Syndrome Decoding in the Head: Shorter Signatures from Zero-Knowledge Proofs

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Zero-Knowledge Proofs for Syndrome Decoding

Syndrome Decoding Problem

From (H, y), find $x \in \mathbb{F}^m$ such that

$$y = Hx \quad \text{and} \quad \mathrm{wt}_H(x) \leq w.$$

$\mathrm{wt}_H(x) := nb\ of\ non\text{-}zero\ coordinates\ of\ x$

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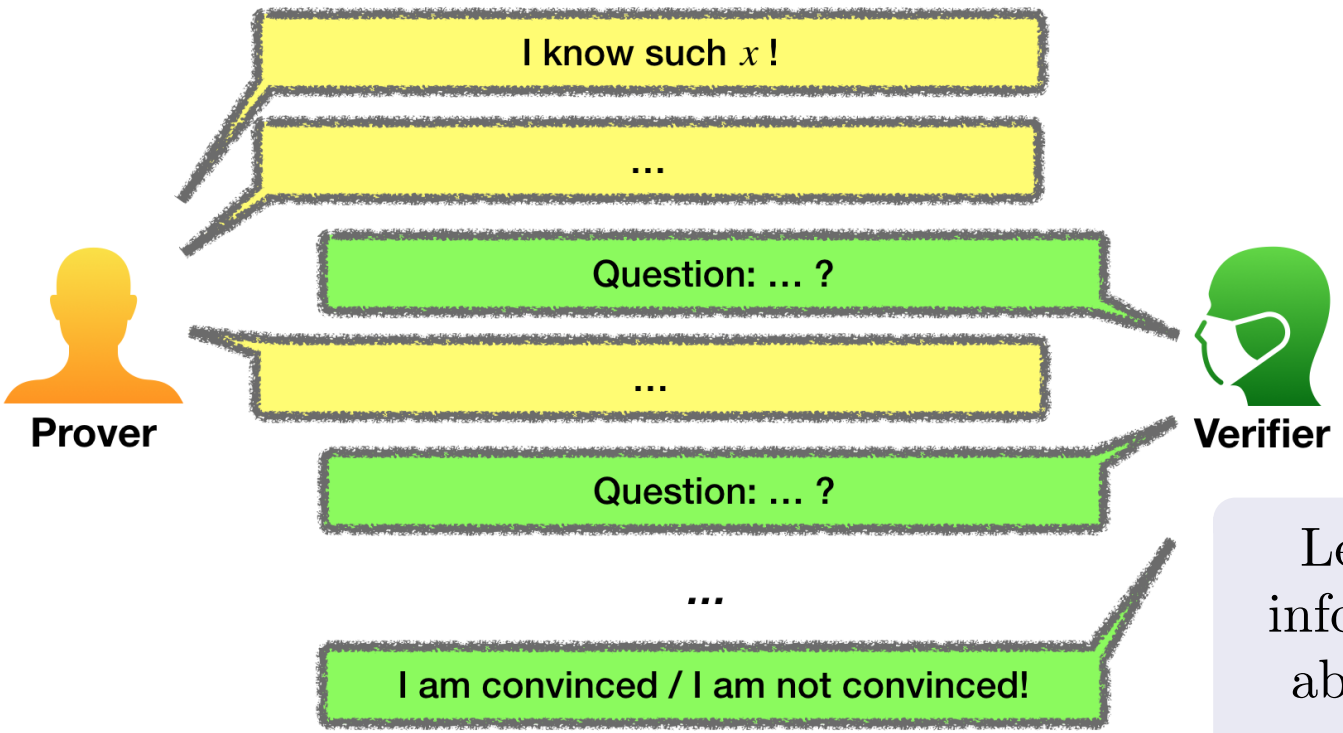


Learn no information about the secret x.

MPC-in-the-Head Paradigm

MPC-in-the-Head Paradigm

- Generic technique to build *zero-knowledge protocols* using *multi-party computation*.

- Introduced in 2007 by:

[IKOS07] Yuval Ishai, Eyal Kushilevitz, Rafail Ostrovsky, and Amit Sahai. *Zero-knowledge from secure multiparty computation.* STOC 2007.

Sharing of the secret

The secret x satisfies

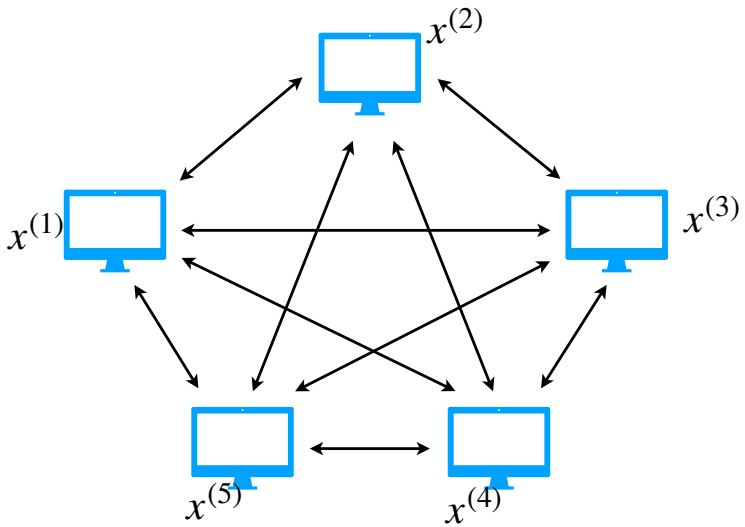
$$y = Hx \qquad \text{and} \qquad \mathrm{wt}_H(x) \leq w.$$

We share it in N parts:

$$x = x^{(1)} + x^{(2)} + \ldots + x^{(N-1)} + x^{(N)}.$$

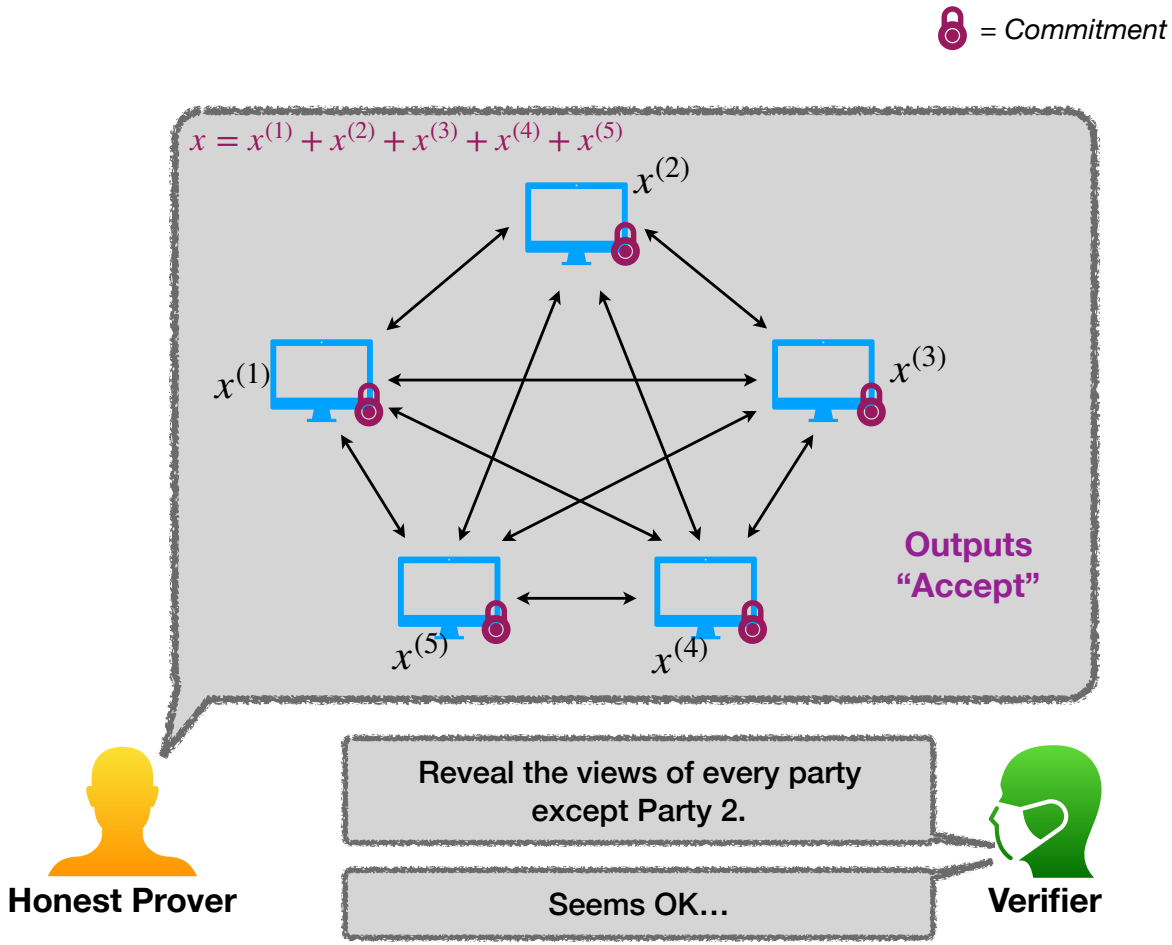
MPC-in-the-Head Paradigm

$$x = x^{(1)} + x^{(2)} + x^{(3)} + x^{(4)} + x^{(5)}$$



The multi-party computation outputs
- *Accept* if x is a syndrome decoding solution,
- *Reject* otherwise.

MPC-in-the-Head Paradigm



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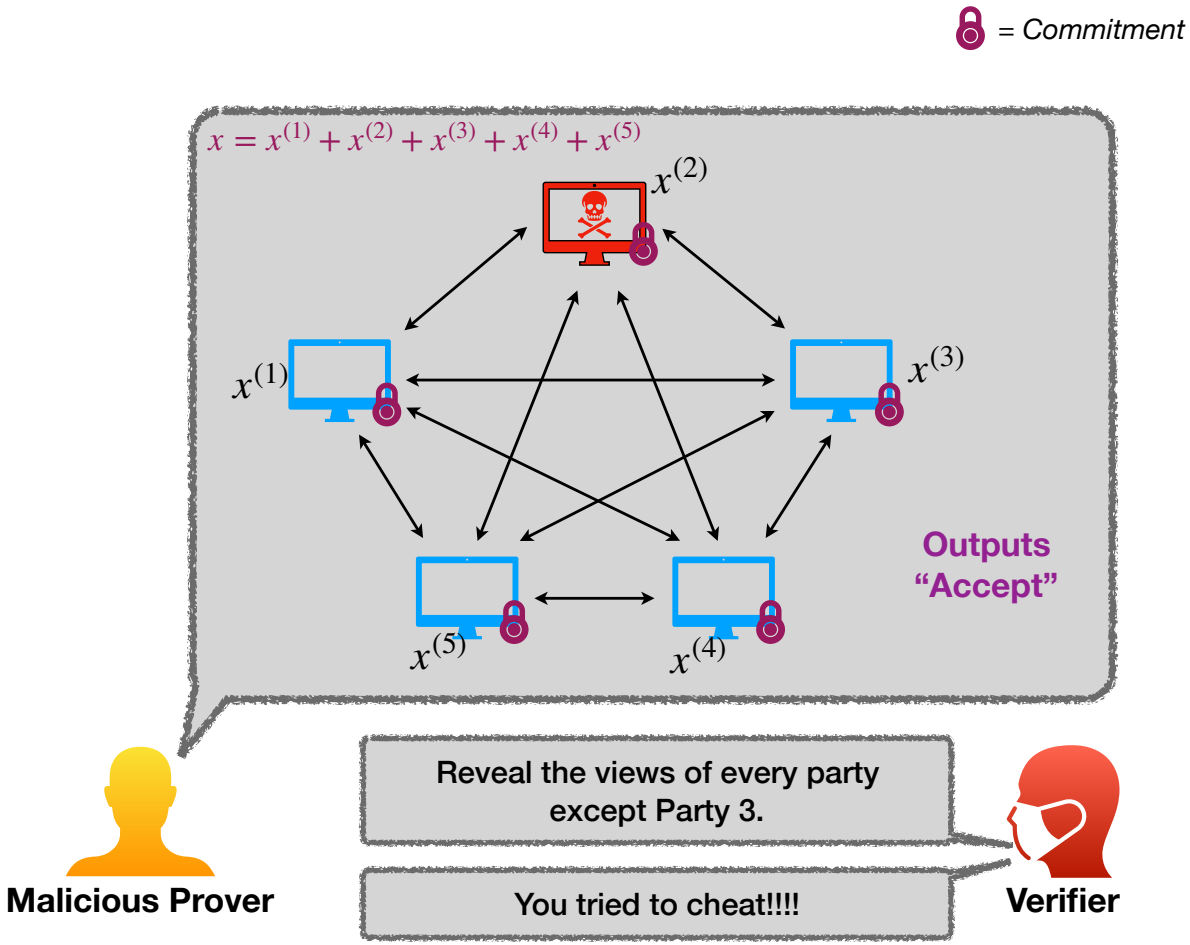

Rephrase the constraint

The multi-party computation must check that the vector x satisfies

$$\underbrace{y = Hx}_{\text{linear, easy to check}} \quad \text{and} \quad \underbrace{\mathrm{wt}_H(x) \leq w}_{\text{non-linear, hard to check}}$$

Rephrase the constraint

The multi-party computation must check that the vector x satisfies

$$y = Hx$$

and

$$\exists\, Q, P \text{ two polynomials} : SQ = PF \text{ and } \deg Q = w$$

where

S is defined by interpolation such that $\forall i,\ S(\gamma_i) = x_i$,

$F := \prod_{i=1}^{m}(X - \gamma_i)$.

Rephrase the constraint

Let us assume that there exists $Q, P \in \mathbb{F}_{\text{poly}}[X]$ s.t.

$$S \cdot Q = P \cdot F \qquad \text{and} \qquad \deg Q = w$$

where

S is built by interpolation such that $\forall i,\ S(\gamma_i) = x_i$,

$F := \prod_{i=1}^{m}(X - \gamma_i)$,

then, the verifier deduces that

$$\forall i \leq m,\ (Q \cdot S)(\gamma_i) = P(\gamma_i) \cdot F(\gamma_i) = 0$$
$$\Rightarrow\ \forall i \leq m,\ Q(\gamma_i) = 0 \ \text{ or } \ S(\gamma_i) = x_i = 0$$

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i.e.

$$\text{wt}_H(x) \leq w$$

Rephrase the constraint

Such polynomial Q can be built as

$$Q := Q' \cdot \underbrace{\prod_{i:x_i \neq 0} (X - \gamma_i)}$$

The non-zero positions of x are encoding as roots.

And $P := \frac{S \cdot Q}{F}$ since F divides $S \cdot Q$.

Guidelines for the MPC Protocol

We want to build a MPC protocol which check if some vector is a syndrome decoding solution.

Let us assume $H = (H'|I)$. We split x as $\begin{pmatrix} x_A \\ x_B \end{pmatrix}$.

We have $y = Hx$, so

$$x_B = y - H'x_A.$$

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Inputs of the MPC protocol: x_A, Q, P.

Aim of the MPC protocol:

Check that x_A corresponds to a syndrome decoding solution.

Guidelines for the MPC Protocol

Inputs: x_A, Q, P.

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2. Build the polynomial S by interpolation such that

$$\forall i \in \{1, \ldots, m\}, S(\gamma_i) = x_i.$$

Interpolation Formula:

$$S(X) = \sum_i x_i \cdot \prod_{\ell \neq i} \frac{X - \gamma_\ell}{\gamma_i - \gamma_\ell}.$$

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3. Check that $S \cdot Q = P \cdot F$.

Guidelines for the MPC Protocol

Inputs: x_A, Q, P.

1. Build $x_B := y - H' x_A$ and deduce $x := \begin{pmatrix} x_A \\ x_B \end{pmatrix}$.
2. Build the polynomial S by interpolation such that

$$\forall i \in \{1, \ldots, m\}, S(\gamma_i) = x_i.$$

3. Get a random point r from $\mathbb{F}_{\text{points}}$ (field extension of $\mathbb{F}_{\text{poly}}$).
4. Compute $S(r)$, $Q(r)$ and $P(r)$.
5. Using [BN20], check that $S(r) \cdot Q(r) = P(r) \cdot F(r)$.

[BN20] Carsten Baum and Ariel Nof. *Concretely-efficient zero-knowledge arguments for arithmetic circuits and their application to lattice-based cryptography.* PKC 2020.

Analysis

Even if x_A does not describe a SD solution (implying that $S \cdot Q \neq P \cdot F$), the MPC protocol can output Accept if

Case 1 :

$$S(r) \cdot Q(r) = P(r) \cdot F(r)$$

which occurs with probability (Schwartz-Zippel Lemma)

$$\Pr_{r \xleftarrow{\$} \mathbb{F}_{\text{points}}} [S(r) \cdot Q(r) = P(r) \cdot F(r)] \leq \frac{m + w - 1}{|\mathbb{F}_{\text{points}}|}$$

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Case 2 : the [BN20] protocol fails, which occurs with probability

$$\frac{1}{|\mathbb{F}_{\text{points}}|}.$$

Summary

The MPC protocol π checks that (x_A, Q, P) describes a solution of the SD instance (H, y).

	Output of π	
	ACCEPT	REJECT
A good witness	1	0
Not a good witness	p	$1-p$

where

$$p = \underbrace{\frac{m+w-1}{|\mathbb{F}_{\text{points}}|}}_{\substack{\text{false positive} \\ \text{from Schwartz-Zippel}}} + \left(1 - \frac{m+w-1}{|\mathbb{F}_{\text{points}}|}\right) \cdot \underbrace{\frac{1}{|\mathbb{F}_{\text{points}}|}}_{\substack{\text{false positive} \\ \text{from [BN20]}}}$$

MPC-in-the-Head paradigm

Prover $\mathcal{P}$		Verifier $\mathcal{V}$
H, y, x such that $y = Hx$ and $\mathrm{wt}_H(x) \leq w$		H, y
Prepare Q, P.		
$\textsc{Com}_i \leftarrow \mathrm{Com}(\text{inputs of } \mathcal{P}_i)$	$\xrightarrow{\textsc{Com}_1,\ldots,\textsc{Com}_N}$	$r \in \mathbb{F}_{\text{points}}$
Run the MPC protocol π for each party.	$\xleftarrow{r}$	
	$\xrightarrow{\text{broadcast messages}}$	$i^* \xleftarrow{\$} \{1, \ldots, N\}$
	$\xleftarrow{i^*}$	
	$\xrightarrow{\text{all } V_i \text{ for } i \neq i^*}$	
		Check that the views are consistent Check that the MPC output is Accept

Zero-Knowledge Protocol

Soundness error:

$$p + (1 - p) \cdot \frac{1}{N}$$

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Proof size:

- Inputs of $N - 1$ parties:

		$\mathcal{P}_1$		$\mathcal{P}_2$		$\cdots$		$\mathcal{P}_{N-1}$		$\mathcal{P}_N$
x_A	$=$	$[\![x_A]\!]_1$	$+$	$[\![x_A]\!]_2$	$+$	$\cdots$	$+$	$[\![x_A]\!]_{N-1}$	$+$	$[\![x_A]\!]_N$
Q	$=$	$[\![Q]\!]_1$	$+$	$[\![Q]\!]_2$	$+$	$\cdots$	$+$	$[\![Q]\!]_{N-1}$	$+$	$[\![Q]\!]_N$
P	$=$	$[\![P]\!]_1$	$+$	$[\![P]\!]_2$	$+$	$\cdots$	$+$	$[\![P]\!]_{N-1}$	$+$	$[\![P]\!]_N$
		$\uparrow$		$\uparrow$				$\uparrow$		
		seed_1		seed_2				seed_{N-1}		

Zero-Knowledge Protocol

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Proof size:

- Inputs of $N - 1$ parties:
  - Party $i < N$: a seed of λ bits
  - Last party: $[\![x_A]\!]_N$, $[\![Q]\!]_N$, $[\![P]\!]_N$

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- Inputs of $N - 1$ parties:
  - Party $i < N$: a seed of λ bits
  - Last party: $[\![x_A]\!]_N$, $[\![Q]\!]_N$, $[\![P]\!]_N$
- Extra cost due to [BN20] protocol.
- Use of several optimisations.

Fiat-Shamir Transform

Signature algorithm:

Inputs:

- x such that $y = Hx$ and $\mathrm{wt}_H(x) \leq w$
- the message mess to sign

1. Prepare the witness, *i.e.* the polynomials P and Q.
2. Commit to party's inputs in distinct commitments $\mathrm{COM}_1, \ldots, \mathrm{COM}_N$.
3. $r = \mathrm{Hash}(\mathsf{mess}, \mathsf{salt}, \mathrm{COM}_1, \ldots, \mathrm{COM}_N)$.
4. Run the MPC protocol π for each party.
5. $i^* = \mathrm{Hash}(\mathsf{mess}, \mathsf{salt}, r, \text{broadcast messages})$.
6. Build the signature with the views of all the parties except the party i^*.

Security of the signature

5-round Identification Scheme $\Rightarrow$ Signature

Attack of [KZ20]:

$$\mathrm{cost}_{\mathrm{forge}} := \min_{\tau_1,\tau_2:\tau_1+\tau_2=\tau} \left\{ \frac{1}{\sum_{i=\tau_1}^{\tau} \binom{\tau}{i} p^i (1-p)^{\tau-i}} + N^{\tau_2} \right\}$$

[KZ20] Daniel Kales and Greg Zaverucha. *An attack on some signature schemes constructed from five-pass identification schemes.* CANS 2020.

Parameters selected

Variant 1: SD over $\mathbb{F}_2$,

$$(m, k, w) = (1280, 640, 132)$$

We have $\mathbb{F}_{poly} = \mathbb{F}_{2^{11}}$.

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Variant 2: SD over $\mathbb{F}_2$,

$$(m, k, w) = (1536, 888, 120)$$

but we split $x := (x_1 \mid \ldots \mid x_6)$ into 6 chunks and we prove that $\mathrm{wt}_H(x_i) \leq \frac{w}{6}$ for all i.

We have $\mathbb{F}_{poly} = \mathbb{F}_{2^8}$.

Parameters selected

Variant 3: SD over $\mathbb{F}_{2^8}$,

$$(m, k, w) = (256, 128, 80)$$

We have $\mathbb{F}_{poly} = \mathbb{F}_{2^8}$.

Obtained Performances

Scheme Name	$\lvert\mathsf{sgn}\rvert$	$\lvert\mathsf{pk}\rvert$	t_{sgn}	t_{verif}
FJR22 - $\mathbb{F}_2$ (fast)	15.6 KB	0.09 KB	-	-
FJR22 - $\mathbb{F}_2$ (short)	10.9 KB	0.09 KB	-	-
FJR22 - $\mathbb{F}_2$ (fast)	17.0 KB	0.09 KB	13 ms	13 ms
FJR22 - $\mathbb{F}_2$ (short)	11.8 KB	0.09 KB	64 ms	61 ms
FJR22 - $\mathbb{F}_{256}$ (fast)	11.5 KB	0.14 KB	6 ms	6 ms
FJR22 - $\mathbb{F}_{256}$ (short)	8.26 KB	0.14 KB	30 ms	27 ms

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Comparison Code-based Signatures (1/2)

Scheme Name	$\|\mathsf{sgn}\|$	$\|\mathsf{pk}\|$	t_{sgn}	t_{verif}
BGS21	24.1 KB	0.1 KB	-	-
BGS21	22.5 KB	1.7 KB	-	-
GPS21 - 256	22.2 KB	0.11 KB	-	-
GPS21 - 1024	19.5 KB	0.12 KB	-	-
FJR21 (fast)	22.6 KB	0.09 KB	13 ms	12 ms
FJR21 (short)	16.0 KB	0.09 KB	62 ms	57 ms
BGKM22 - Sig1	23.7 KB	0.1 KB	-	-
BGKM22 - Sig2	20.6 KB	0.2 KB	-	-
FJR22 - $\mathbb{F}_2$ (fast)	15.6 KB	0.09 KB	-	-
FJR22 - $\mathbb{F}_2$ (short)	10.9 KB	0.09 KB	-	-
FJR22 - $\mathbb{F}_2$ (fast)	17.0 KB	0.09 KB	13 ms	13 ms
FJR22 - $\mathbb{F}_2$ (short)	11.8 KB	0.09 KB	64 ms	61 ms
FJR22 - $\mathbb{F}_{256}$ (fast)	11.5 KB	0.14 KB	6 ms	6 ms
FJR22 - $\mathbb{F}_{256}$ (short)	**8.26 KB**	0.14 KB	30 ms	27 ms

Comparison Code-based Signatures (2/2)

Scheme Name	$\|\mathsf{sgn}\|$	$\|\mathsf{pk}\|$	t_{sgn}	t_{verif}
Durandal - I	3.97 KB	14.9 KB	4 ms	5 ms
Durandal - II	4.90 KB	18.2 KB	5 ms	6 ms
LESS-FM - I	15.2 KB	9.78 KB	-	-
LESS-FM - II	5.25 KB	205 KB	-	-
LESS-FM - III	10.39 KB	11.57 KB	-	-
Wave	**2.07 KB**	3.1 MB	$\geq$ 300 ms	2 ms
Wavelet	**0.91 KB**	3.1 MB	$\geq$ 300 ms	$\leq$ 1 ms
FJR22 - $\mathbb{F}_2$ (fast)	15.6 KB	0.09 KB	-	-
FJR22 - $\mathbb{F}_2$ (short)	10.9 KB	0.09 KB	-	-
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Conclusion

Summary

- ☞ New signature scheme with Syndrome Decoding
- ☞ Conservative scheme (based on a NP-Hard problem)
- ☞ Small "signature size + public key size"

Future Work

- ☞ Optimize the signature implementation.
- ☞ Search parameter sets which provide better performances.

More details in `https://eprint.iacr.org/2022/188`.