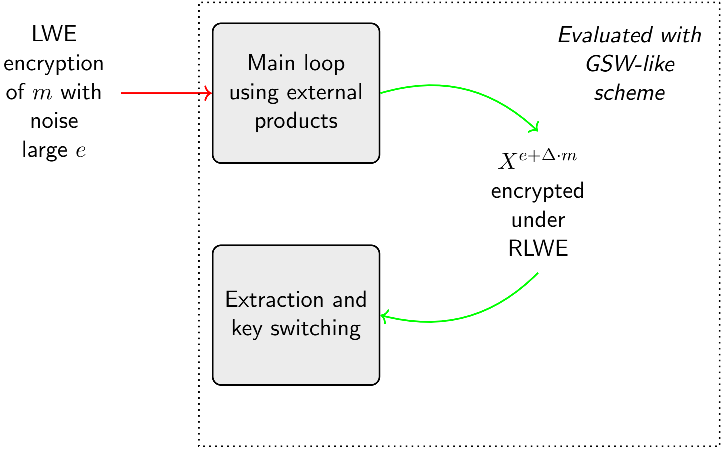

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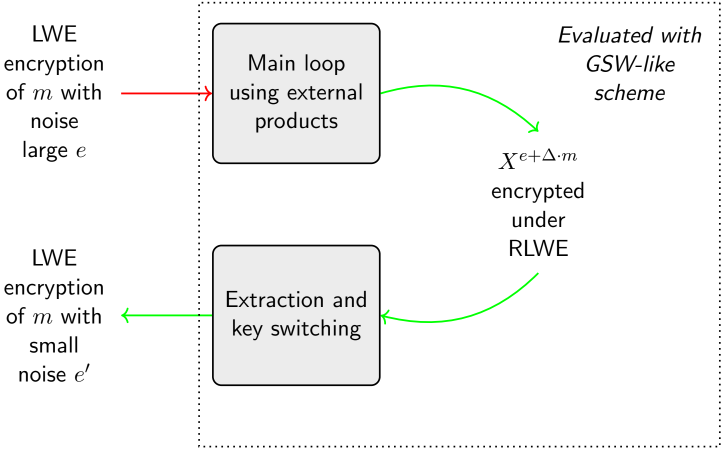

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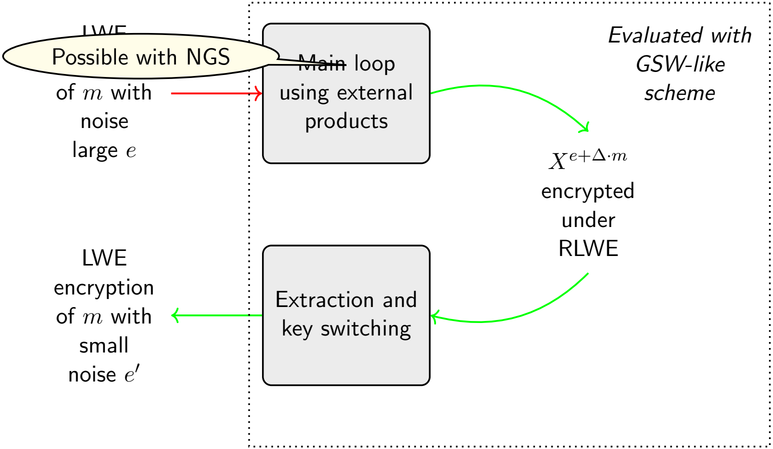

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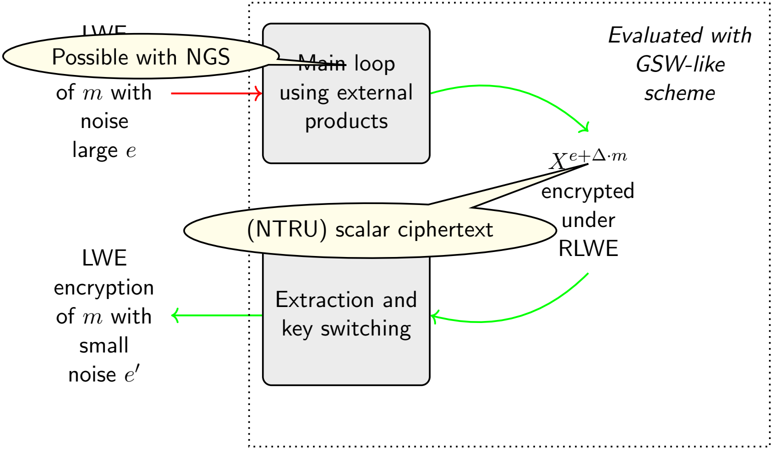

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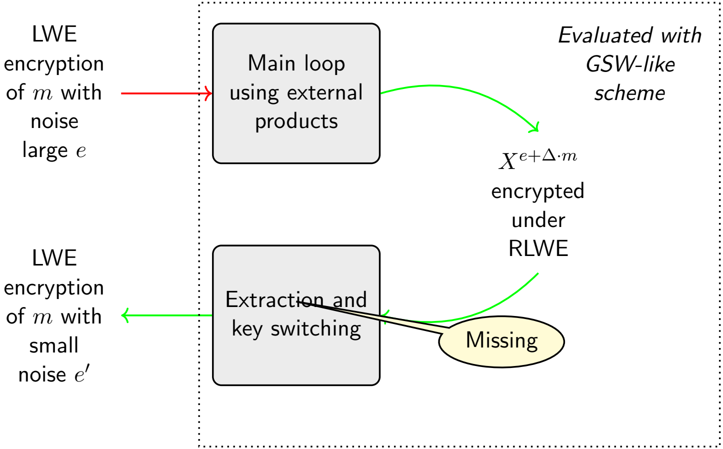

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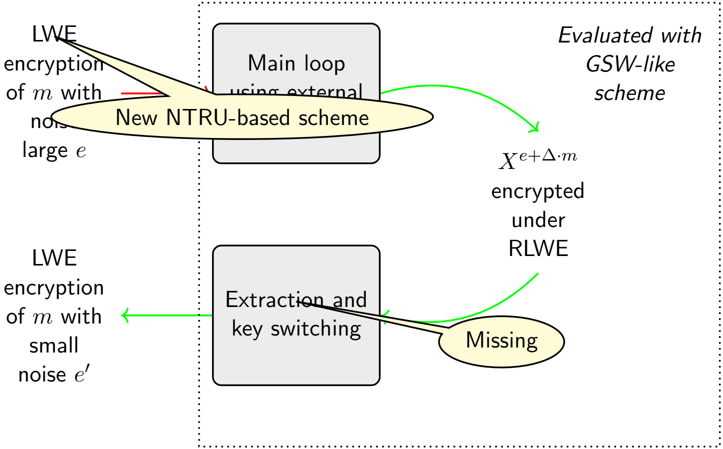

Matrix-NTRU (MNTRU) scheme

- We propose a base scheme whose security relies on the MNTRU problem.
- $\mathbf{C} := \mathbf{G} \cdot \mathbf{F}^{-1} \bmod q \ \approx_C \ \mathcal{U}(\mathbb{Z}_q^{n\times n})$.

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- So, we define a ciphertext by using just the first row of $\mathbf{C}$

$$\mathsf{row}_1(\mathbf{C}) = \mathbf{G} \cdot \mathsf{col}_1(\mathbf{F}^{-1})$$

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- Then, ciphertext is a vector $\mathbf{c} \in \mathbb{Z}_q^n$.
- Let $\mathbf{f}$ be the first column of $\mathbf{F}$.
- To decrypt, we compute $\mathbf{c} \cdot \mathbf{f}$.
- Thus, we can also use external products to compute $X^{\mathbf{c} \cdot \mathbf{f}}$.

Extraction and key switching

- We start with $\mathsf{Enc}_{\mathbf{f}}(m) \in \mathbb{Z}_q^n$.
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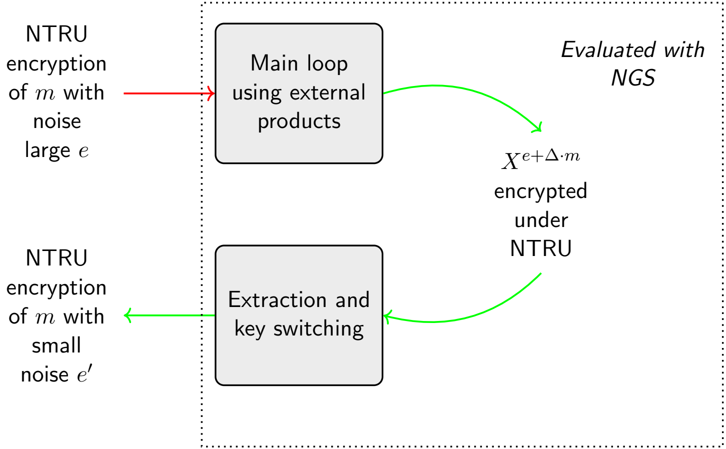
We can set $q \in \tilde{O}(N)$, thus, below the fatigue point $O(N^{2.48})$.

FHE based on NTRU



Using NGS to bootstrap LWE-based scheme

Considering a base scheme over the LWE problem, as in FHEW and TFHE, we have

- $\text{Enc}_{\mathbf{s}}(m) = (\mathbf{a}, b = \mathbf{a} \cdot \mathbf{s} + e + \Delta m) \in \mathbb{Z}_q^{n+1}$

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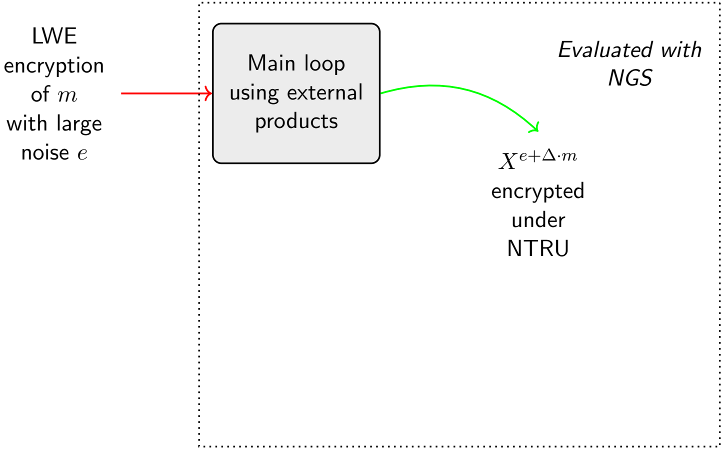
- $\mathsf{Enc}_{\mathbf{s}}(m) = (\mathbf{a}, b = \mathbf{a} \cdot \mathbf{s} + e + \Delta m) \in \mathbb{Z}_q^{n+1}$
- We can already use NGS external products to compute

$$\mathsf{Enc}_f(X^{b-\mathbf{a}\cdot\mathbf{s}}) = \mathsf{Enc}_f(X^{e+\Delta m}) \in R_q$$

FHE based on LWE and NTRU



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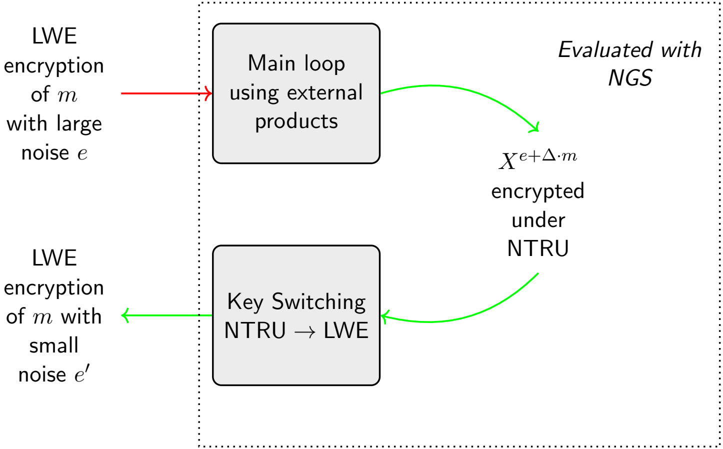

Practical results and conclusion

Proof-of-concept implementation

C++ implementation available in
`https://github.com/KULeuven-COSIC/FINAL`.

For a fair comparison, we used the same FFT library as TFHE and compiled our code with the same compilation flags.

Parameter selection

For each basis B_i we have a different dimension $\ell_i := \lceil \log_{B_i}(Q) \rceil$ for n_i bootstrapping keys. For the first n_1 external products, we use the decomposition base B_1, then we use B_2 for the remaining n_2 external products.

Base scheme	n	q	N	Q	(B_1, n_1)	(B_2, n_2)	ℓ_1	ℓ_2
MNTRU	800	$\approx 2^{17}$	2^{10}	$\approx 2^{19.8}$	$(8, 750)$	$(16, 50)$	7	5
LWE	610	$\approx 2^{16.5}$	2^{10}	$\approx 2^{19.8}$	$(8, 140)$	$(16, 470)$	7	5

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Bootstrapping executes n external products...

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	Key switching key	Bootstrapping key	Mult. on R_Q	FFTs	Run. time
TFHE	40 MB	31 MB	7560	6300	66 ms
MNTRU	34.4 MB	43 MB	11000	6300	92 ms
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LWE-NTRU FHE:

45% less key material and bootstrapping 28% faster than TFHE.

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  - We hope that NGS will be used to replace GSW in other applications apart from the bootstrapping presented here.
- We proposed an NTRU-to-LWE key switching and showed how to use it in combination with NGS to bootstrap LWE schemes.
- Therewith, we ran bootstrapping faster than in TFHE.

Thanks!

Any question or comment?

Please, feel free to contact!

`https://hilder-vitor.github.io`