Moz$\mathbb{Z}_{2^k}$arella: Efficient Vector-OLE and Zero-Knowledge Proofs Over $\mathbb{Z}_{2^k}$

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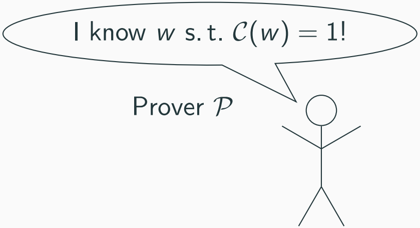
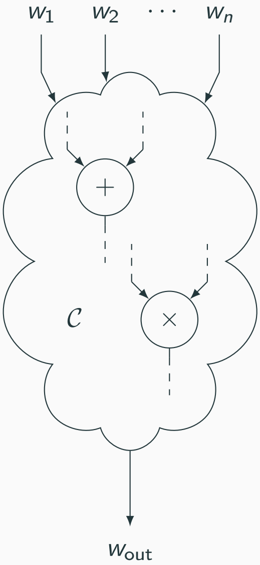
Zero-Knowledge Proofs for Arithmetic Circuits



$\mathcal{C}$ is an arithmetic circuit over the ring $\mathbb{Z}_{2^k}$



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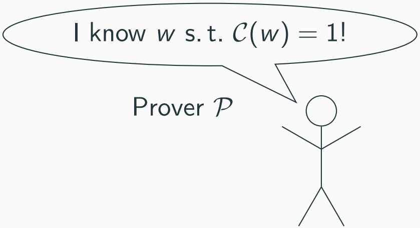

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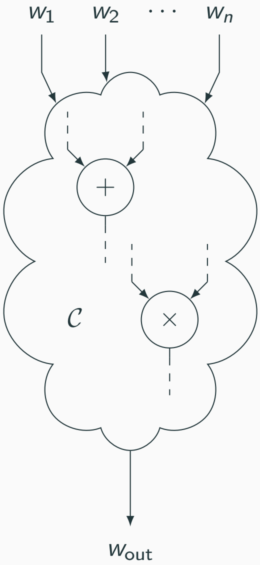
Security Properties:

- Soundness
- Zero-Knowledge
- Completeness



Computation over $\mathbb{Z}_{2^k}$ vs. Finite Fields $\mathbb{F}_{p^r}$

- $\mathbb{Z}_{2^k} = \mathbb{Z}/2^k\mathbb{Z} = (\{0, \ldots, 2^k - 1\}, +, \cdot)$ – the ring of integers modulo 2^k

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Advantages

- maps naturally to data types used by CPUs and programming languages
- e.g., `uint32_t`, `uint64_t` in C

$\rightsquigarrow$ easier to convert programs to corresponding circuits

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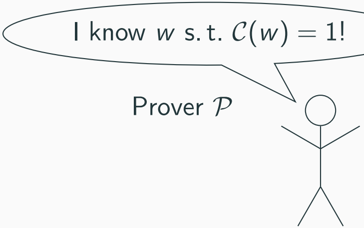
Disadvantages

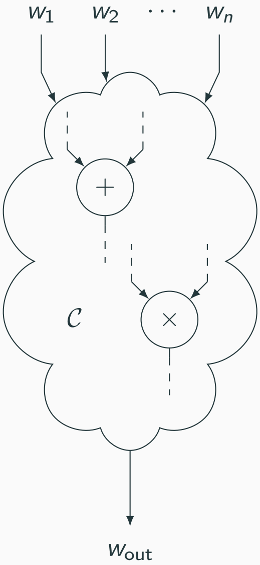
- $\mathbb{Z}_{2^k}$ is not a field
- zero-divisors
- no division by multiples of 2
- polynomials can have lots of roots

$\rightsquigarrow$ common tricks don't work and protocols get more complicated

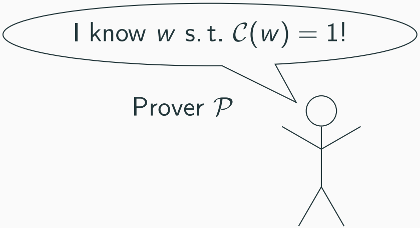
$\rightsquigarrow$ proofs of security are harder

Zero-Knowledge Proofs for Arithmetic Circuits



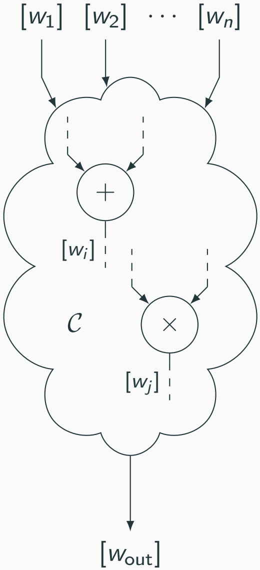


Zero-Knowledge Proofs for Arithmetic Circuits via Commit & Prove



Ingredients:

1. linearly homomorphic commitments $[\,\cdot\,]$
   - can compute $[z] \leftarrow a \cdot [x] + [y] + b$



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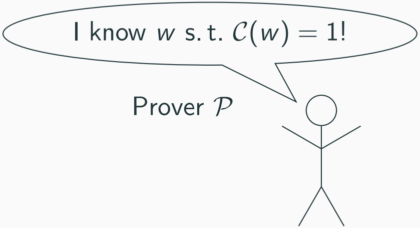

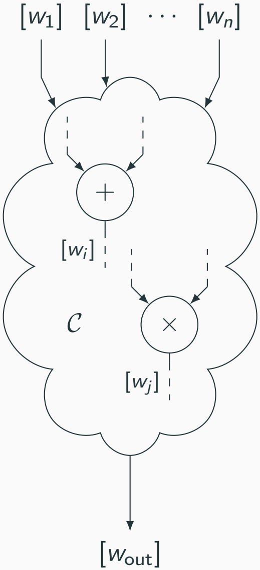

Ingredients:

1. linearly homomorphic commitments $[\cdot]$
   - can compute $[z] \leftarrow a \cdot [x] + [y] + b$
2. multiplication check
   - given $([a], [b], [c])$, verify $a \cdot b \stackrel{?}{=} c$

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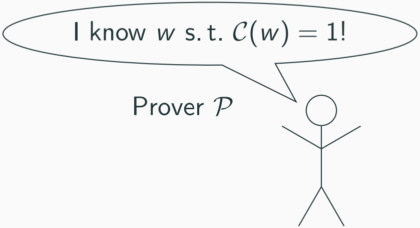
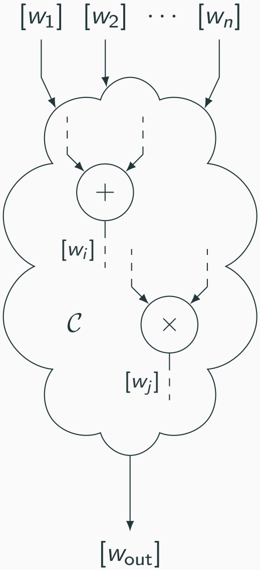


Setting:

- designated verifier
- linear communication
- linear time prover and verifier
- minimal overhead compared to circuit evaluation
  - computation and memory



Linearly Homomorphic Commitments from Vector OLE

For large fields: authenticate $x \in \mathbb{F}$ with information-theoretic MAC:

$$M[x] = \Delta \cdot x + K[x] \in \mathbb{F}$$

Prover holds value x and tag $M[x]$

Verifier holds global key $\Delta \in_R \mathbb{F}$ and key $K[x] \in_R \mathbb{F}$

(cf. Mac'n'Cheese [BMRS21], Wolverine [WYKW21])

Linearly Homomorphic Commitments from Vector OLE

For ring $\mathbb{Z}_{2^k}$: authenticate $x \in \mathbb{Z}_{2^k}$ over the larger ring $\mathbb{Z}_{2^{k+s}}$

$$M[x] = \Delta \cdot \tilde{x} + K[x] \pmod{2^{k+s}} \qquad \text{with } x = \tilde{x} \pmod{2^k}$$

Prover holds value $\tilde{x}$ and tag $M[x]$ Verifier holds global key $\Delta \in_R \mathbb{Z}_{2^s}$ and key $K[x] \in_R \mathbb{Z}_{2^{k+s}}$

(cf. SPD$\mathbb{Z}_{2^k}$ [CDESX18], Appenzeller2Brie [BBMRS21])

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Vector Oblivious Linear Evaluation:

such that $\vec{M} = \Delta \cdot \vec{x} + \vec{K} \pmod{2^\ell}$

VOLE for $\mathbb{Z}_{2^k}$

How to instantiate VOLE?

From Oblivious Transfer or Homomorphic Encryption

$\rightsquigarrow$ communication at least linear in output size

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Via Pseudorandom Correlation Generators (PCGs)

- interactive generation of a short seed $\rightarrow$ non-interactive expansion to long correlated string
- communication <u>sublinear</u> in vector length n
- based on variants of Learning Parity with Noise (LPN)
- active security only for fields [WYKW21; Boy+19]

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Here: actively secure VOLE for rings $\mathbb{Z}_{2^k}$ with sublinear communication

Learning Parity with Noise (LPN) Assumption

$$\begin{bmatrix} \\ \vec{s} \\ \\ \end{bmatrix}$$

$\vec{s}$: short, uniform seed

Learning Parity with Noise (LPN) Assumption

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A: generating matrix of a random linear code

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VOLE Extension Protocol

Goal: expand m seed/base VOLEs into $n \gg m$ VOLEs: $\vec{z} = \Delta \cdot \vec{x} + \vec{y}$

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based on Wolverine [WYKW21]

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1. start with $\vec{w} = \Delta \cdot \vec{u} + \vec{v}$ base VOLEs of length m
2. generate and concatenate t single-point VOLEs of length n/t

   $\rightsquigarrow$ obtain $\vec{c} = \Delta \cdot \vec{e} + \vec{b}$ where $\vec{e}$ has t non-zero entries

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3. apply the generating matrix A to all seed VOLEs and add the error VOLEs

Sender computes $\begin{cases} \vec{x} := A \cdot \vec{u} + \vec{e} \\ \vec{z} := A \cdot \vec{w} + \vec{c} \end{cases} \implies \vec{z} := \Delta \cdot \vec{x} + \vec{y}$

Receiver computes $\vec{y} := A \cdot \vec{v} + \vec{b}$

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crucial step: need communication sublinear in n

Sender computes $\vec{z} := A \cdot \vec{w} + \vec{c}$ $\Longrightarrow$ $\vec{z} := \Delta \cdot \vec{x} + \vec{y}$

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GGM Construction for a PRF

$\boxed{s_0}$

$G(x) = G(x)_0 \mid\mid G(x)_1$ length-doubling PRG

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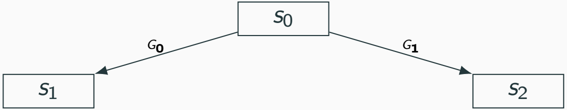

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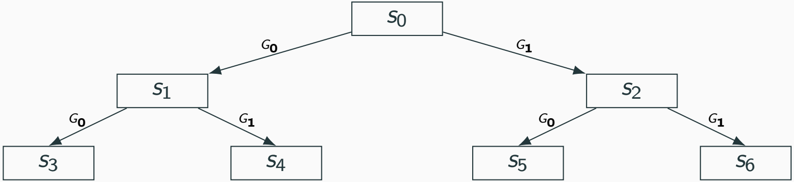

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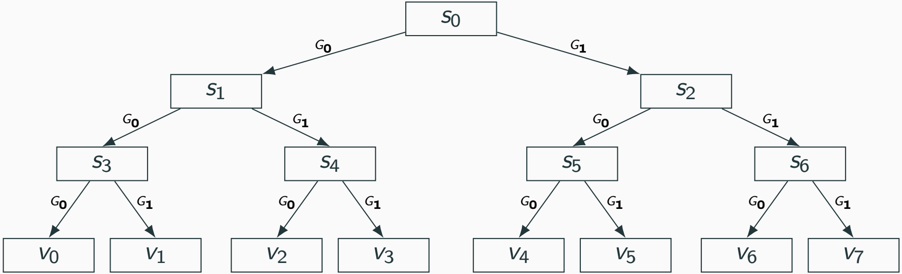

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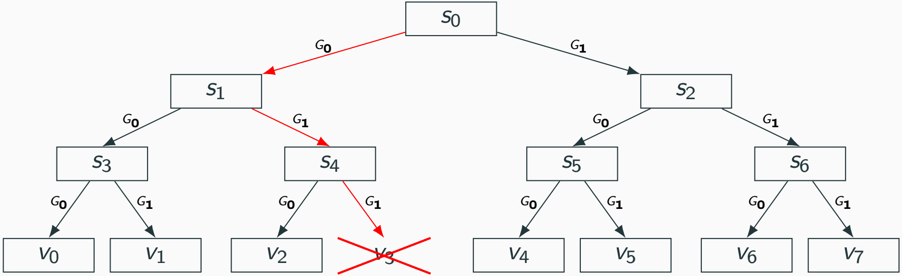

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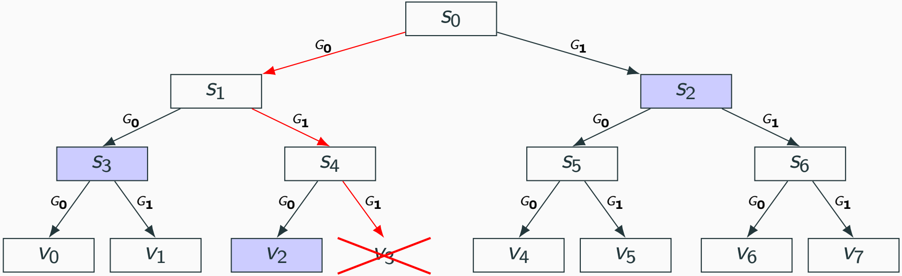

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   - $\mathcal{R}$ samples a PRF key k and
     use $\log(n/t)$ OTs to obliviously transfer a punctured key $k\{\alpha\}$ to $\mathcal{S}$
   - set $w_i := v_i := F(k, i)$ for all $i \neq \alpha$ and $v_\alpha := F(k, \alpha)$

   $\rightsquigarrow$ $\mathcal{R}$ does not learn α and $\mathcal{S}$ does not learn v_α

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2. compute $w_\alpha = \Delta \cdot \beta + v_\alpha$
   - use one base VOLE $\delta = \Delta \cdot \beta + \gamma$
   - $\mathcal{R}$ sends $d := \gamma - \sum_j v_j$
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a malicious receiver $\mathcal{R}$ can
- use inconsistent values in the OTs
- send an incorrect d

$\rightsquigarrow$ leakage on the noise coordinate α

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3. ensure consistency

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1. distribute long random vector $\vec{v}$ or $\vec{w}$ with ... PRF F
   - $\mathcal{R}$ sa...
2. com...
   - use one bas...
   - $\mathcal{R}$ sends $d := \gamma - \sum_j v_j$
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3. ensure consistency
   - Wolverine's [WYKW21] approach: use a random linear combination over the field $\mathbb{F}_q$

random linear combinations are tricky over $\mathbb{Z}_{2^k}$

– need to enlarge ring by s bit to ensure consistency of lower bits! cf. SPD$\mathbb{Z}_{2^k}$ [CDESX18]

$\rightsquigarrow$ cannot do multiple iterations

Single-Point VOLE Protocol

Goal: generate $\vec{w} = \Delta \cdot \vec{u} + \vec{v}$ where $u_\alpha = \beta \in_R \mathbb{Z}_{2^\ell} \setminus \{0\}$ and $u_j = 0$ for $j \neq \alpha$

1. distribute long random vector $\vec{w} \approx \vec{v}$ with a punctured PRF F
   - $\mathcal{R}$ samples a PRF key k and use $\log(n/t)$ OTs to obliviously transfer a punctured key $k\{\alpha\}$ to $\mathcal{S}$
   - set $w_i := v_i := F(k, i)$ for all $i \neq \alpha$ and $v_\alpha := F(k, \alpha)$

   $\rightsquigarrow$ $\mathcal{R}$ does not learn α and $\mathcal{S}$ does not learn v_α
2. compute $w_\alpha = \Delta \cdot \beta + v_\alpha$
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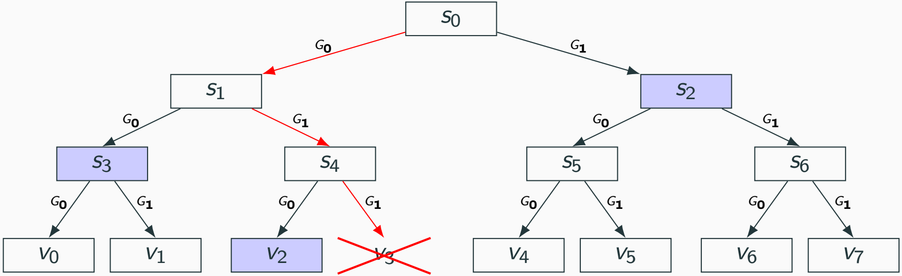
GGM Construction for a Punctured PRF



$G(x) = G(x)_0 \mid\mid G(x)_1$ length-doubling PRG

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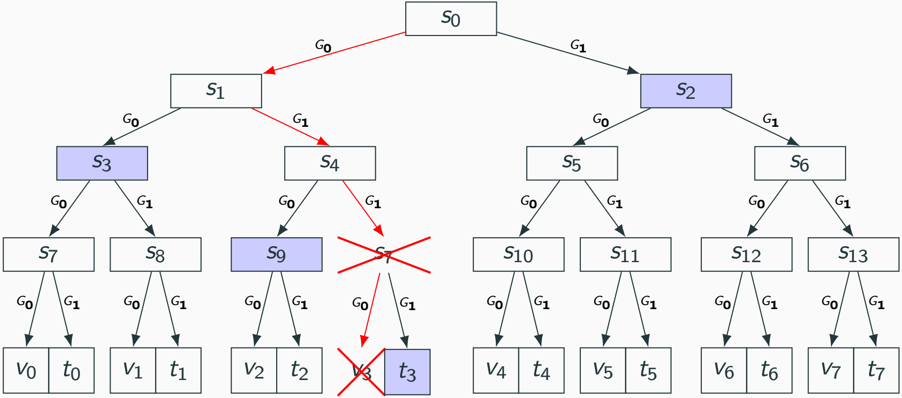

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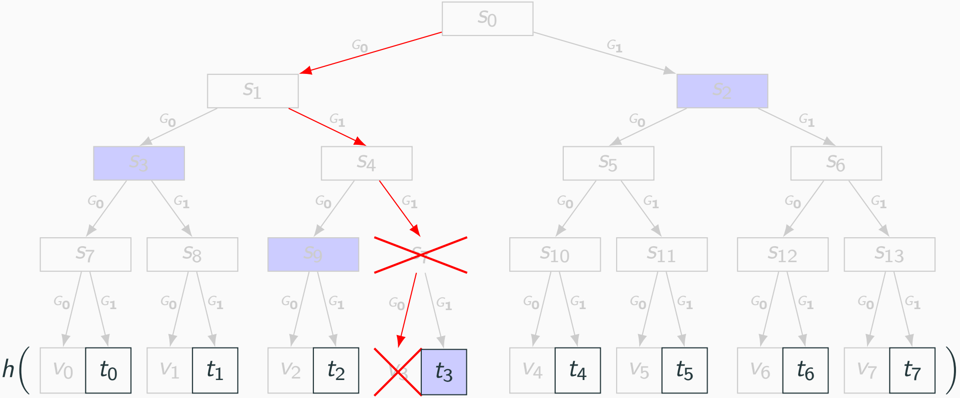

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$\rightsquigarrow$ compute and verify universal hash $h(t_0, \ldots, t_7)$

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– malicious $\mathcal{R}$ can guess χ_α with probability $1/2$

$\rightsquigarrow$ adjust functionality to allow leakage of α with probability $1/2$

$\rightsquigarrow$ increase noise rate of the error to compensate

QuarkSilver – More Efficient Multiplication Check for $\mathbb{Z}_{2^k}$

Verifying Multiplications for $\mathbb{Z}_{2^k}$ – QuarkSilver

Goal: Given $([a], [b], [c])$, verify that $\tilde{a} \cdot \tilde{b} = \tilde{c} \pmod{2^k}$

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use a random linear combination to verify many multiplications

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- **Ring** $\mathbb{Z}_{2^{k+s}}$: polynomial p has at most $2^{s/2}$ roots in the range $\{0, \ldots, 2^s - 1\}$

Performance & Summary

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QuarkSilver Zero-Knowledge

- $\approx$ one ring element per multiplication
- $\approx$ 1.3 million multiplications per second
- QuickSilver [YSWW21]: $\approx 5\times$ more for 64 bit field
- QuarkSilver
  - larger rings
    $\rightsquigarrow$ more expensive arithmetic and communication
  - but provides native 64 bit arithmetic

Summary

Efficient VOLE for $\mathbb{Z}_{2^k}$

- practical actively secure protocol with sublinear communication
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by Popo le Chien (CC BY-SA 3.0), https://commons.wikimedia.org/wiki/File:Mozzarella_di_bufala2.jpg

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Full version of Mo$\mathbb{Z}_{2^k}$zarella on ePrint: `https://ia.cr/2022/819`

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Prover holds value x and tag $M[x]$

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- authenticates the lower k bits
- increases storage and communication costs

Linear Operations and Openings [BBMRS21]

Linear Operations: to compute $[z] \leftarrow \alpha \cdot [x] + [y] + c$, set

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Batched Open: e.g., send $H(M[x_1], \ldots, M[x_n])$ instead of all $M[x_1], \ldots, M[x_n]$

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- bucketing with untrusted multiplication triples $\rightsquigarrow$ Beaver multiplication

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- bucketing with untrusted multiplication triples $\rightsquigarrow$ Beaver multiplication

Variant 2: Adaption of the Mac'n'Cheese [BMRS21] check

- needs large message space $\rightsquigarrow$ authenticate $\mathbb{Z}_{2^{k+s}}$ elements over $\mathbb{Z}_{2^{k+\mathbf{2}s}}$

Verifying Multiplications for $\mathbb{Z}_{2^k}$ [BBMRS21]

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	$\xrightarrow{\text{Random}() \to [x]}$	
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	$\mathrm{Random}() \to [x]$ $\longrightarrow$	
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	$\longleftarrow$ $\eta \in_R \mathbb{Z}_{2^s}$	

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Prover $\mathcal{P}$ — Verifier $\mathcal{V}$

$\mathcal{P} \rightarrow \mathcal{V}$: $\mathrm{Random}() \to [x]$

$\mathcal{P} \rightarrow \mathcal{V}$: $\mathrm{Input}(x \cdot b) \to [z]$

$\mathcal{V} \rightarrow \mathcal{P}$: $\eta \in_R \mathbb{Z}_{2^s}$

$\mathcal{P} \rightarrow \mathcal{V}$: $\mathrm{Open}(\eta \cdot [a] - [x]) \to \epsilon$

$\mathcal{P} \rightarrow \mathcal{V}$: $\mathrm{CheckZero}(\eta \cdot [c] - [z] - \epsilon \cdot [b])$

VOLE Performace ($\sigma = 40$)

Table 1: Run-time of the Extend operation in ns per VOLE and the communication cost in bit per VOLE. The benchmarks are parametrized by the ring size ℓ (i.e., using $\mathbb{Z}_{2^\ell}$). The computational security parameter is set to $\kappa = 128$. For statistical security $\sigma \in \{40, 80\}$, we target batch sizes of $n_o = 10^7$ and $n_o = 10^8$, and use LPN parameters (m, t, n).

σ	ℓ	Run-time LAN	Run-time WAN	Communication $\mathcal{S} \to \mathcal{R}$	Communication $\mathcal{R} \to \mathcal{S}$	Communication total
	$m = 553\,600,\ t = 2\,186,\ n = 10\,558\,380$					
	64	27.3	190.8	0.467	0.927	1.394
	104	40.7	186.7	0.509	0.955	1.464
	144	55.2	212.6	0.551	0.983	1.534
40	244	80.7	255.0	0.593	1.011	1.604
	$m = 773\,200,\ t = 15\,045,\ n = 100\,816\,545$					
	64	20.1	46.0	0.318	0.636	0.954
	104	33.2	58.9	0.347	0.655	1.002
	144	46.7	75.1	0.376	0.674	1.050
	244	76.7	102.8	0.405	0.694	1.098

VOLE Performace ($\sigma = 80$)

Table 2: Run-time of the Extend operation in ns per VOLE and the communication cost in bit per VOLE. The benchmarks are parametrized by the ring size ℓ (i.e., using $\mathbb{Z}_{2^\ell}$). The computational security parameter is set to $\kappa = 128$. For statistical security $\sigma \in \{40, 80\}$, we target batch sizes of $n_o = 10^7$ and $n_o = 10^8$, and use LPN parameters (m, t, n).

σ	ℓ	Run-time		Communication		
		LAN	WAN	$\mathcal{S} \to \mathcal{R}$	$\mathcal{R} \to \mathcal{S}$	total
	$m = 830\,800,\ t = 2\,013,\ n = 10\,835\,979$					
	64	27.6	171.9	0.431	0.853	1.284
	104	42.6	194.1	0.469	0.879	1.349
	144	59.4	217.1	0.508	0.905	1.413
80	244	89.3	277.4	0.547	0.931	1.477
	$m = 866\,800,\ t = 18\,114,\ n = 100\,913\,094$					
	64	21.4	48.2	0.383	0.765	1.148
	104	34.3	61.0	0.418	0.789	1.206
	144	49.2	76.0	0.453	0.812	1.264
	244	79.8	106.8	0.487	0.835	1.322

VOLE Perfomance Comparison with Wolverine [WYKW21]

Table 3: Run-times in ns per VOLE in different bandwidth settings, when generating ca. 10^7 VOLEs with 5 threads and statistical security $\sigma \geq 40$. The parameter ℓ denotes the size of a ring or field element. The numbers for Wolverine are taken from [WYKW21].

	ℓ	20 Mbit/s	50 Mbit/s	100 Mbit/s	500 Mbit/s	1 Gbit/s	10 Gbit/s
this work	64	110.0	68.7	55.0	50.2	50.6	50.4
	104	142.0	95.2	80.1	73.2	71.5	73.6
	144	178.6	134.7	119.3	111.6	112.6	113.3
	244	266.3	219.1	201.7	194.5	193.7	196.5
Wolverine	61	101.0	87.0	85.0	85.0	85.0	—

QuarkSilver Performance

Table 4: We measure the run-time of a batch of $\approx 10^7$ multiplications and their verification in ns per multiplication and the communication cost in bit per multiplication. The benchmarks are parametrized by the statistical security parameter σ, and the computational security parameter is set to $\kappa = 128$. For $\sigma = 40$, we use the ring of size $\ell = 162$, for $\sigma = 80$, we use $\ell = 244$.

σ		Run-time		Communication		
		LAN	WAN	$\mathcal{S} \to \mathcal{R}$	$\mathcal{R} \to \mathcal{S}$	total
40	vole	78.5	265.5	0.5	1.0	1.5
	mult	663.2	2 101.5	192.0	0.0	192.0
	check	28.2	38.2	0.0	0.0	0.0
	total	769.9	2 405.2	192.5	1.0	193.5
80	vole	125.3	345.6	0.5	0.9	1.5
	mult	680.7	2 767.2	256.0	0.0	256.0
	check	42.3	52.4	0.0	0.0	0.0
	total	848.3	3 165.2	256.5	0.9	257.5