A new isogeny representation and application to cryptography.

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A little bit of isogeny-based cryptography recent history

1. (2011) SIDH
2. (2016) SIKE
3. (2018) CSIDH
4. (End of 2021) This work (pSIDH): new directions to explore.
5. (Summer of 2022) Attacks on SIDH.

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- CSIDH and variants, SQISign are safe.
- SIDH, B-SIDH and Séta broken.
- pSIDH??

Mathematical Background

Elliptic curve and isogeny notations

Elliptic Curve over $\mathbb{F}_{p^k}$: $y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_{p^k}$

Isogeny: rational map between elliptic curves.

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Best known attacks: requires random walk in the isogeny graph.
Complexity is polynomial in the size of the graph.

Quaternion algebra definitions

The quaternion algebra $\mathcal{B}(a, b)$ over $\mathbb{Q}$ with $a, b \in \mathbb{Z}$ is

$$\mathcal{B}(a, b) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q}$$

with $i^2 = a$, $j^2 = b$ and $k = ij = -ji$.

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Orders are rings: so we have ideals. In a non-commutative algebra, ideals have distinct left and right orders.

The Deuring Correspondence

p : prime characteristic, $\mathcal{B}(-q, -p)$ where $q > 0$ depends only on p.

Supersingular elliptic curves over $\mathbb{F}_{p^2}$ E (up to Galois conjugacy)	Maximal Orders in $\mathcal{B}(-q, -p)$ $\mathcal{O} \cong \mathrm{End}(E)$
Isogeny with $\varphi : E \to E_1$	Ideal I_φ left $\mathcal{O}$-ideal
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$$E_0 : y^2 = x^3 + x$$

$$\mathrm{End}(E_0) = \langle 1, \iota, \frac{\iota + \pi}{2}, \frac{1 + \iota\pi}{2} \rangle \cong \langle 1, i, \frac{i + j}{2}, \frac{1 + k}{2} \rangle$$

$\pi : (x, y) \mapsto (x^p, y^p)$ is the Frobenius morphism with $\pi \circ \pi = [-p]$.

$\iota : (x, y) \mapsto (-x, \sqrt{-1}y)$ is a twisting automorphism with $\iota \circ \iota = [-1]$.

Making the Deuring correspondence effective

Original motivation: cryptanalysis.

We end up with a bunch of nice algorithmic tools:

- Convert a maximal order in a supersingular curve.
- Translate an ideal into an isogeny.
- Find isogenies between curves of known endomorphism rings.

These algorithms are used in GPS and SQISign, and our goal is to explore the new possibilities they offer.

Isogeny representations.

Definition

Let us take $\varphi : E_1 \to E_2$ of degree D. By definition, we have:

$$\varphi : (x, y) \mapsto \left(\frac{f_1(x, y)}{f_2(x, y)}, \frac{g_1(x, y)}{g_2(x, y)} \right)$$

Isogeny representation: a string s_φ for which there exists algorithms

- $\texttt{Verify}(s_\varphi, E_1, E_2, D) \to b \in \{0, 1\}$ to tell if s_φ is valid.
- $\texttt{Evaluate}(s_\varphi, P)$ to compute $\varphi(P)$.

The standard and kernel representation

Standard rep: rational maps f_1, f_2, g_1, g_2 (ok when degree is small but that's all).

Kernel rep: from a generator of the kernel. Uses the Vélu Formulas!

1. Compact and efficient, when the degree is "nice" = smooth and torsion pts defined over small extension. Most often used in isogeny crypto.
2. Complexity polynomial in D: not adapted for isogenies of arbitrary degrees.

The ideal representation

Ideal rep: the ideal I_φ associated to φ.

Algorithmic study of Deuring correspondence $\Rightarrow$ the ideal rep is both compact and efficient for any degree D and prime p.

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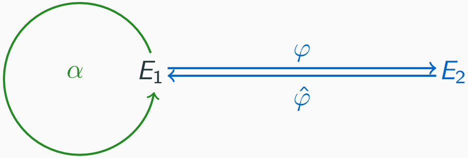
It is almost too good! It reveals every information about φ, E_1, E_2! The ideal representation cannot be anything else than a secret in cryptography.

Can we have a representation "in the middle"?

The suborder representation

Lollipop endomorphisms for proving isogeny existence.

Lollipop endomorphisms for $\varphi : E_1 \to E_2$ of degree D:



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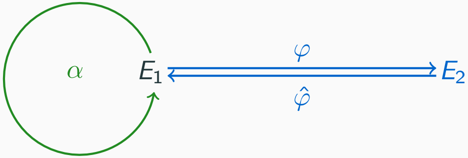

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Conversely: the existence of $\mathbb{Z} + D\mathrm{End}(E_1) \hookrightarrow \mathrm{End}(E_2)$ proves the existence of $\varphi : E_1 \to E_2$ of degree D!

The suborder representation

Let us take $\varphi : E_1 \to E_2$ of degree D. The suborder representation π_φ of φ is made of:

1. D, E_1, E_2
2. $\mathrm{End}(E_1)$ (≈ 16 coefficients in $\mathbb{Z}$ for a basis).
3. s_1, s_2, s_3 where s_i is the kernel representation of an endomorphism $\theta_i : E_2 \to E_2$. Such that $\mathbb{Z} + D\mathrm{End}(E_1) \hookrightarrow \mathrm{End}(E_2)$ is generated by $\{[1], \theta_1, \theta_2, \theta_3\}$.

We can derive polynomial time algorithms `Verify`,`Evaluate` for this representation for any isogeny φ from several new algorithm tools.

Cryptography

A new hard problem?

Going back to our motivation: need a rep not equivalent to the ideal rep.

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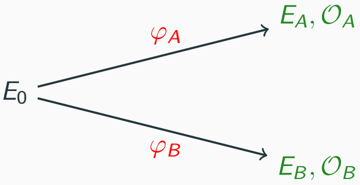
Thm: SOIP $\Leftrightarrow$ SOERP $\Leftrightarrow$ T-SIP.

Best attack: quantum subexponential in D: because we reveal some endomorphisms, but they are chosen as not to give the full endomorphism rings!

pSIDH: a new key exchange?

pSIDH: A new SIDH-like key exchange with public keys as suborders and secrets keys as ideals for big prime degree, $\mathrm{GCD}(\deg \varphi_A, \deg \varphi_B) = 1$.



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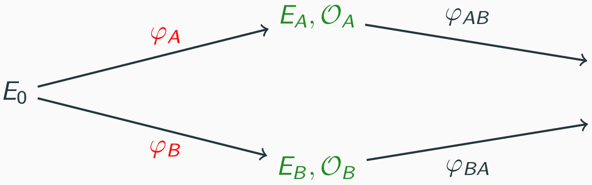

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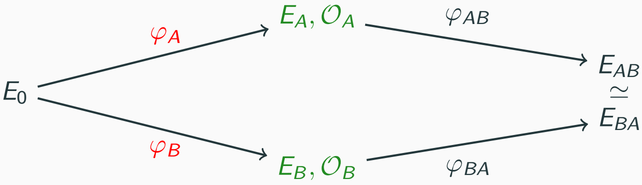

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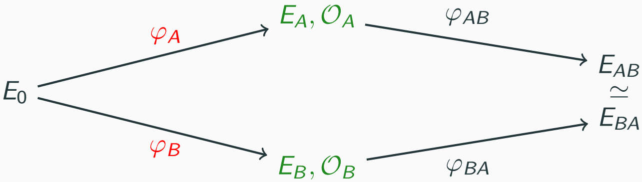

Performance profile similar to CSIDH: quantum subexp attack and verifiable public keys.

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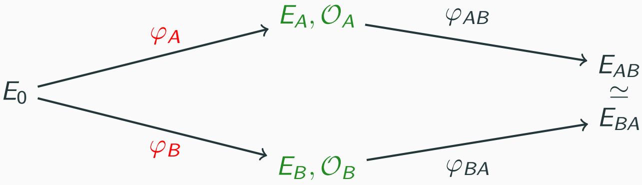

Performance profile similar to CSIDH: quantum subexp attack and verifiable public keys.

In practice not as good... But the structure and the hard problem are different!

Future work and open problems

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`https://eprint.iacr.org/2021/1600`

Questions?