

iO from PRGs in NC^0 , LPN, and Bilinear
Maps

Or:

On The Power of Lattice-Free Cryptography

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Joint work with:



Crypto from LWE/Lattices

Homomorphic Encryption [Gen 09, BV 11, BGV 12...]

Attribute-Based Encryption/Predicate Encryption [GVW 12, BGG+13, GVW 15]

MKFHE [CM 15, MW 16]

Succinct Functional Encryption [GKPVZ 13]

Universal Thresholdizers [BGG+18]

Lockable Obfuscation [GKW 17, WZ 17]

CI Hash/ NIZK [CCHLRW 19, PS 19]

SNARGS for P [CJJ 21]

Quantum FHE [M 18]

CVQC [M 18]

.....

Crypto from LWE/Lattices

Homomorphic Encryption [Gen 09, BV 11, BGV 12...]

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Fundamental Question

Are Lattice based hardness assumptions essential to building these primitives?

CI Hash/ NIZK [CCHLRW 19, PS 19]

SNARGS for P [CJJ 21]

Quantum FHE [M 18]

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Our Result in a Nutshell

We can base not only these primitives but almost all known cryptographic primitives from sub exponential security of following “trio”:

Decisional Linear assumption over Symmetric Bilinear Maps

Field LPN with noise rate $n^{-\delta}$

PRGs in NC^0 with stretch $\kappa^{1+\epsilon}$

Question 1: Are these assumptions connected to lattices?

Question 2: How do you show such a result?

Are the assumptions truly incomparable to Lattices?

Decisional Linear assumption over Symmetric Bilinear Maps

Field LPN with noise rate $n^{-\delta}$

PRGs in NC^0 with stretch $\kappa^{1+\epsilon}$

LPN and PRGs aren't even known to imply PKE.

Don't know if they are in coAM

DLIN has no known reductions to/from Lattices.

Exciting questions in themselves!

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Perfectly re-randomizable encryption
[Canetti-Lin-Tessaro-Vaikunthanathan]

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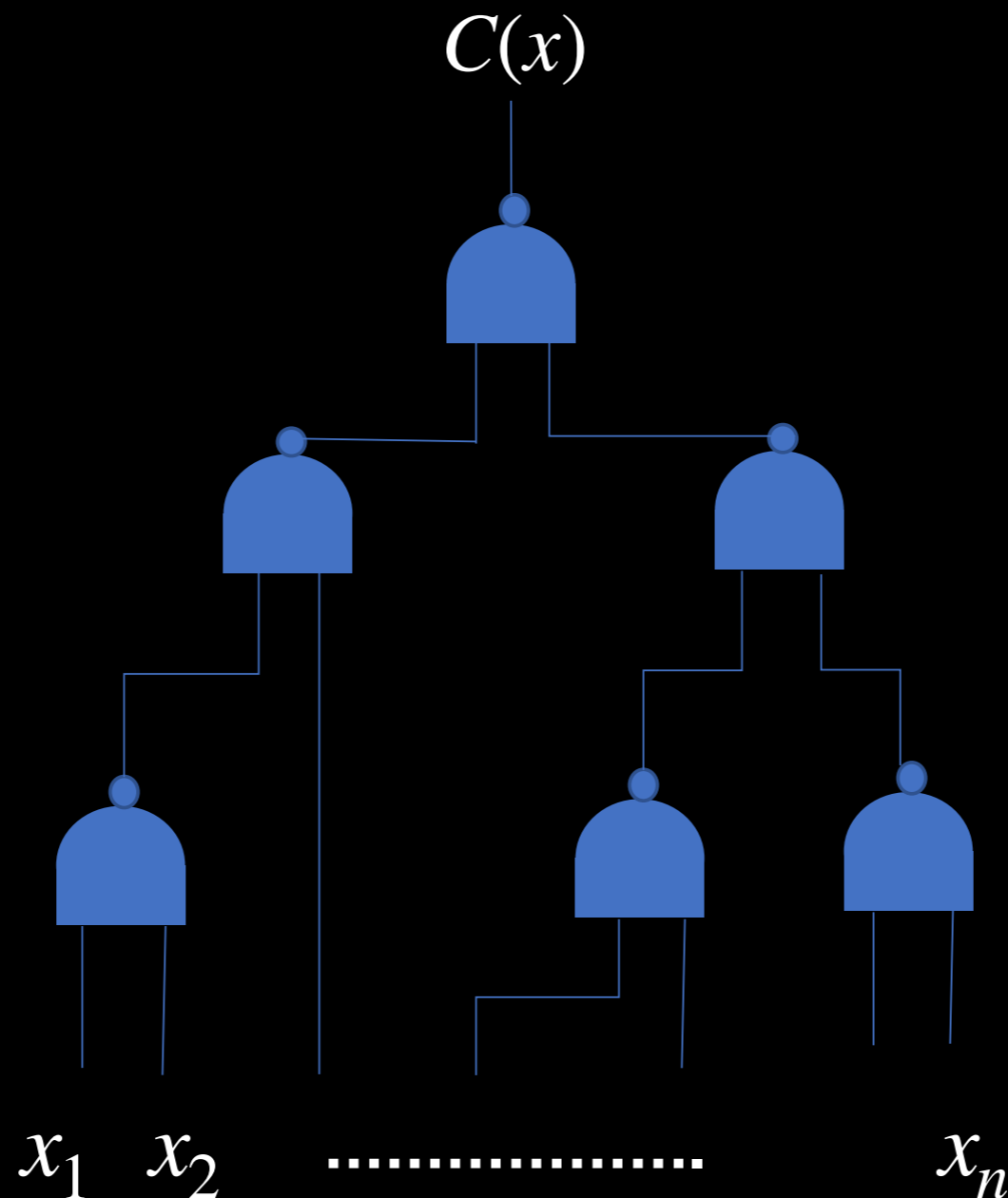
and

Perfectly re-randomizable encryption
[Canetti-Lin-Tessaro-Vaikunthanathan]

And, a number of other applications previously known only via lattices.

Obfuscation Goal

Let $C : [N = 2^n] \rightarrow \{0,1\}$ be a boolean circuit to be obfuscated.



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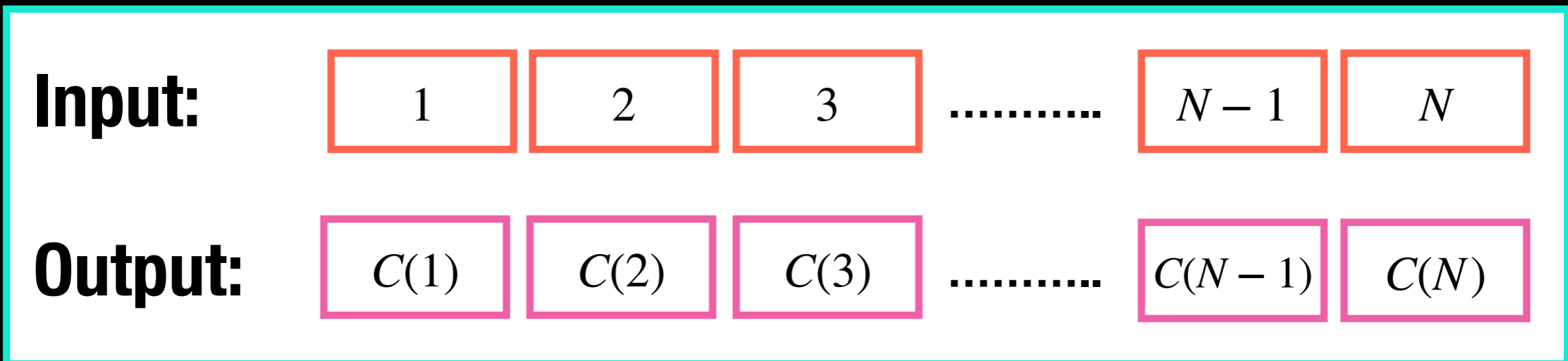
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A “Trivial” $i\mathcal{O}$ Scheme:

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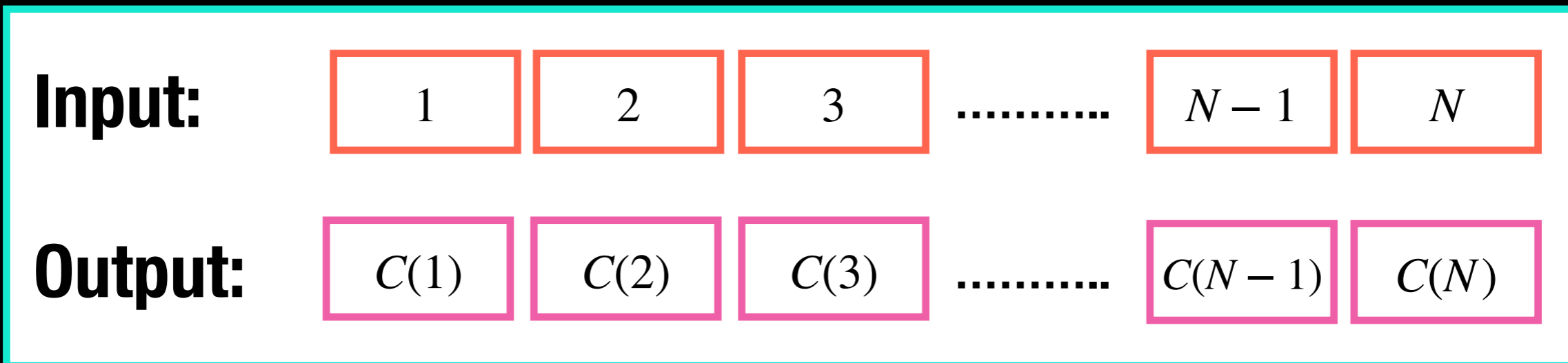


The Truth-Table!

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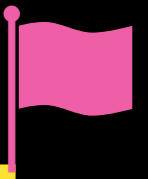
The Truth-Table!

Problem:
Obfuscation takes time! $|T_{i\mathcal{O}(C)}| \propto N$



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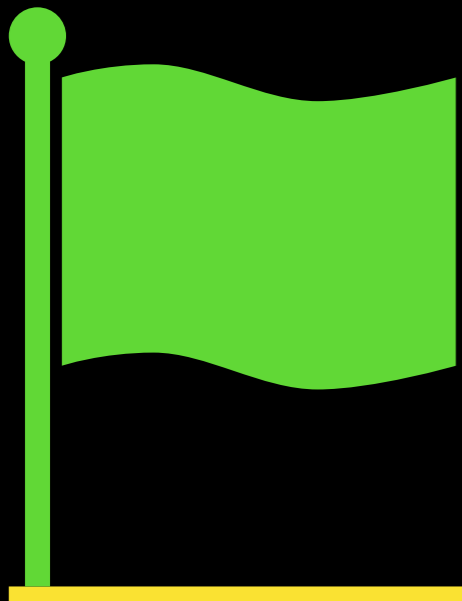


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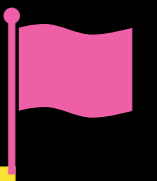
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$$|T_{i\mathcal{O}}(C)| \propto \text{poly}(|C|)$$

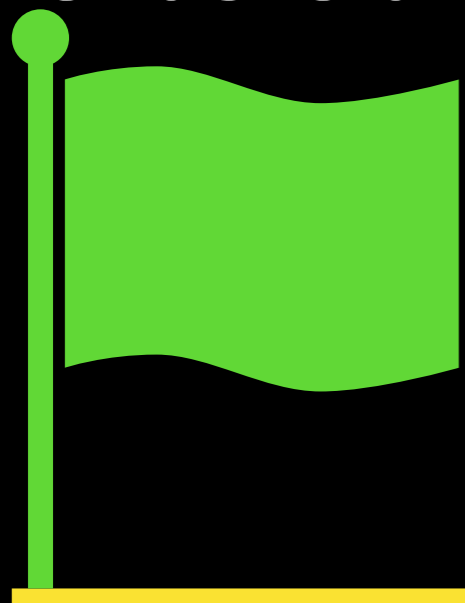


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Non-Trivial $i\mathcal{O}$
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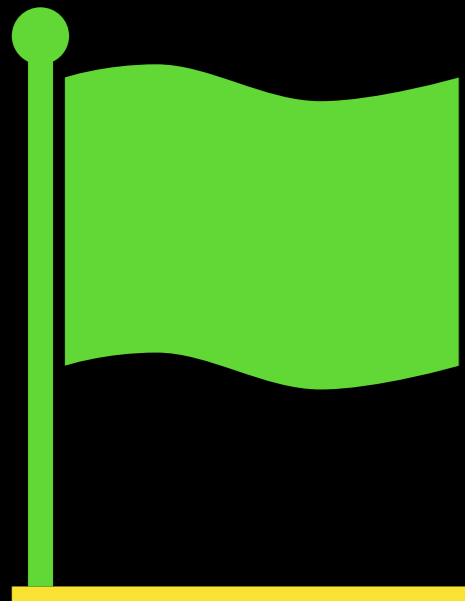
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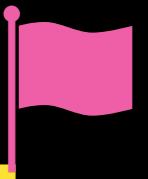


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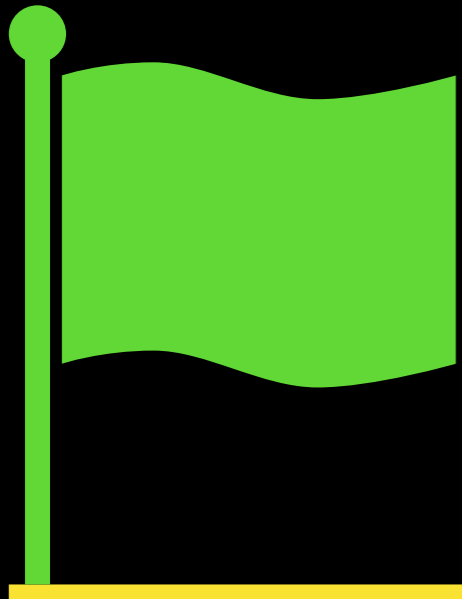
Non-Trivial $i\mathcal{O}$ + DLIN \implies $i\mathcal{O}$!

[BV 15, AJ 15, LPST 16, BNPW 17]



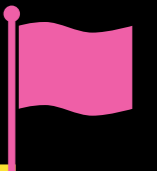
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Previous work:

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The Truth-Table!

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Non-Trivial $i\mathcal{O} + \text{LWE} \implies i\mathcal{O}!$

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Our Approach

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Special Encryption Scheme

Encrypt($\tilde{C} = (C, r)$)



Size of \tilde{C} :

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Can learn functions

$$\{U_x(\tilde{C}) = C(x)\}_{x \in [N]}$$

$$U_1(\tilde{C}) \ U_2(\tilde{C}) \ \cdots \ U_x(\tilde{C}) \ \cdots \ U_N(\tilde{C})$$

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Problem:

$U_x(C, r) = C(x)$ is
too complex!

**Learn
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Our Approach

How simple can U_x be?

Application of [Yao 86, AIK 04, L 17, AS 17]:

If PRGs with locality d exist

$3d + 1$ -Local: $U_x(\tilde{C})$ depends on $3d + 1$ bits of \tilde{C} .

Degree-16 polynomial!

Truth-table!

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Can learn “degree-16” functions

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Good News:

Can handle

quadratic functions

[Lin 17, AJLMS 19, JLMS 19, GJLS 20, Wee 21].

**Based on
DLIN**

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Goal: Replace U_x by quadratic functions.

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Good M

Public $P(\tilde{C})$

Encrypt($S(\tilde{C})$)



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[Lin 17, AJLMS 16
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Coefficients constant
degree polynomial over

P

Bas

DLIN

“degree-16” functions

$\{U_x(\tilde{C})\}_{x \in [N]}$

$$U_x(\tilde{C}) = \sum_{i,j} q_{x,i,j} S_i \cdot S_j \pmod p \quad U_1(\tilde{C}) \quad U_2(\tilde{C}) \quad \dots \quad U_x(\tilde{C}) \quad \dots \quad U_N(\tilde{C})$$

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Use of LPN

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Step 1: Approximate $U_x(\tilde{C})$ by quadratic $f_x(S)$,

$$f_x(S) = U_x(\tilde{C}) \quad \text{for most inputs } x \in [N]$$

Main Idea: Variable change via LPN.

Use of LPN

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$$h_x(S, M) = f_x(S) + \text{Corr}_x(M) = U_x(\tilde{C}) \quad \forall x \in [N]$$

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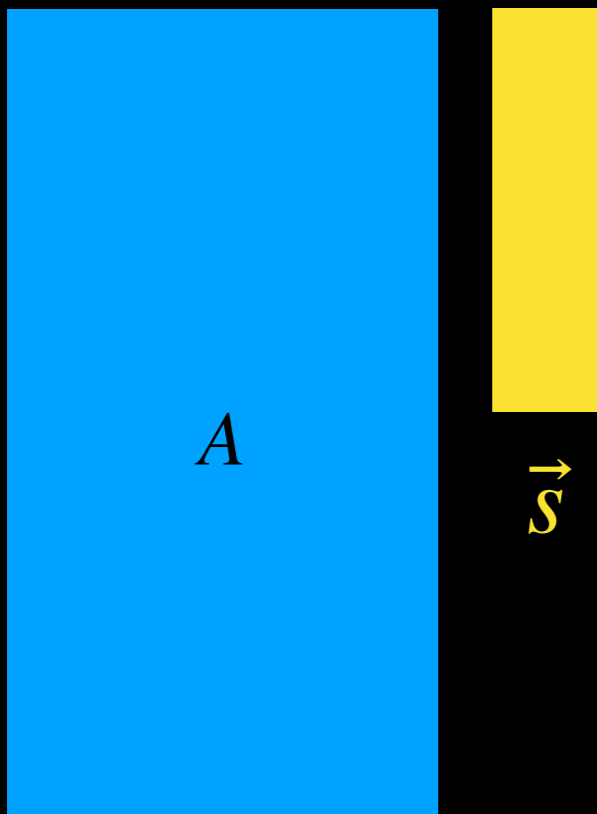
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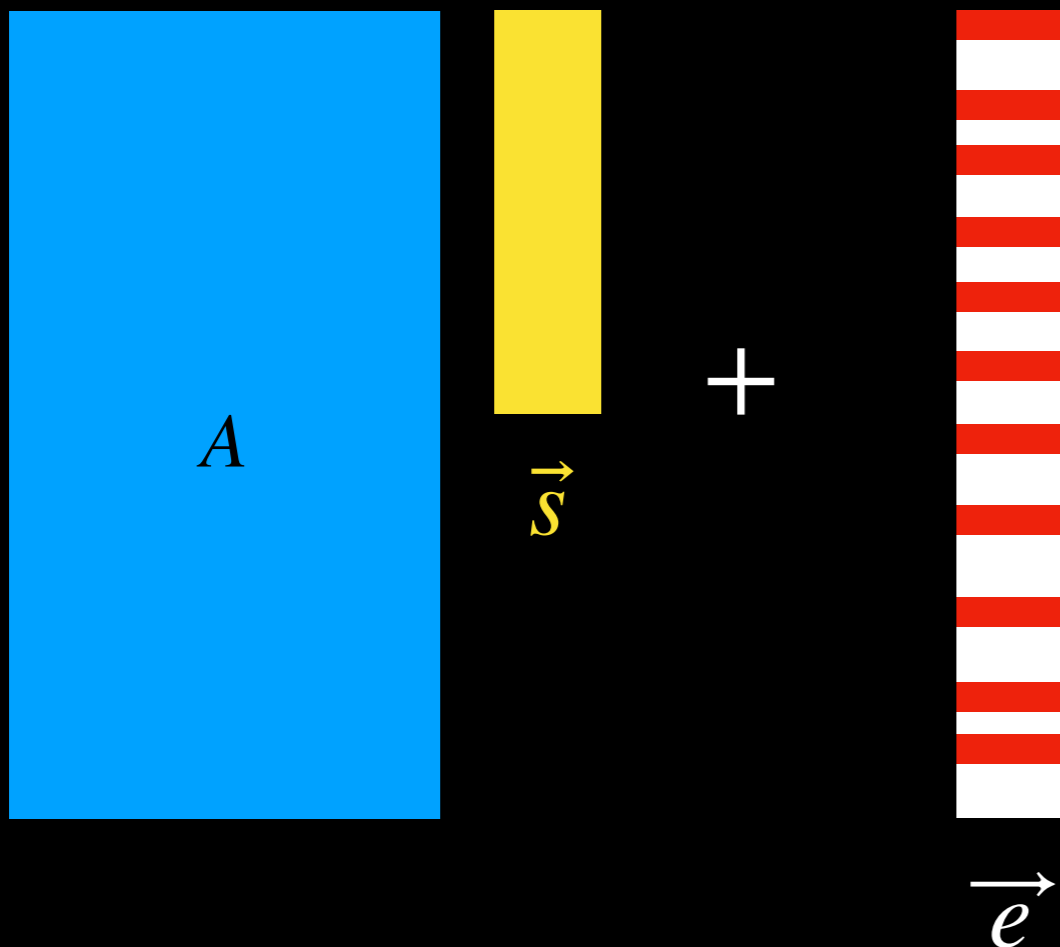
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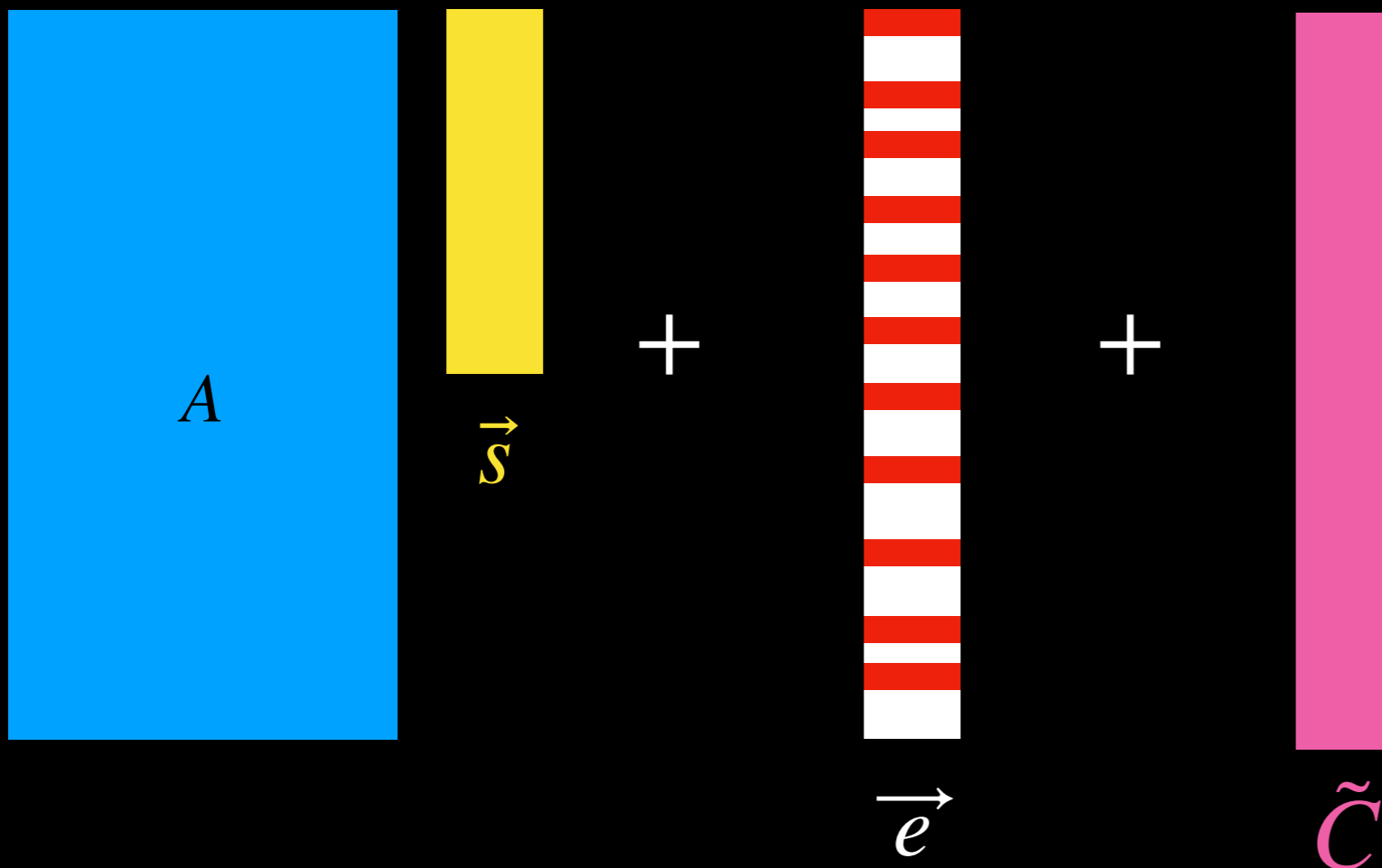
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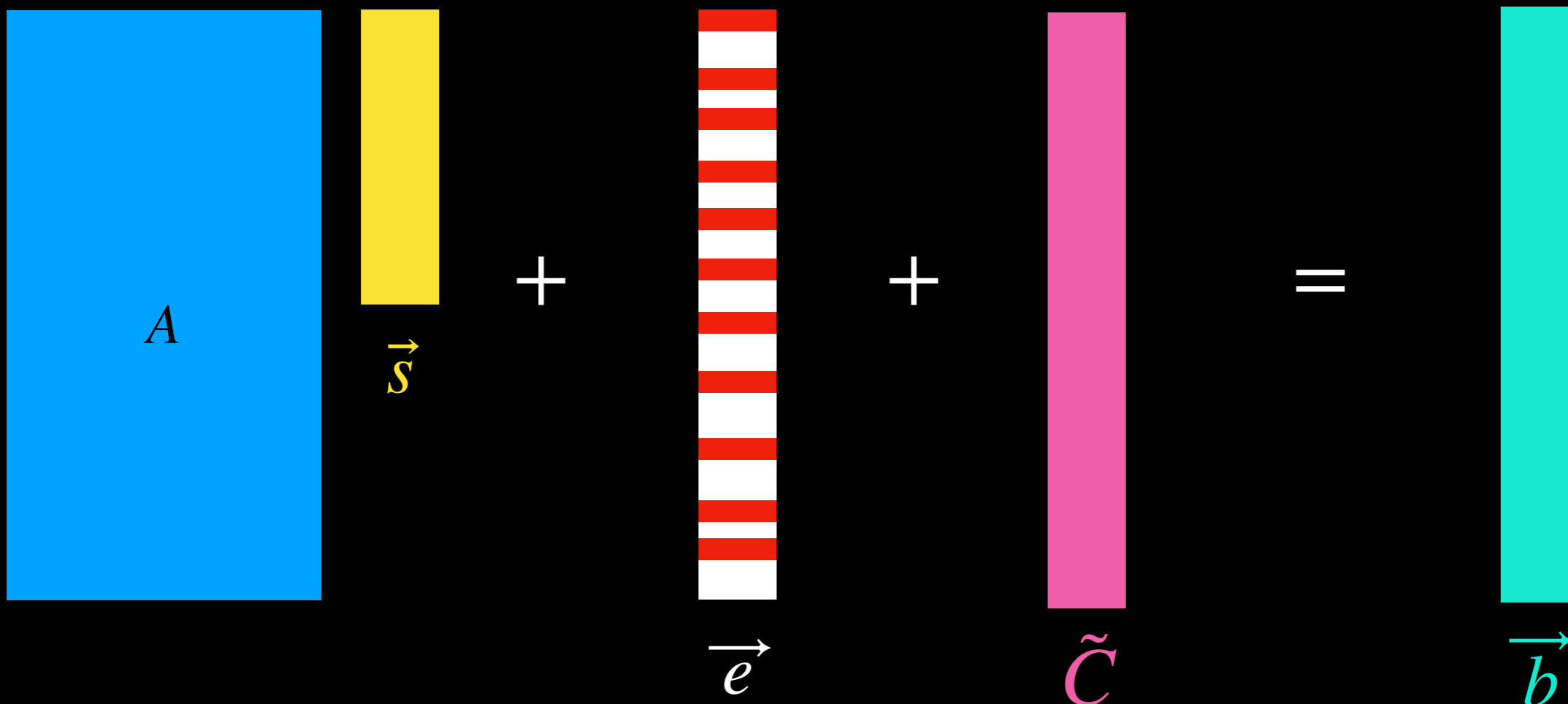
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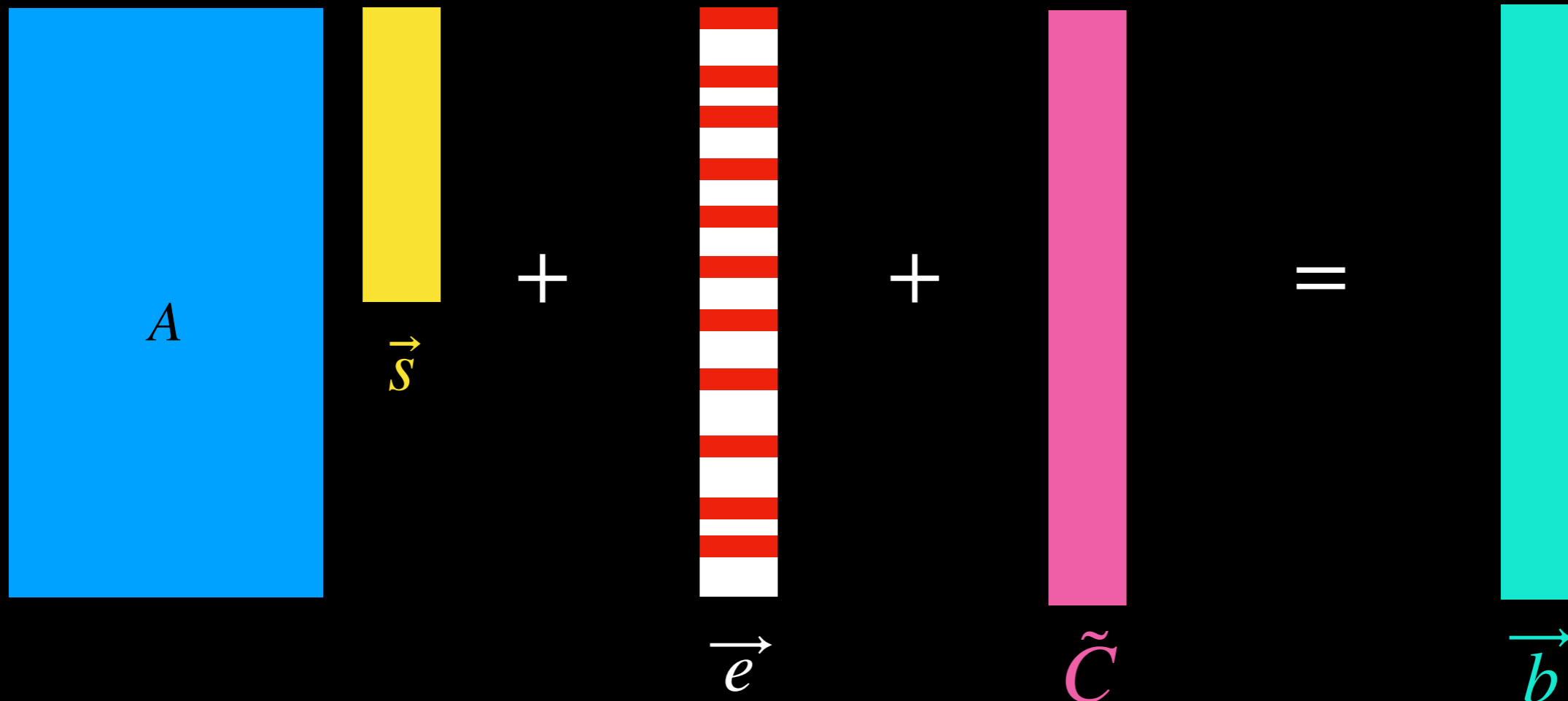


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\vec{A}, \vec{b} encrypts \tilde{C} !



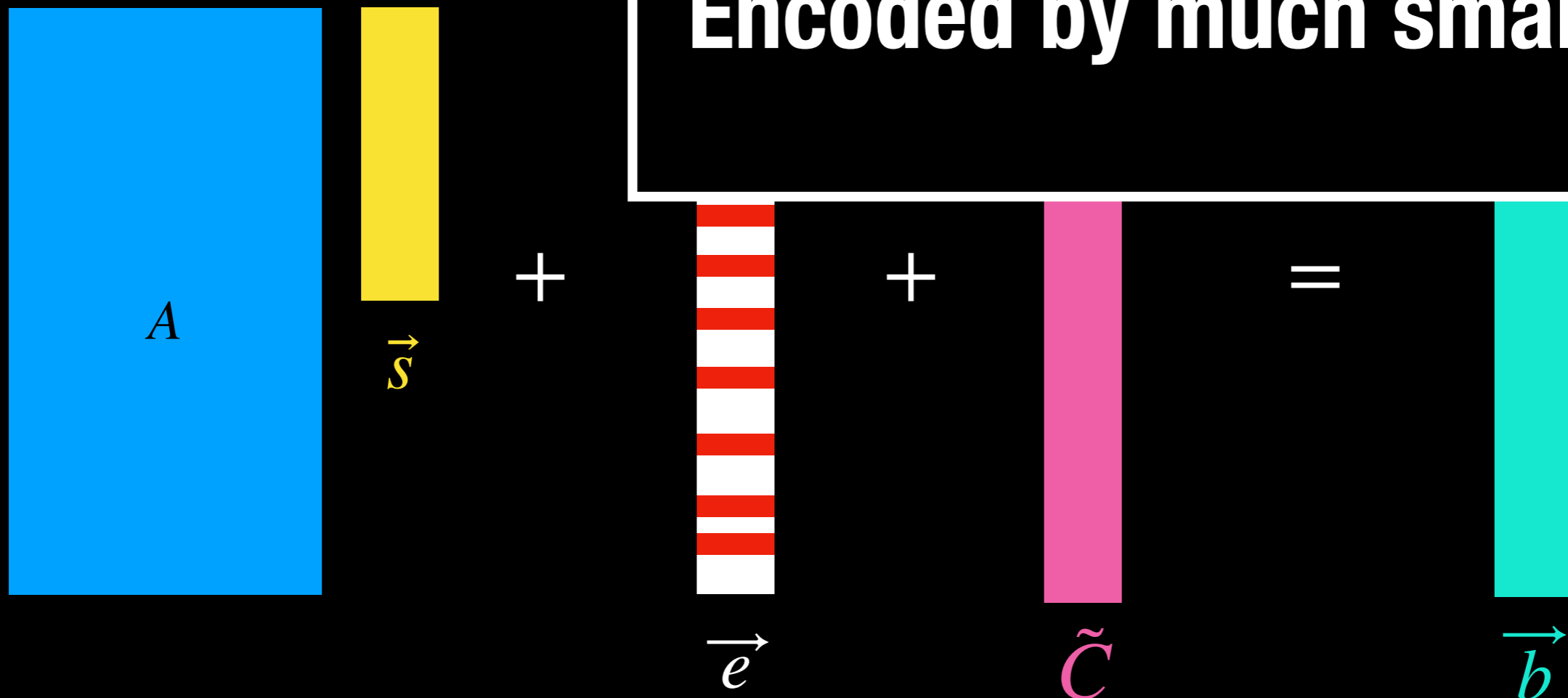
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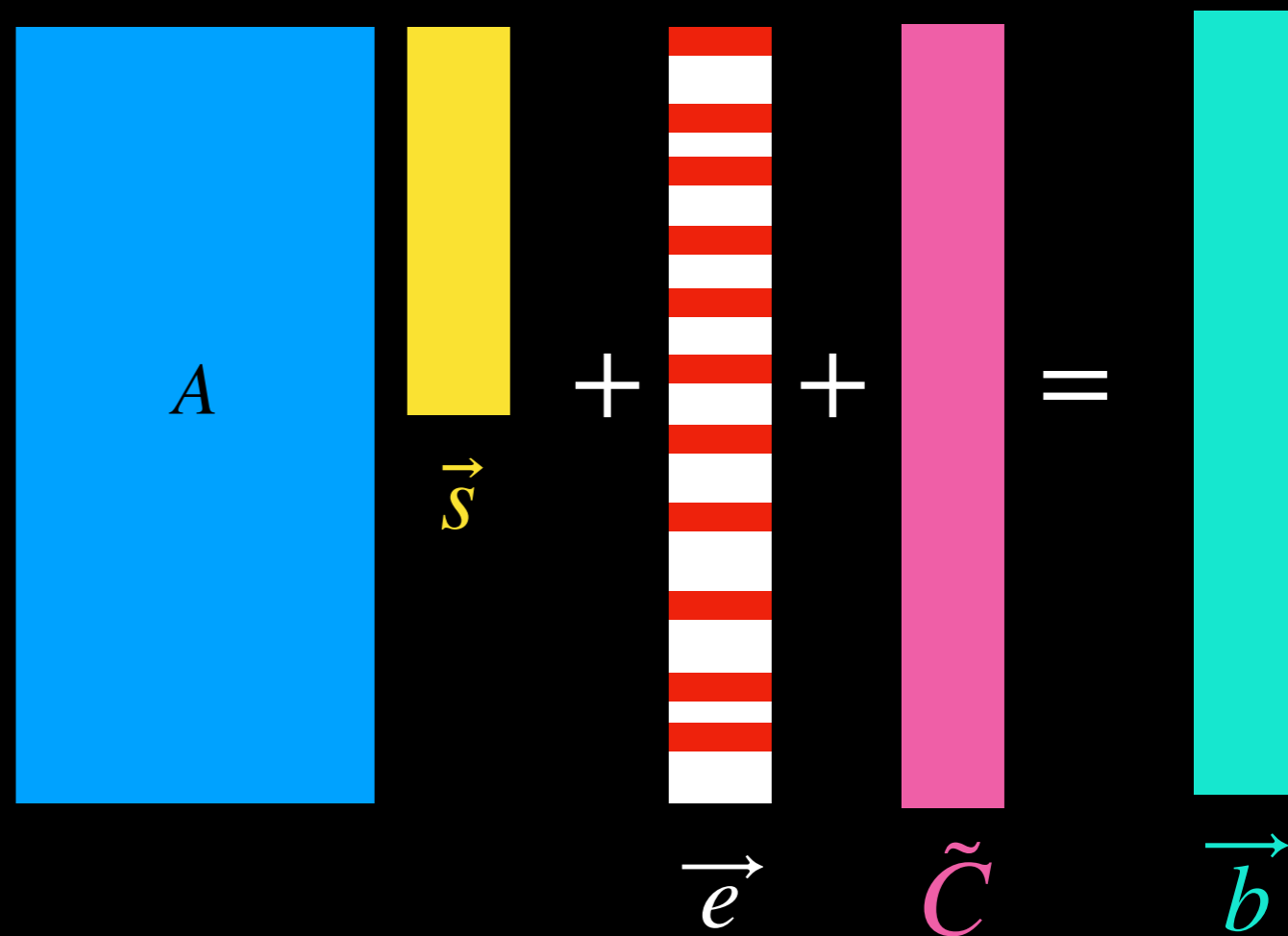
\vec{A}, \vec{b} encrypts \tilde{C} !

Can be made public!

Encoded by much smaller \vec{s}



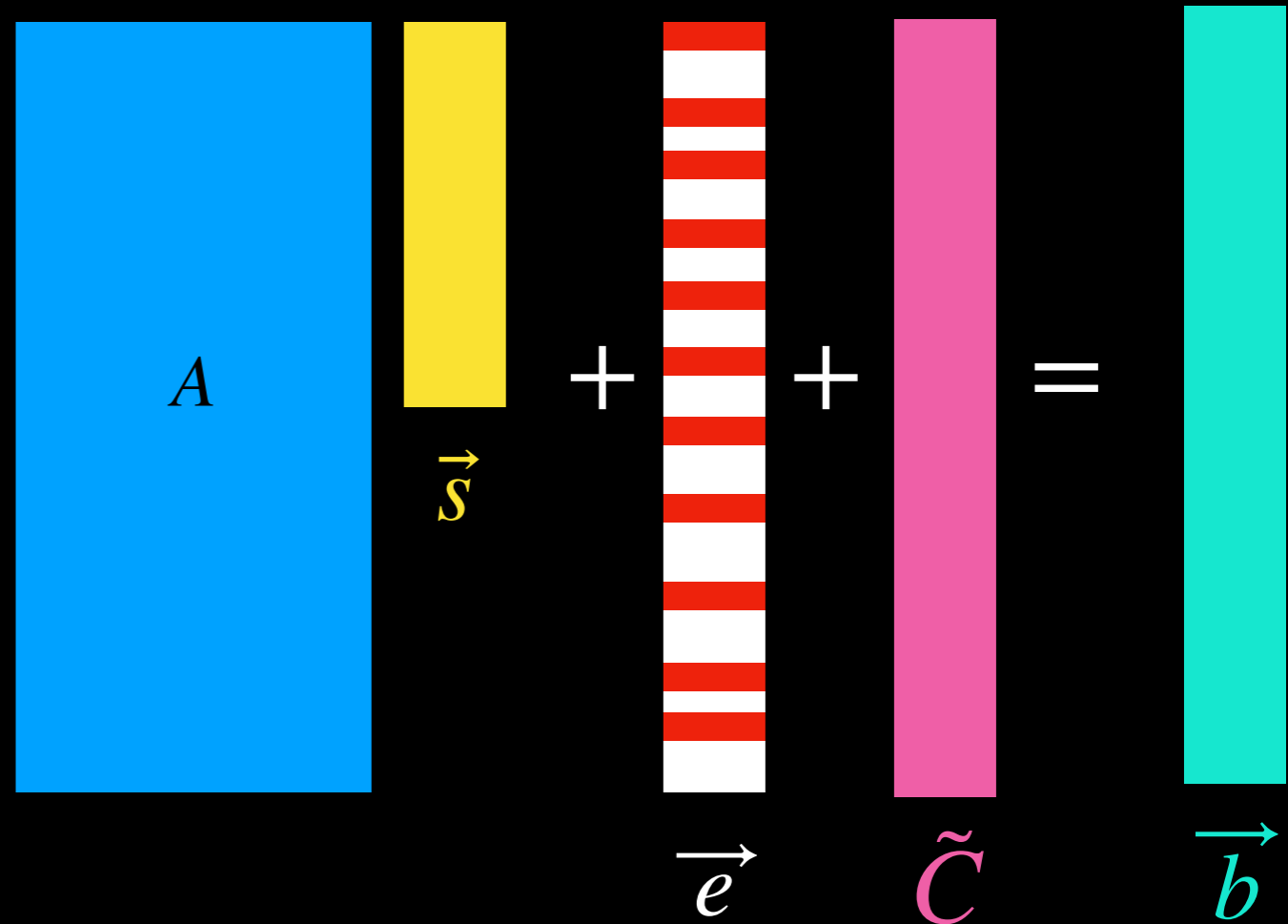
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Applying LPN

Goal: Find f_x that is:

- Quadratic in short S ,
- For most x , $f_x(S) = U_x(\tilde{C})$

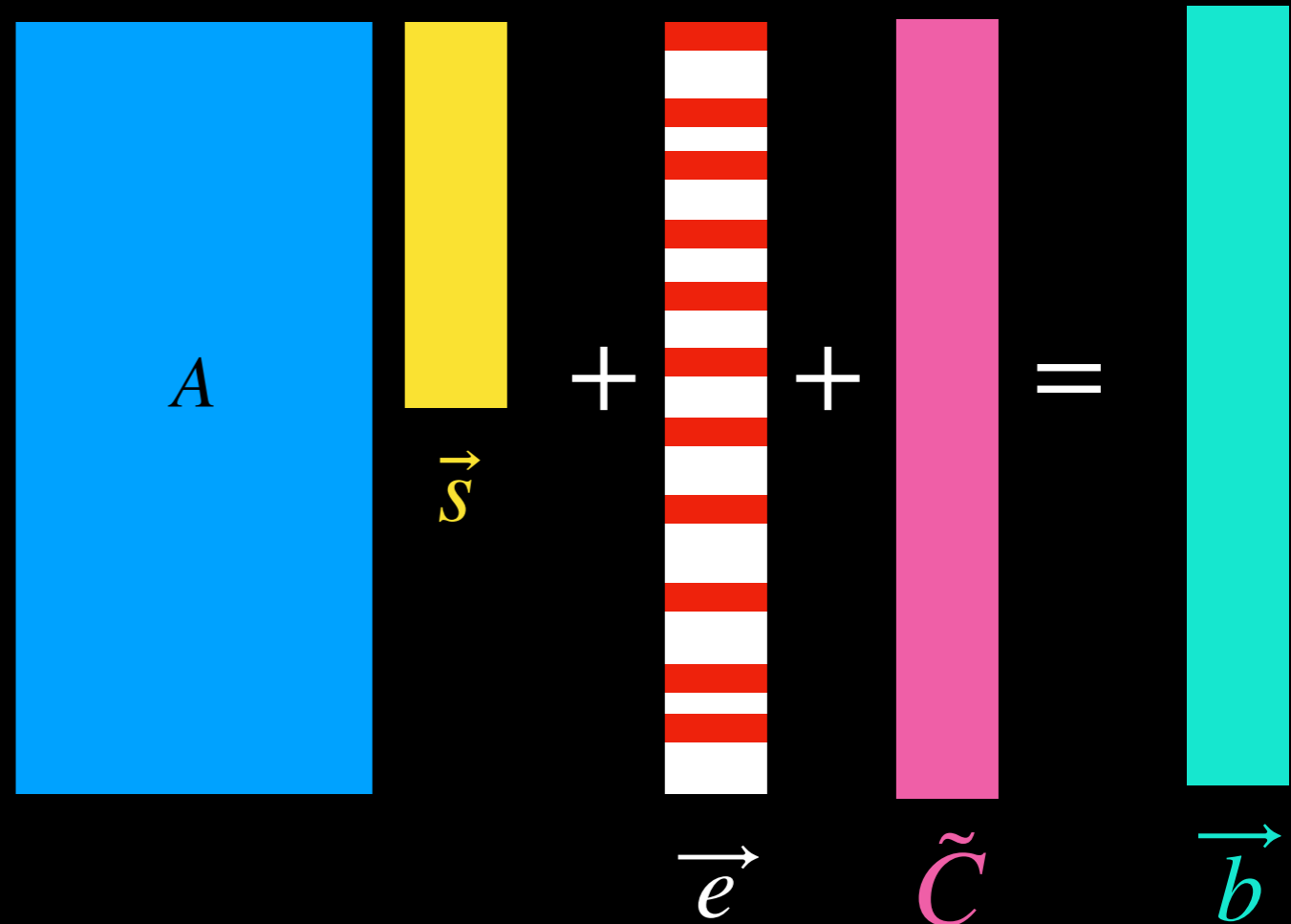


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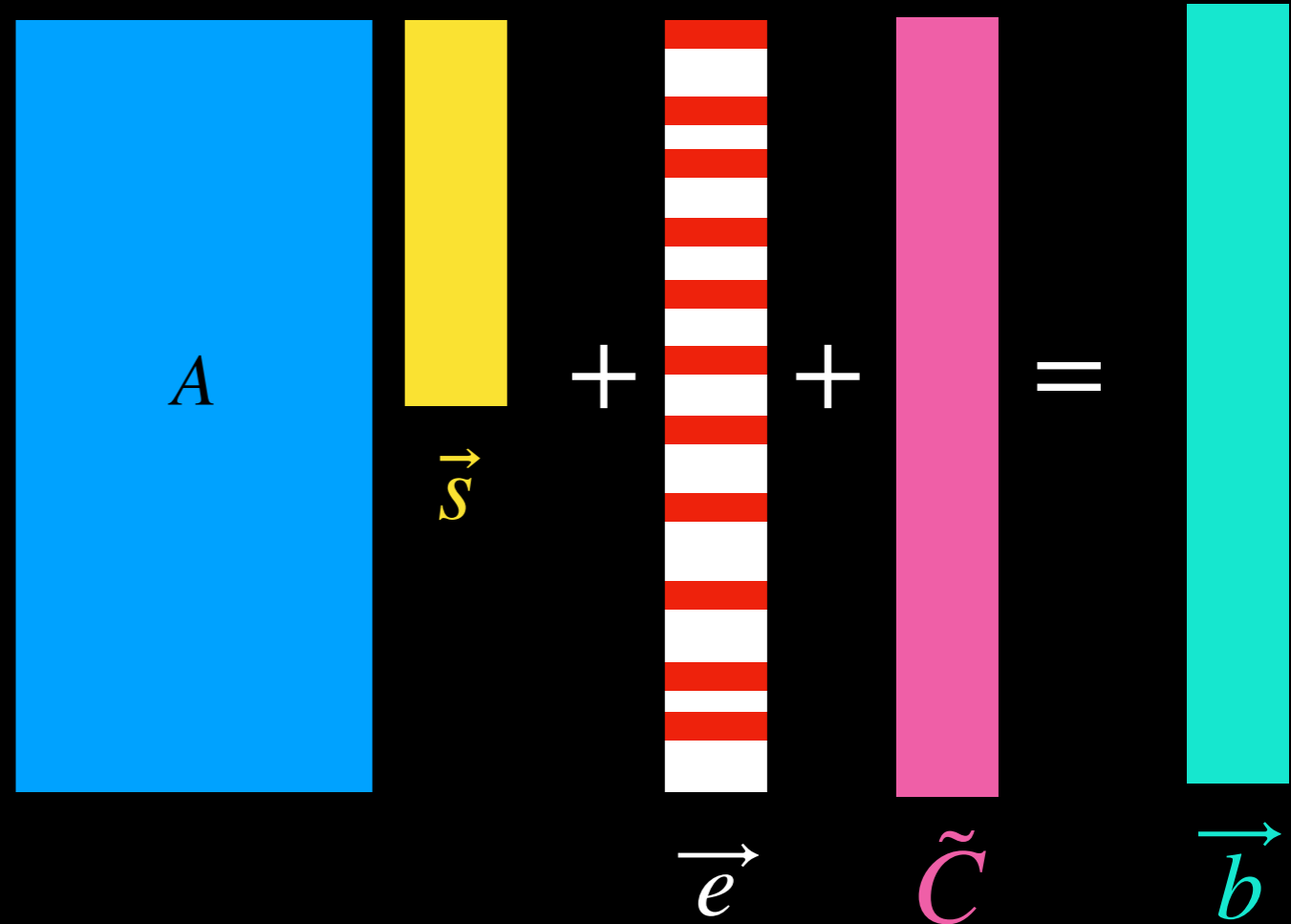
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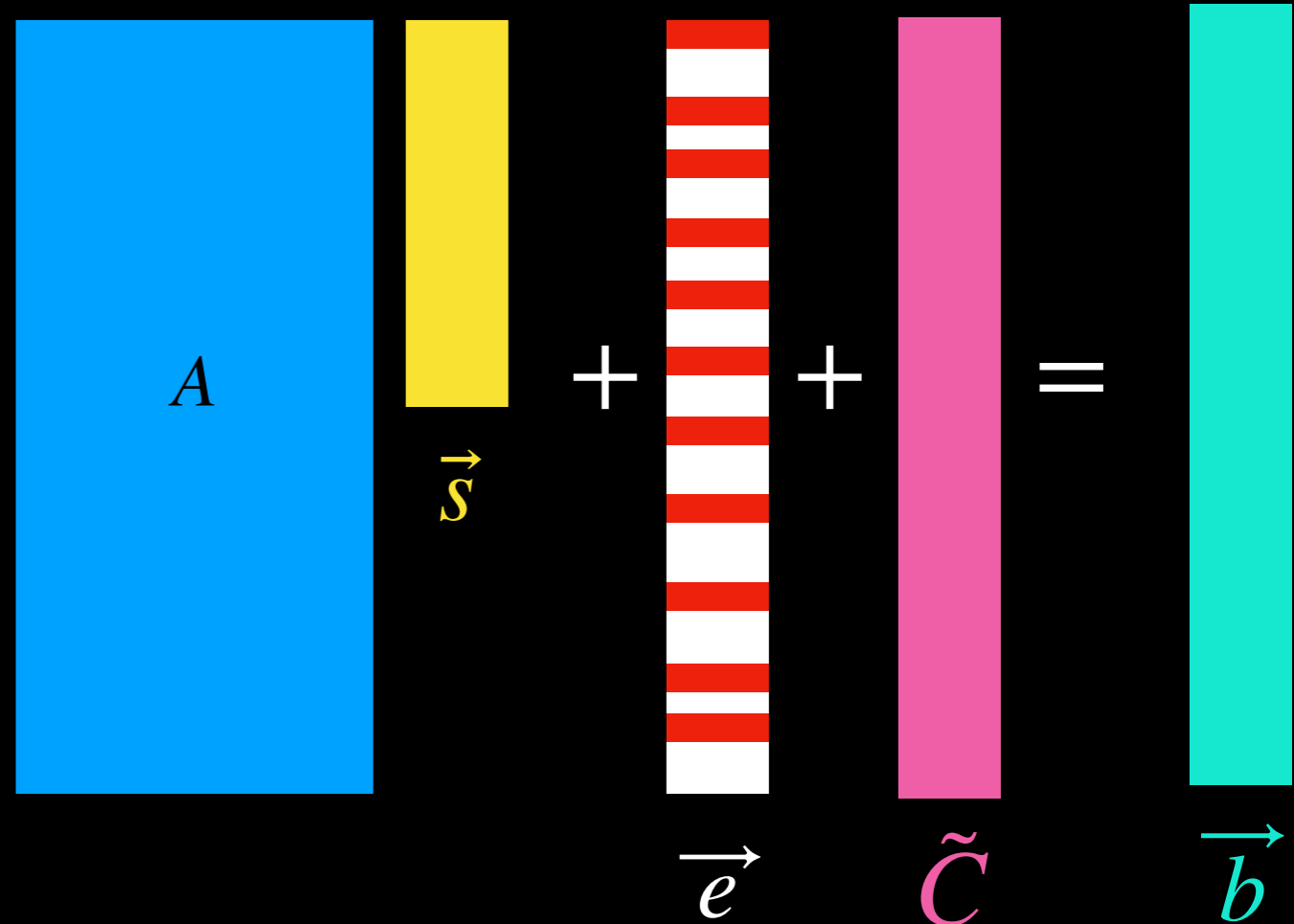
Degree - 16 in \vec{s}



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$$U_x(\tilde{C} + \vec{e}) = U_x(\vec{b} - A\vec{s})$$

Degree - 16 in \vec{s}

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Goal: Find f_x that is:

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$\{f_x\}_x$ approximates $\{U_x\}_x$

\vec{e} is sparse, U_x depends on 16 bits, for any x

$$U_x(\tilde{C} + \vec{e}) = U_x(\tilde{C})$$

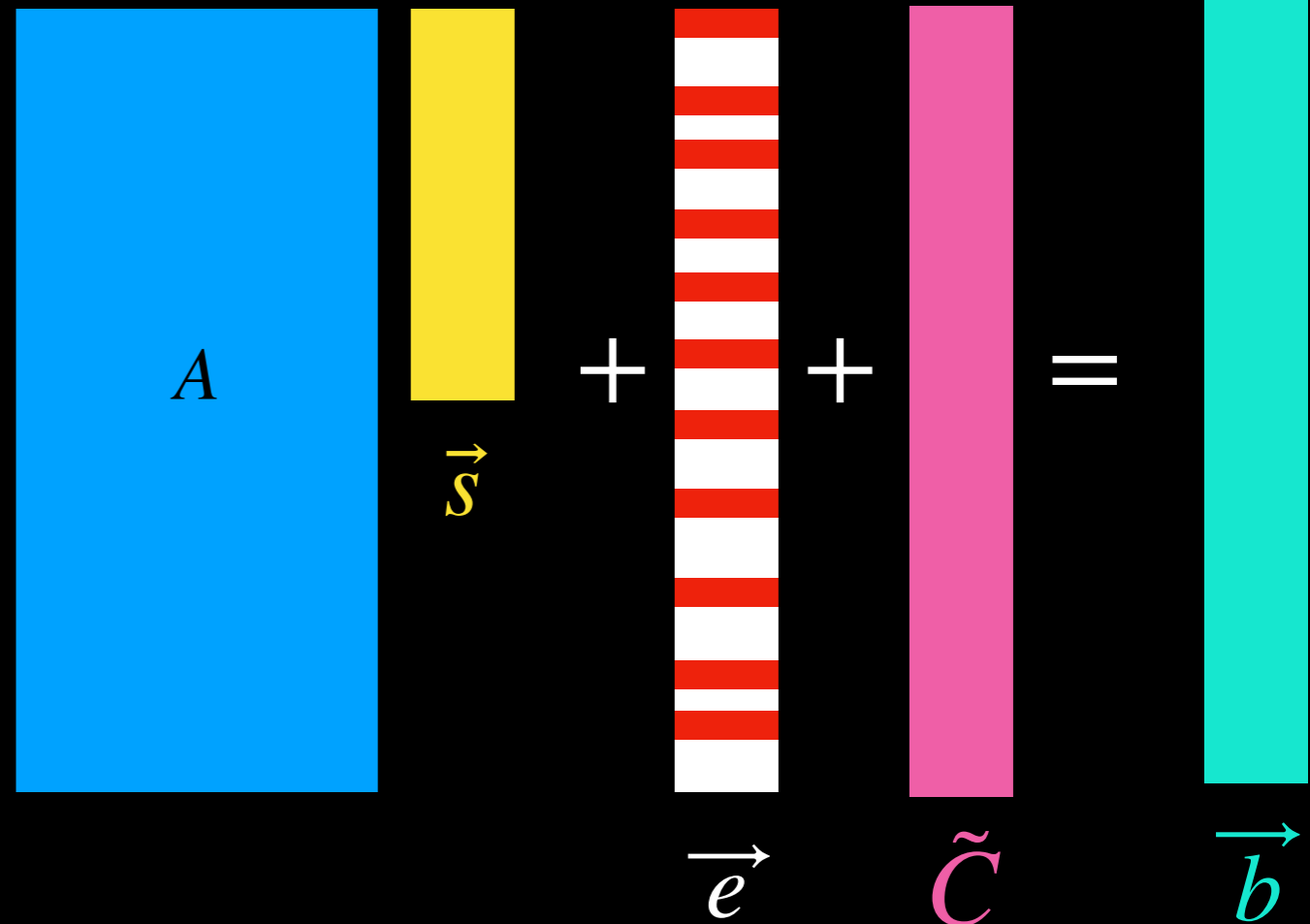
with high probability (over \vec{e}).

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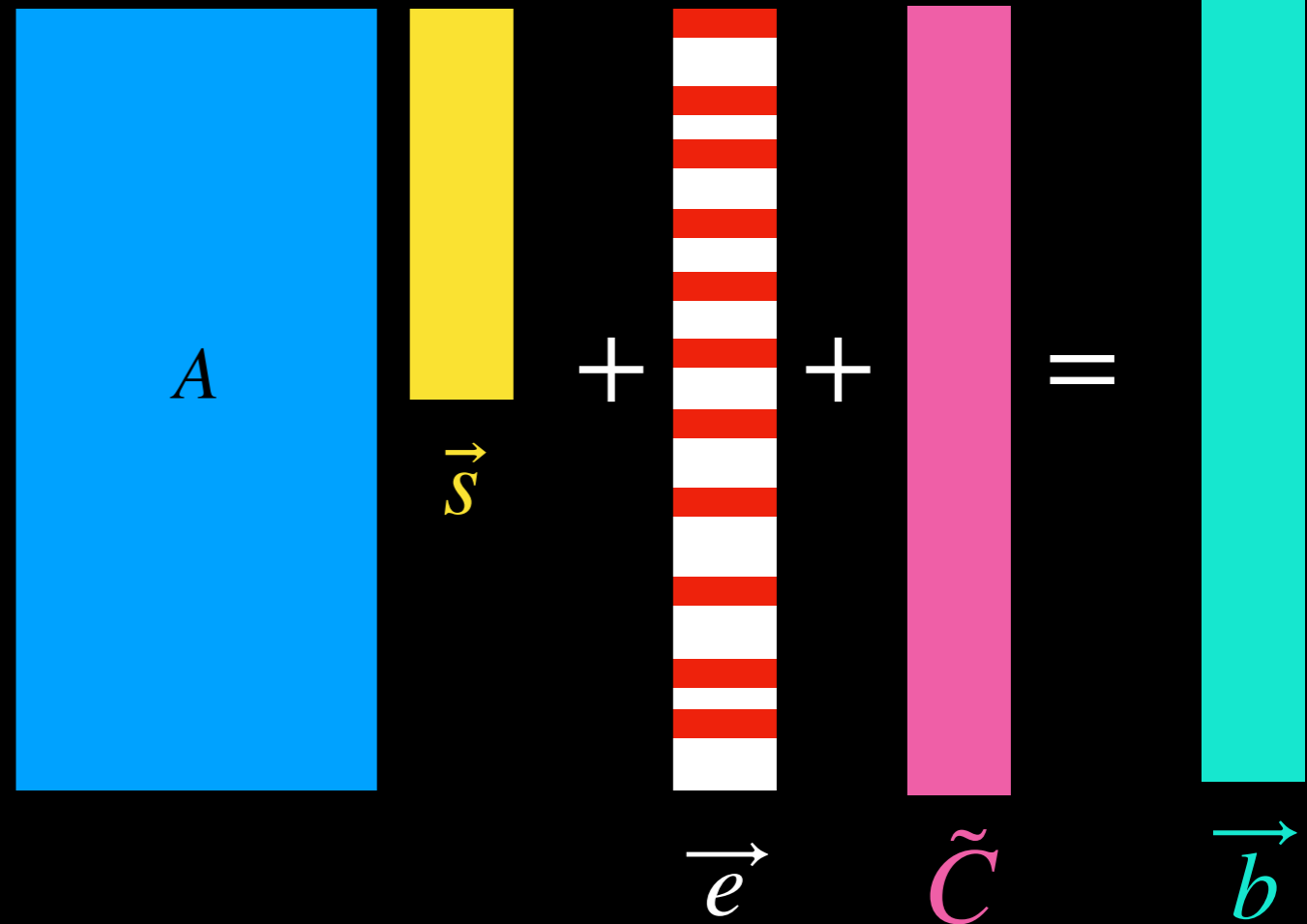
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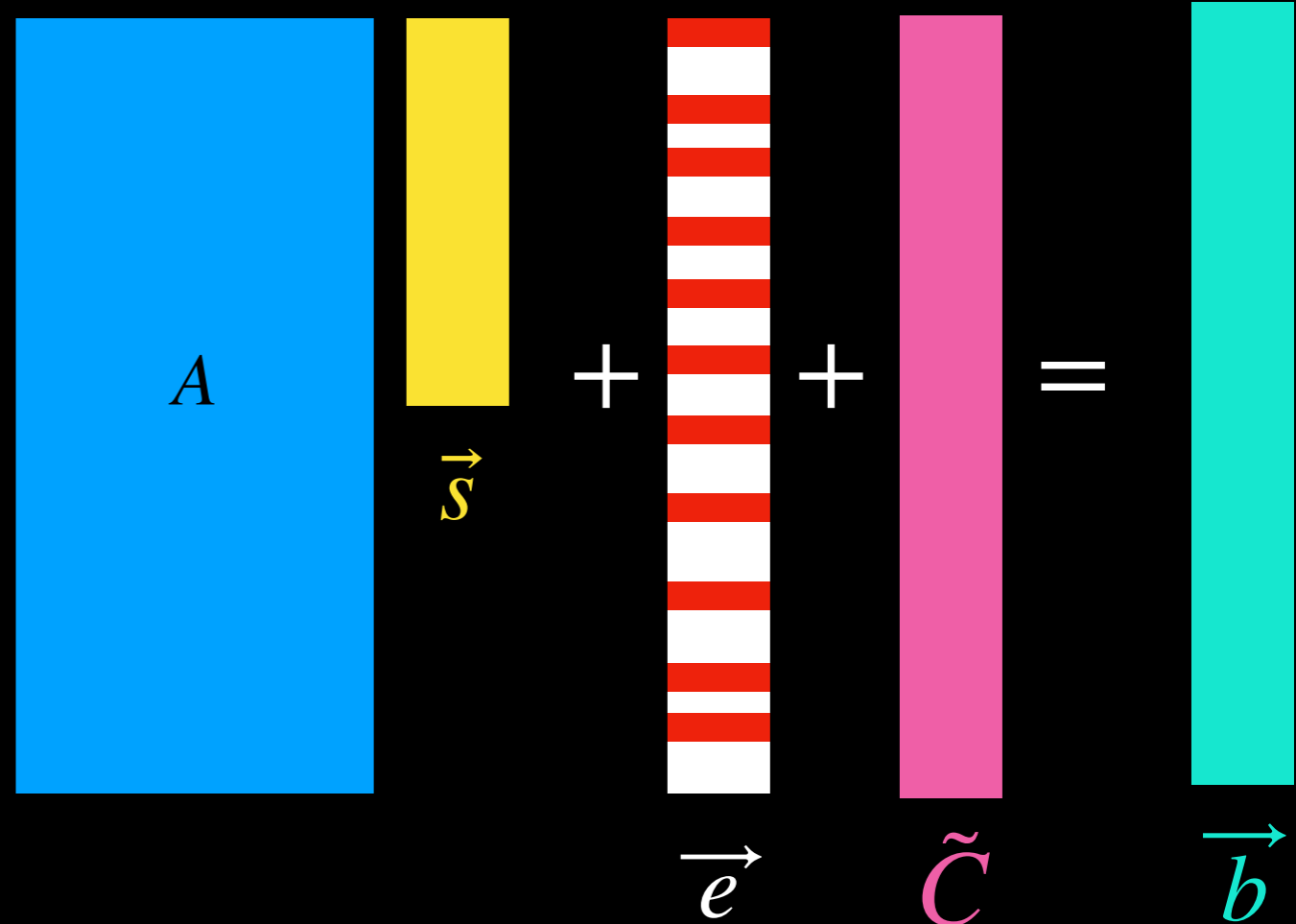
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$|\vec{s}|$ is very small

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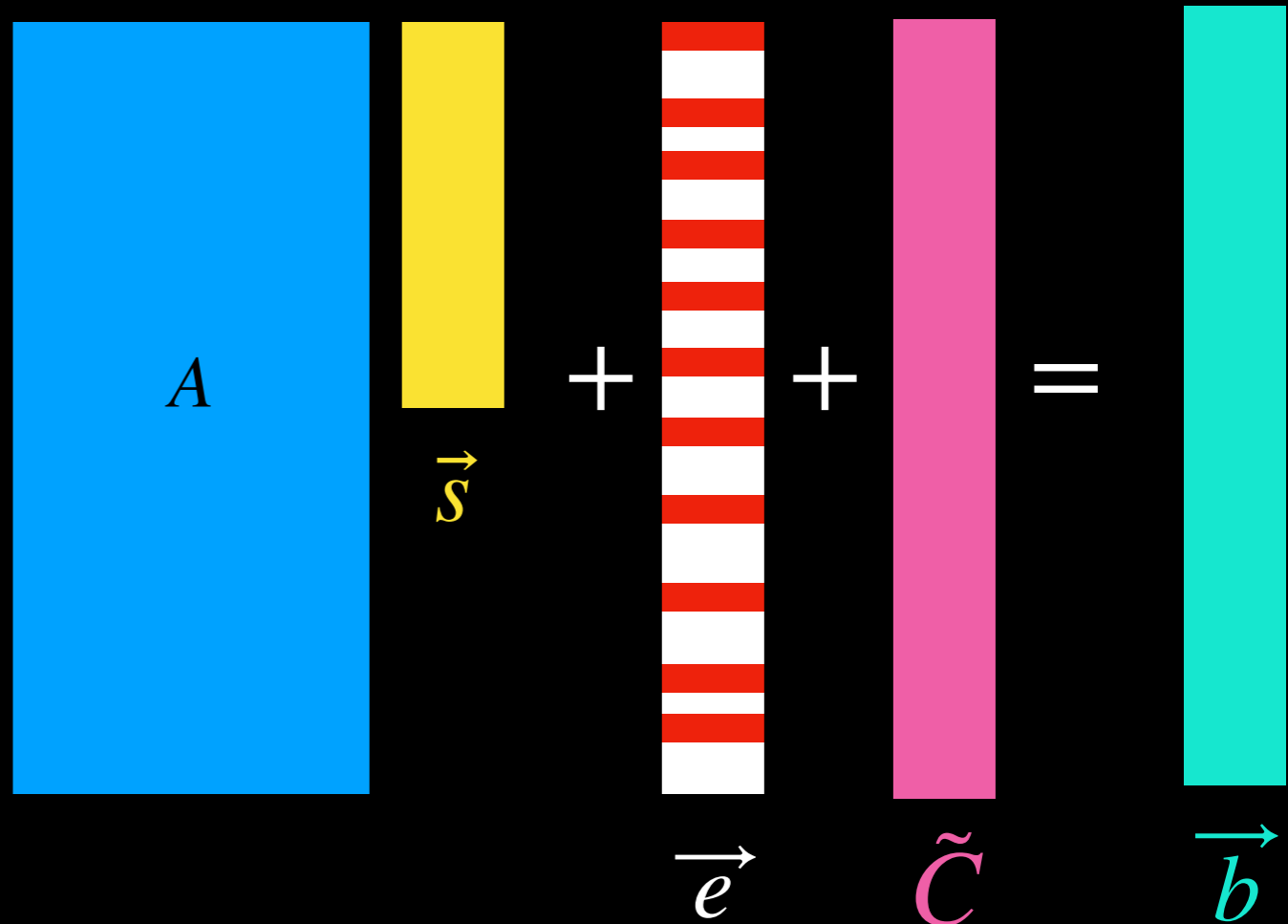
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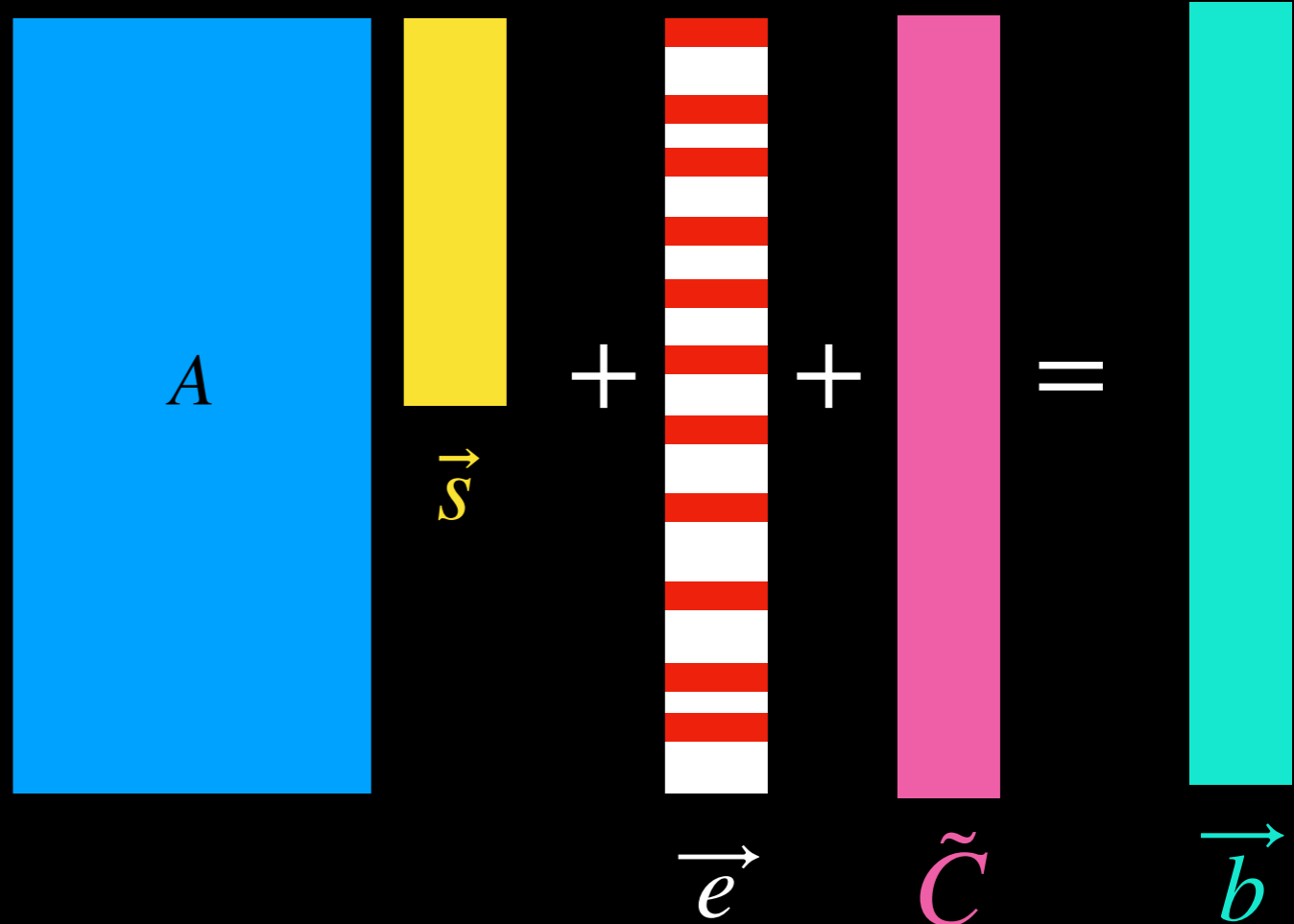
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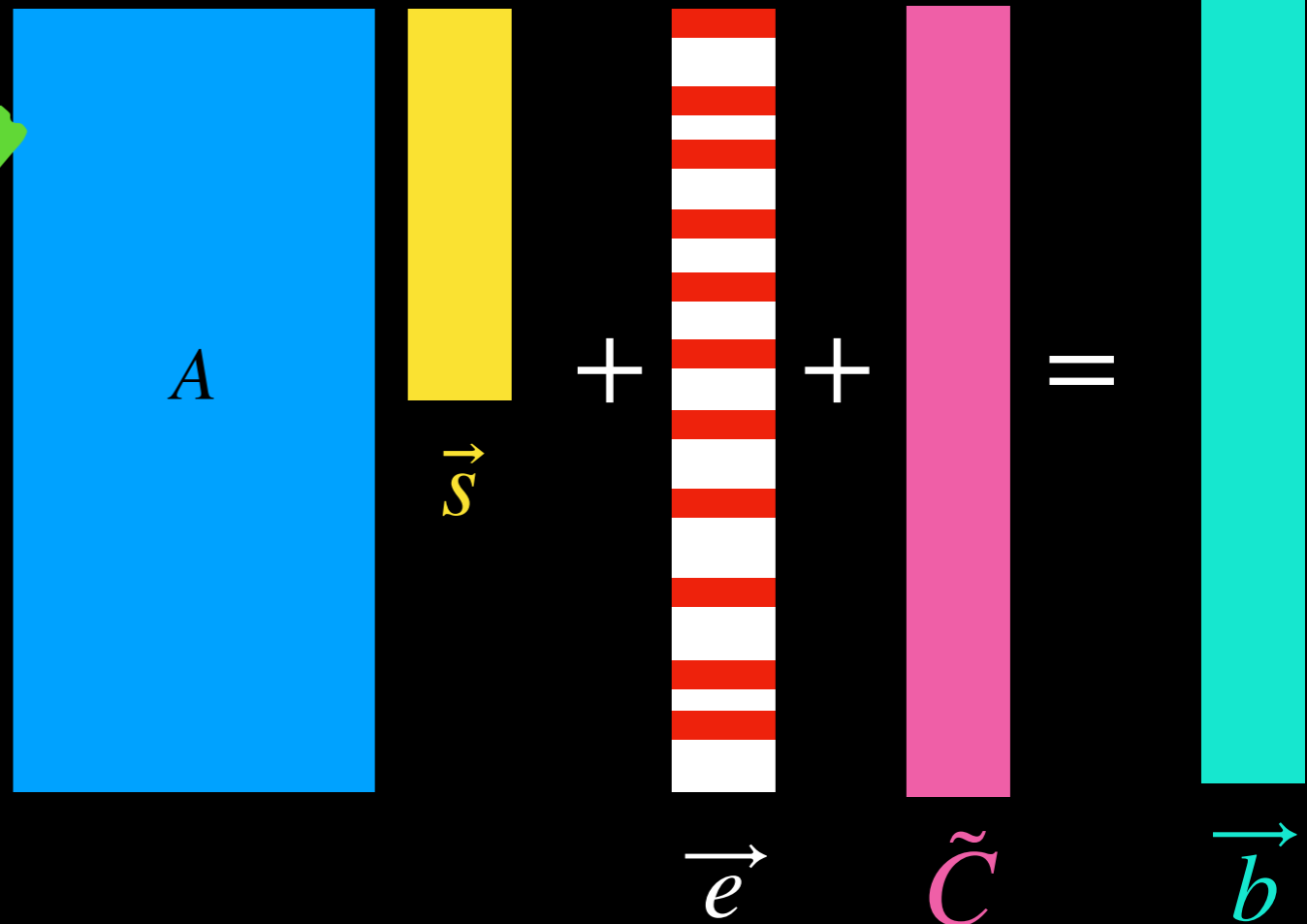
$$|\vec{s}| \ll N^{0.10}$$

$$|S| \ll N^{0.80}$$

Applying LPN

Goal: Find f_x that is:

- Quadratic in short S ,
- For most x , $f_x(S) = U_x(\tilde{C})$



Consider:

$$U_x(\tilde{C} + \vec{e}) = U_x(\vec{b} - A\vec{s}) = f_x(S)$$

Degree - 16 in \vec{s}

Quadratic in $S = (\vec{s}, 1)^{\otimes 8}$

$|\vec{s}|$ is very small $\implies |S|$ is small

$$|\vec{s}| \ll N^{0.10}$$

$$|S| \ll N^{0.80}$$

Use of LPN

Goal: Replace U_x by quadratic functions.

Step 1: Approximate $U_x(\tilde{C})$ by quadratic $f_x(S)$,

$$f_x(S) = U_x(\tilde{C}) \quad \text{for most inputs } x \in [N]$$



Main Idea: Variable change via Random Linear Codes.

Step 2: Error correction via quadratic $\text{Corr}_x(M)$

$$h_x(S, M) = f_x(S) + \text{Corr}_x(M) = U_x(\tilde{C}) \quad \forall x \in [N]$$

Main Idea: Compression via Matrix Factorization.

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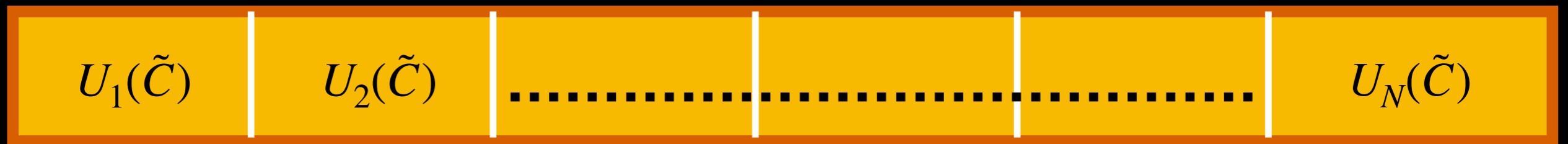
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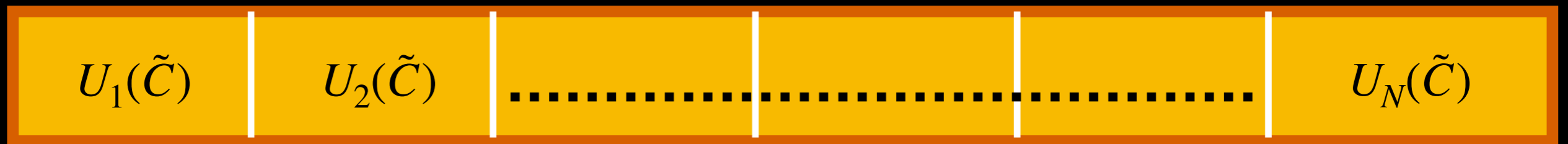
Step 2: Error Correction

Target

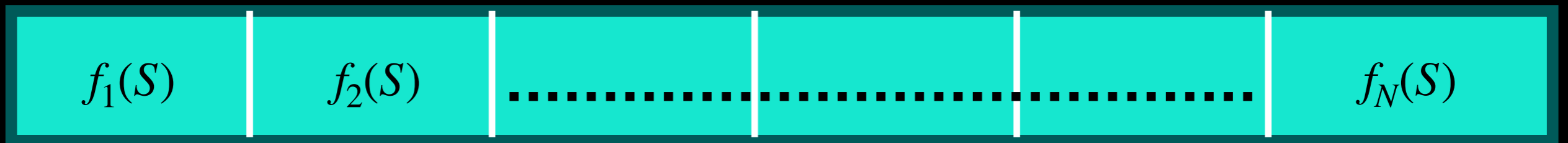


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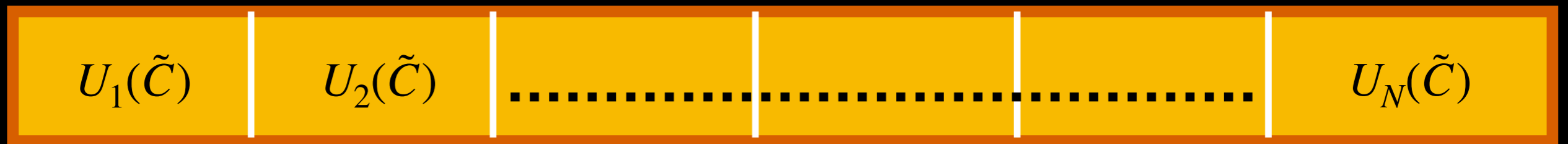


Actual

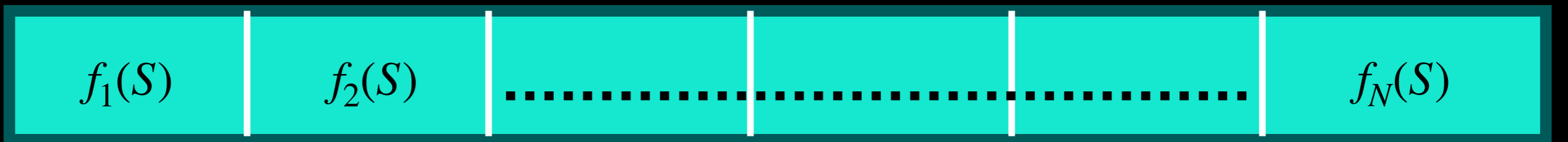


Step 2: Error Correction

Target



Actual



Correction vector =

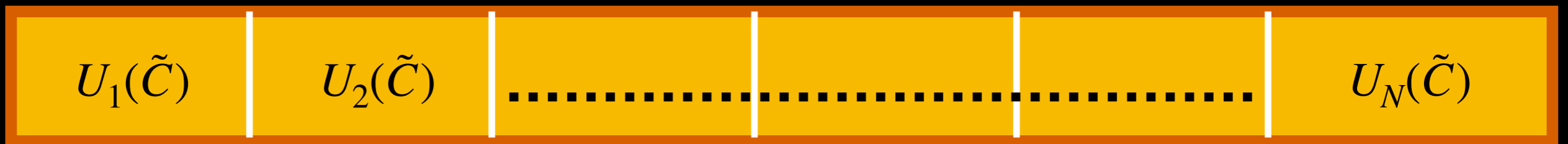
Target

-

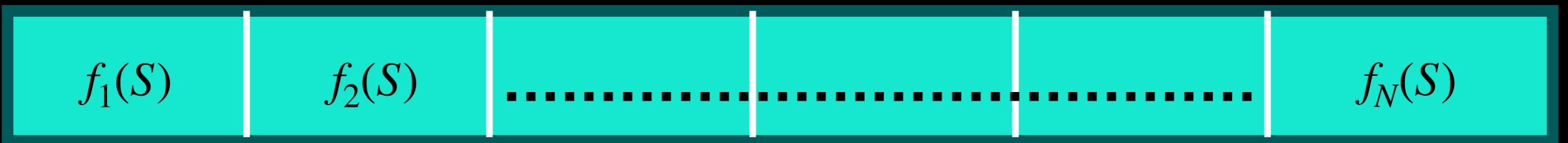
Actual

Step 2: Error Correction

Target



Actual



Correction vector =

Target

-

Actual



Takeaway: Correction vector is sparse!!

Amortization

Previously we showed that for any circuit C

Map: $\tilde{C} \rightarrow P, S$

Size = $\tilde{O}(N^{0.99})$

Time = $\tilde{O}(N)$

Requires LWE [GKPVZ 13, BV 15, AJ 15, LPST 16, BNPW 16]

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Requires LWE [GKPVZ 13, BV 15, AJ 15, LPST 16, BNPW 16]

Main Lemma:

$$\text{Map: } (\tilde{C}_1, \dots, \tilde{C}_k) \rightarrow (P_1, P_2, \dots, P_k, S_1, \dots, S_k)$$

$$\text{Time} = \tilde{O}(Nk^{1-\epsilon} + k^c)$$

Sublinear in Nk

Time Succinctness

Show that this suffices for $i\mathcal{O}$

Efficient circuit implementations for special RAM programs such as lookups and sorting.

Thank you!

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FHE directly from these assumptions?

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Complexity/algorithm questions:

Reductions to LWE/GAP-SVP for these assumptions?

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LPN/PRG: Build PKE/ show that they are in CoAM