Accelerating the Delfs–Galbraith Algorithm with Fast Subfield Root Detection

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Based on joint work with Craig Costello and Jia Shi

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Motivation

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- The security of **SQISign** (signature scheme) and **B-SIDH** (key exchange) relies on its difficulty.
- The best known classical attack against this general problem is the **Delfs–Galbraith algorithm**.

Contributions

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- Provide an optimised implementation of the Delfs–Galbraith algorithm: Solver.
- Develop an efficient method to detect if a polynomial $f(X) \in \mathbb{F}_{p^d}[X]$ has a root in $\mathbb{F}_p$.
- Use this to introduce an improved attack, SuperSolver, with lower concrete complexity.

Elliptic Curves and j-invariants

An elliptic curve E over $\mathbb{F}_{p^2}$ ($p \neq 2, 3$) is a smooth curve given by the equation

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Definition

Let $E : y^2 = x^3 + ax + b$ supersingular. Then, the j-invariant of E is

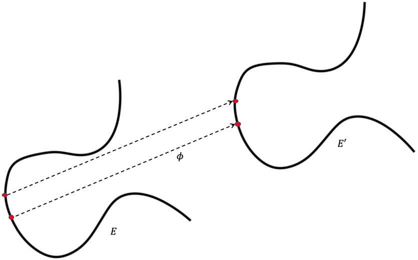
$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2} \in \mathbb{F}_{p^2}.$$

Isogenies

Definition

Let E and E' be two elliptic curves, and let $\phi : E \rightarrow E'$ be a map between them. Then, ϕ is an *isogeny* if it is non-constant and $\phi(\mathcal{O}_E) = \mathcal{O}_{E'}$.

Isogenies are *group homomorphisms*, meaning that for $P, Q \in E$ we have $\phi(P \oplus_E Q) = \phi(P) \oplus_{E'} \phi(Q)$.



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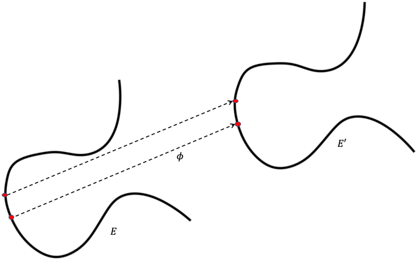

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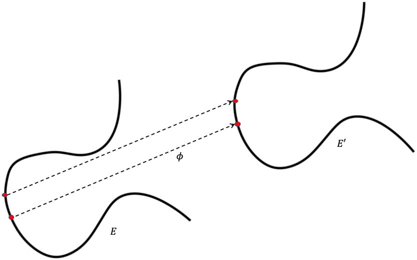

For (seperable) isogenies, the degree $\deg(\phi) = \#\ker(\phi)$. We call an isogeny of degree ℓ an ℓ-isogeny.

The Supersingular Isogeny Problem

In its most general form, the *supersingular isogeny problem* asks to find an isogeny

$$\phi : E_1 \longrightarrow E_2,$$

between two given supersingular elliptic curves $E_1/\mathbb{F}_{p^2}$ and $E_2/\mathbb{F}_{p^2}$.

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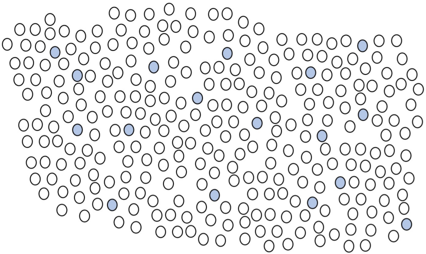
We do not assume:

- torsion point information
- degree of the isogeny
- starting curve is of a special form

The Supersingular Isogeny Graph $\mathcal{X}(\bar{\mathbb{F}}_p, \ell)$

Let p be a large prime, $p \nmid \ell$.



Vertices: Isomorphism classes of supersingular elliptic curves E represented by a j-invariant in $\mathbb{F}_{p^2}$.
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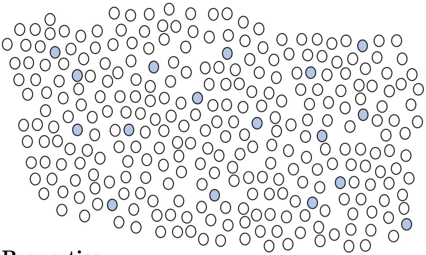

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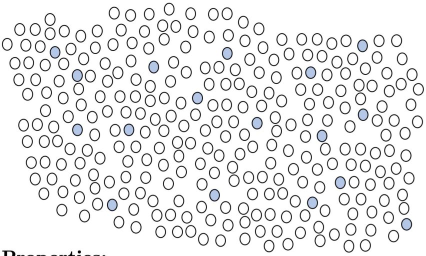

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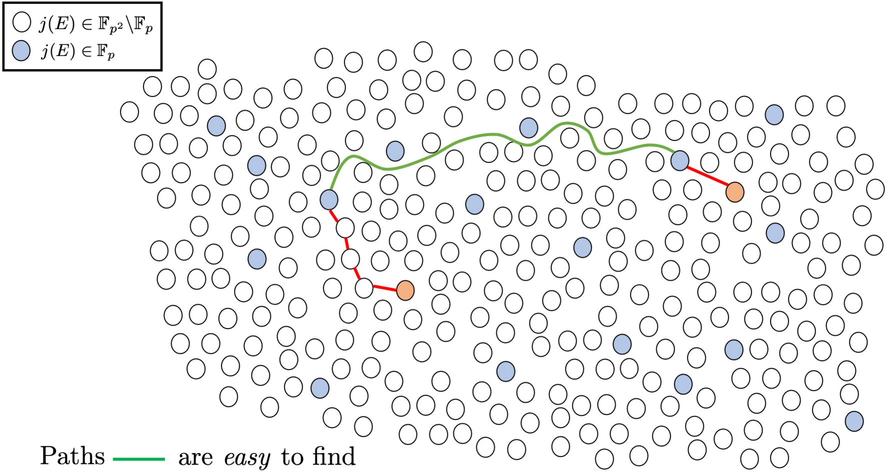
Finding a path between two nodes j_1, j_2 = Finding an isogeny between E_1, E_2

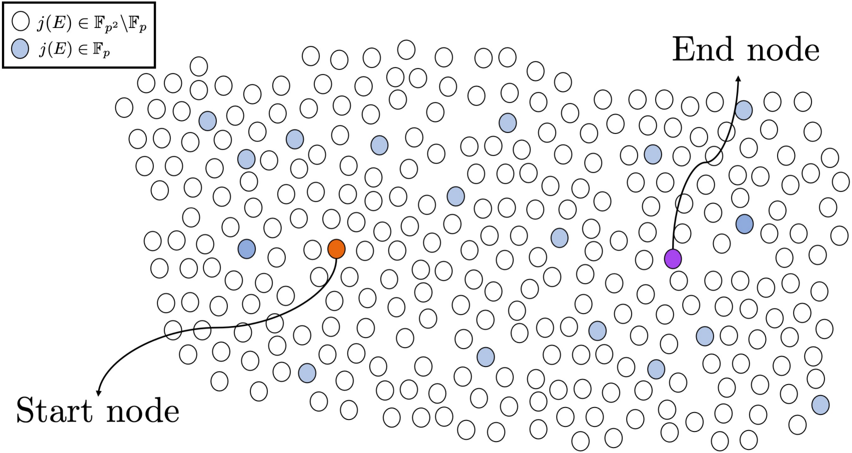
Key Observation



Paths —— are *easy* to find

Finding paths —— is the bottleneck

The Delfs–Galbraith Algorithm



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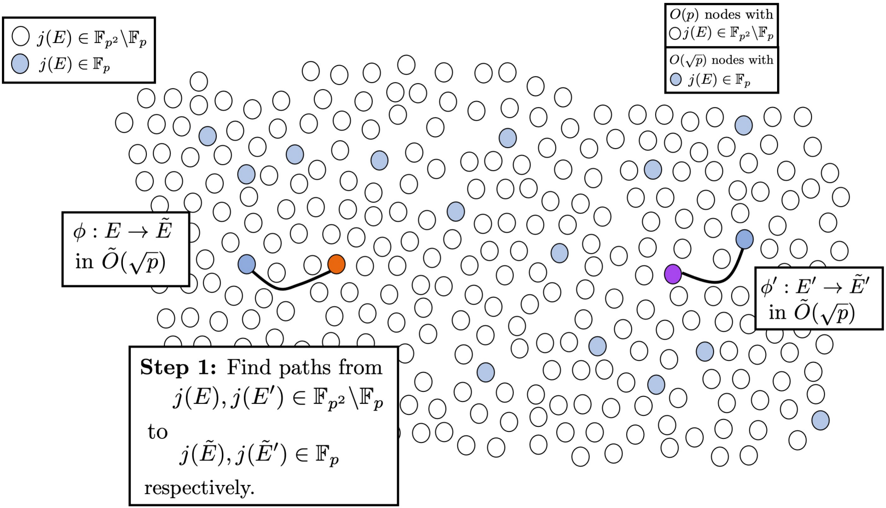

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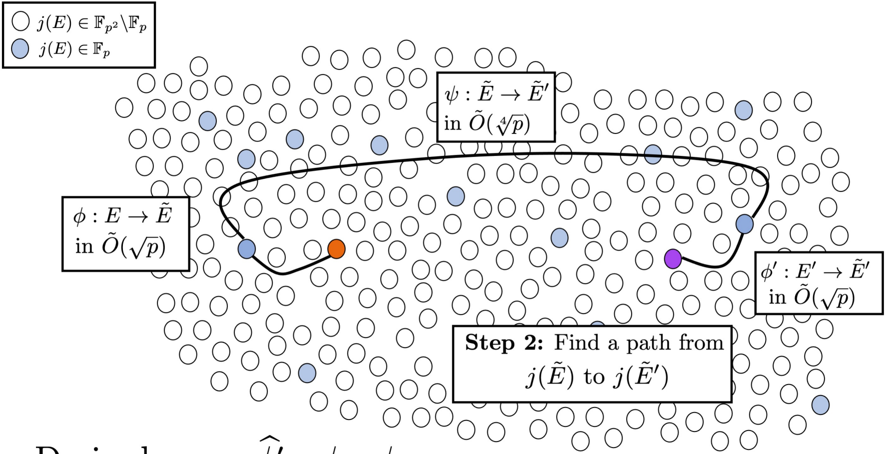

Desired map: $\widehat{\phi'} \circ \psi \circ \phi$

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The modular polynomial $\Phi_\ell(X, Y) \in \mathbb{Z}[X, Y]$ is symmetric of degree N_ℓ in both X and Y where

$$N_\ell := \prod_{i=1}^{n} (\ell_i + 1)\ell_i^{e_i - 1}, \text{ for prime decomposition } \prod_{i=1}^{n} \ell_i^{e_i} \text{ of } \ell.$$

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This tells us that the roots of $\Phi_\ell(X, j)$ are neighbours of j in $\mathcal{X}(\bar{\mathbb{F}}_p, \ell)$. Reducing coefficients $\bmod p$ we can work with $\Phi_{\ell,p}(X, Y) \in \mathbb{F}_p[X, Y]$.

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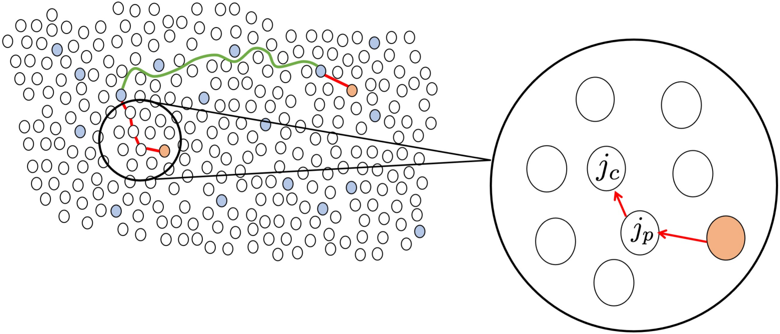

1. Store the current and previous j-invariants j_c and j_p.

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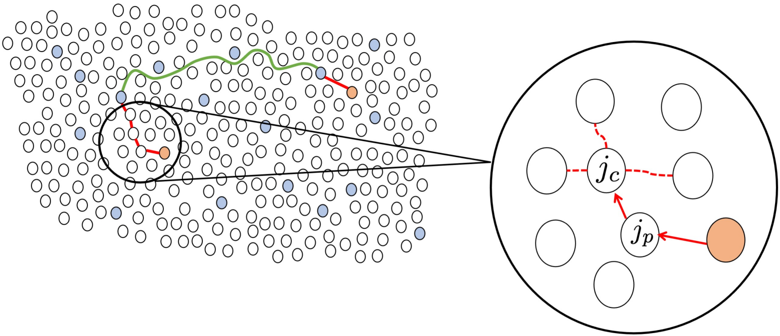

2. Find the $N_\ell - 1$ roots of $\Phi_{\ell,p}(X, j_c)/(X - j_p)$.

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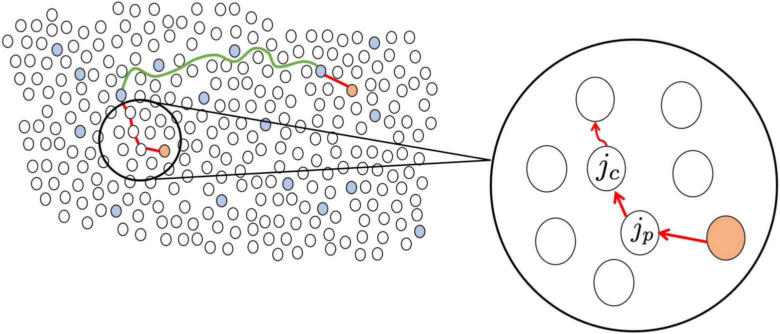

3. Choose one of these and walk to the corresponding node.

Concrete Complexity of Delfs–Galbraith

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Experimentally, given a node $j \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$, the average number of $\mathbb{F}_p$ multiplications needed to find a path to a node $j' \in \mathbb{F}_p$ is

$$c \cdot \sqrt{p} \cdot \log_2 p,$$

with $0.75 \leq c \leq 1.05$.

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At each step, we want to know if the current node j_c is ℓ-isogenous to a $j \in \mathbb{F}_p$.

SuperSolver Overview

At each step of the random walk in $\mathcal{X}(\bar{\mathbb{F}}_p, 2)$, SuperSolver inspects the ℓ-isogeny graph with fast subfield root detection for ℓ in a carefully chosen set, to efficiently detect whether j_c has an ℓ-isogenous neighbour in $\mathbb{F}_p$.

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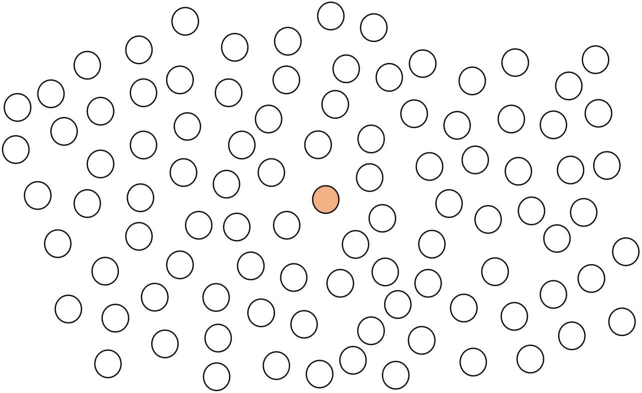

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We also show how to transform $f, \pi(f) \in \mathbb{F}_{p^2}[X]$ to give $g_1, g_2 \in \mathbb{F}_p[X]$ with the same gcd and avoid *all* costly multiplications in $\mathbb{F}_{p^2}$.

List of Optimal ℓ's

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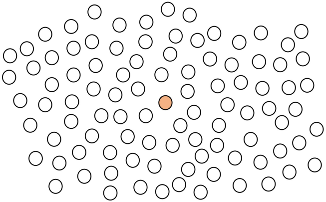

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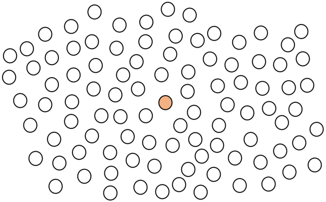

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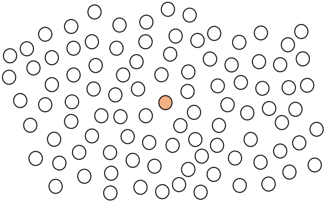

We compute the list of optimal ℓ's to minimise the number of $\mathbb{F}_p$ multiplications per node revealed. The key is that calculating the list of optimal ℓ's can be done in *precomputation*.

Results

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Example: For $p = 2^{24} - 3$, averaging over 5000 pseudo-random supersingular j-invarants in $\mathbb{F}_{p^2}$, we get:

Solver used *112878* $\mathbb{F}_p$ multiplications and walked on *1897* nodes.

SuperSolver used *53900* $\mathbb{F}_p$ multiplications and walked on *318* nodes.

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Experiments on cryptographic sized primes and one j-invariant. We ran SuperSolver and Solver until the number of $\mathbb{F}_p$ multiplications used exceeded 10^8, recording the total number of nodes covered.

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Examples:

- For $p = 2^{50} - 27$, SuperSolver covers between *3 and 4 times* the number of nodes that Solver does.
- For $p = 2^{800} - 105$, SuperSolver covers between *18 and 19 times* the number of nodes.

Conclusions

What does this mean for isogeny-based cryptography?

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- We improve the concrete complexity of Delfs–Galbraith - asymptotic complexity is unchanged.
- Affects schemes such as B-SIDH and SQISign, which have Delfs–Galbraith as their best attack.

For more details, see our full paper at eprint/2021/1488

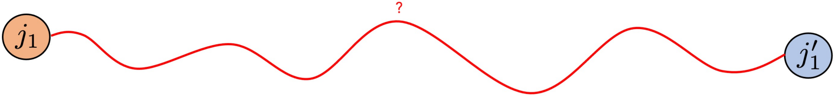
Additional Slides

Worked Example

Let $p = 2^{20} - 3$. Sample our start and end node:

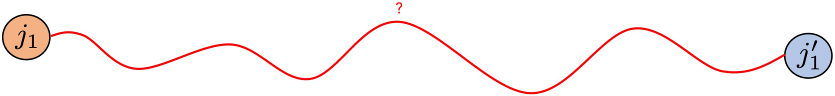
Start Node: $j_1 = 129007\alpha + 818380$

End Node: $j_2 = 97589\alpha + 660383$



Worked Example: Solver

Path from $j_1 = 129007\alpha + 818380$ to subfield node.



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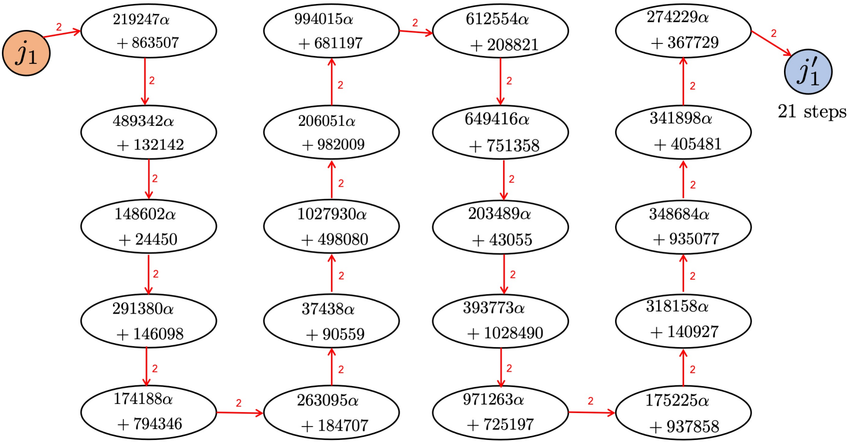
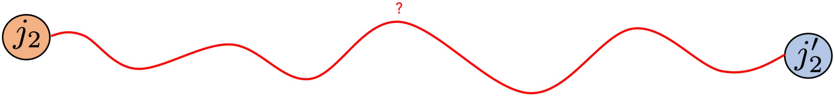
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 760776$.



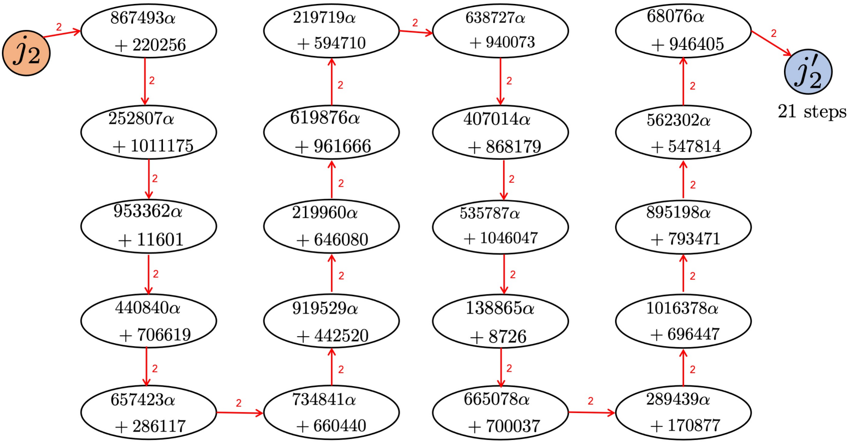
Worked Example: Solver

Path from $j_2 = 97589\alpha + 660383$ to subfield node.



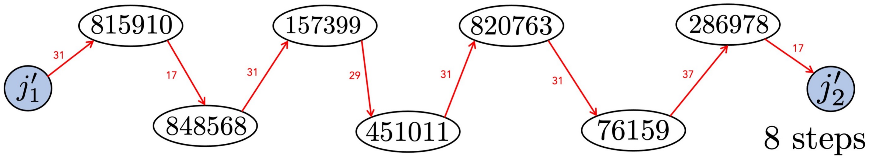
Worked Example: Solver

Path from $j_2 = 97589\alpha + 660383$ to subfield node $j_2' = 35387$.



Worked Example: Solver

Path between subfield nodes $j_1' = 760776$ and $j_2' = 35387$.



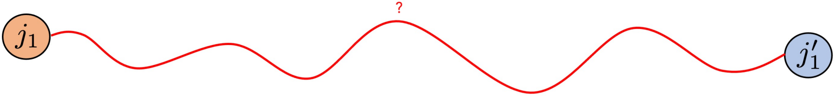
In total, the path has $21 + 21 + 8 =$ **50 steps**.

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.

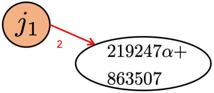
Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node.



Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.

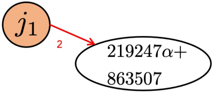


3-isogenous neighbour in $\mathbb{F}_p$?

$$\Phi_{3,p}(X, 219247\alpha + 863507) = X^4 + (212814\alpha + 479338)X^3 + (408250\alpha + 920025)X^2 + (811739\alpha + 93038)X + 942336\alpha + 847782$$

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j'_1 = 35387$.

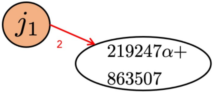


3-isogenous neighbour in $\mathbb{F}_p$?

$$g_1 = X^4 + 479338X^3 + 920025X^2 + 93038X + 847782$$
$$g_2 = 425628X^3 + 816500X^2 + 574905X + 836099$$

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.



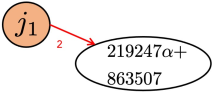
3-isogenous neighbour in $\mathbb{F}_p$?

$$g_1 = X^4 + 479338X^3 + 920025X^2 + 93038X + 847782$$
$$g_2 = 425628X^3 + 816500X^2 + 574905X + 836099$$

$$\boxed{\gcd(g_1, g_2) = 1 \implies \text{no 3-isogenous neighbour in } \mathbb{F}_p}$$

Worked Example: SuperSolver

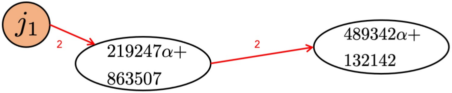
The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.



3-isogenous neighbour in $\mathbb{F}_p$? No.
Similarly, no 5-isogenous neighbour in $\mathbb{F}_p$.

Worked Example: SuperSolver

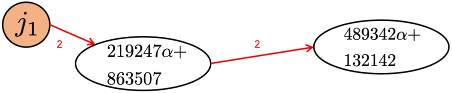
The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.



3-isogenous neighbour in $\mathbb{F}_p$?

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.

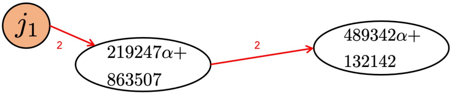


3-isogenous neighbour in $\mathbb{F}_p$?

$$\Phi_{3,p}(X, 489342\alpha + 132142) = X^4 + (872004\alpha + 13960)X^3 + (1031755\alpha + 822066)X^2 + (969683\alpha + 747785)X + 813010\alpha + 255391.$$

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.



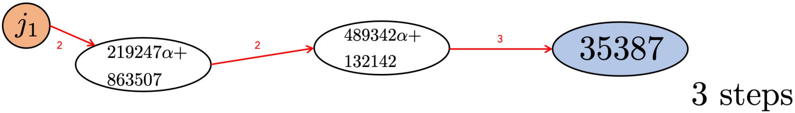
3-isogenous neighbour in $\mathbb{F}_p$?

$$g_1 = X^4 + 13960X^3 + 822066X^2 + 747785X + 255391$$
$$g_2 = 695435X^3 + 1014937X^2 + 890793X + 577447$$

$$\gcd(g_1, g_2) = X + 1013186 \implies \text{3-isogenous neighbour in } \mathbb{F}_p$$
$$-1013186 = 35387$$

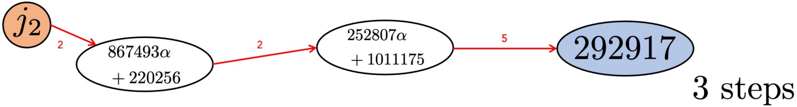
Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.



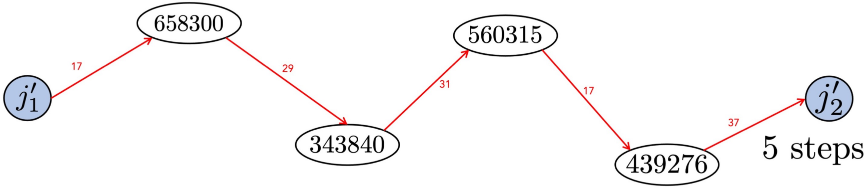
Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_2 = 97589\alpha + 660383$ to subfield node $j_2' = 292917$.



Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path between subfield nodes $j_1' = 35387$ and $j_2' = 292917$.



In total, the path has $3 + 3 + 5 = \mathbf{11}$ **steps**.

Concrete Complexity of Delfs–Galbraith

Solver is an optimised implementation of the Delfs–Galbraith algorithm.

- **Choice of $\ell = 2$:** Taking a step in $\mathcal{X}(\mathbb{F}_p, 2)$ means computing a square root.
- **Square root finding in $\mathbb{F}_{p^2}$:** Use Scott's 'Tricks of the trade' paper to find square roots in $\mathbb{F}_{p^2}$ with only two $\mathbb{F}_p$ exponentiations (and a few $\mathbb{F}_p$ multiplications).
- **Random walks in 2-isogeny graph**: Depth-first search with bounded depth.

We use Solver to find the concrete complexity of Delfs–Galbraith.

Experimentally, given a node $j \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$, the average number of $\mathbb{F}_p$ multiplications needed to find a path to a node $j' \in \mathbb{F}_p$ is

$$c \cdot \sqrt{p} \cdot \log_2 p,$$

with $0.75 \leq c \leq 1.05$.

Fast Subfield Root Detection

Recall to take a step in $\mathcal{X}(\bar{\mathbb{F}}_p, \ell)$ we find the roots of

$$\Phi_{\ell,p}(X, j_c) \in \mathbb{F}_{p^2}[X].$$

We want a fast way of detecting whether it has a root in $\mathbb{F}_p$ *without* finding roots.

Lemma

Let $\pi : a \mapsto a^p$ be the p-power Frobenius map and f a polynomial in $\mathbb{F}_{p^2}[X]$. Then, $\gcd(f, \pi(f))$ is the largest divisor of f defined over $\mathbb{F}_p$. In particular, if

$$\deg\big(\gcd(f, \pi(f))\big) = \begin{cases} 1, & f \text{ has a root in } \mathbb{F}_p \\ 0, & f \text{ does not have a root in } \mathbb{F}_p \end{cases}.$$

Fast Subfield Root Detection

Problem: In general $f, \pi(f) \in \mathbb{F}_{p^2}[X]$ and we want to avoid costly multiplications in $\mathbb{F}_{p^2}$.

Observation

For polynomials $f_1, f_2 \in \mathbb{F}_{p^2}[X]$, if

$$g_1 = af_1 + bf_2, \text{ and } g_2 = cf_1 + df_2,$$

with $a, b, c, d \in \mathbb{F}_{p^2}$ such that $ad - bc \neq 0$ with we have $\gcd(f_1, f_2) = \gcd(g_1, g_2)$.

Solution: Let $\alpha \in \mathbb{F}_{p^2}$ be such that $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$. For $f(X) := \Phi_{\ell,p}(X, j_c)$, if

$$g_1 = \frac{1}{2}\Big(f + \pi(f)\Big), \text{ and } g_2 = \frac{\alpha}{2}\Big(f - \pi(f)\Big),$$

then $g_1, g_2 \in \mathbb{F}_p[X]$ and $\gcd(f, \pi(f)) = \gcd(g_1, g_2)$.

List of Optimal ℓ's

Though the inspection of the neighbours of j_c in the ℓ-isogeny graph increases the total number of $\mathbb{F}_p$ multiplications at each step, more nodes are checked.

We compute the list of ℓ's that minimise $\#\mathbb{F}_p$ multiplications per node inspected.

1. Determine the cost per node revealed of taking a step in the 2-isogeny graph: cost_2
2. Determine the cost per node inspected in the ℓ-isogeny graph: cost_ℓ.
3. Determine a list $L = [\ell_1, \ldots, \ell_n]$ of $\ell_i > 2$ with $\text{cost}_\ell < \text{cost}_2$
4. Find the subset of L that minimises the total cost of each step:

$$\text{cost} = \frac{\text{total } \# \text{ of } \mathbb{F}_p \text{ multiplications}}{\text{total } \# \text{ of nodes revealed}}.$$

Calculating the list of optimal ℓ's can be done in precomputation.