Orientations and the supersingular endomorphism ring problem

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Supersingular elliptic curves

Isogenies and endomorphisms

Elliptic curves

Elliptic curve over $\mathbb{F}_q$: solutions (x,y) in $\mathbb{F}_q$ of

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An **isogeny** in a map

$$\varphi : E \to F$$

which preserves certain structures. In particular, it is a group homomorphism with a finite kernel

Endomorphism ring

An **endomorphism** is an isogeny $\varphi : E \to E$

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They form a **ring** $\mathrm{End}(E)$

- $\varphi + \psi$ is pointwise addition: $(\varphi + \psi)(P) = \varphi(P) + \psi(P)$
- $\varphi\psi$ is the composition: $(\varphi\psi)(P) = \varphi(\psi(P))$

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What is the structure of $\mathrm{End}(E)$?

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What is the structure of $\mathrm{End}(E)$?

- It contains $\mathbb{Z} \subset \mathrm{End}(E)$...
- $(\mathrm{End}(E), +)$ is a **lattice** of dimension 2 or 4

The Endomorphism Ring problem

A curve E is **supersingular** if $(\mathrm{End}(E), +)$ is a lattice of dimension 4

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EndRing: Given a supersingular curve E, compute $\mathrm{End}(E)$. I.e., find 4 endomorphisms that form a basis of $\mathrm{End}(E)$:

$$\mathrm{End}(E) = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4$$

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Isogeny Path Problem

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EndRing $\Updownarrow$ Isogeny Path Problem $\xleftarrow{\text{Often}}$ Breaking isogeny-based cryptosystems

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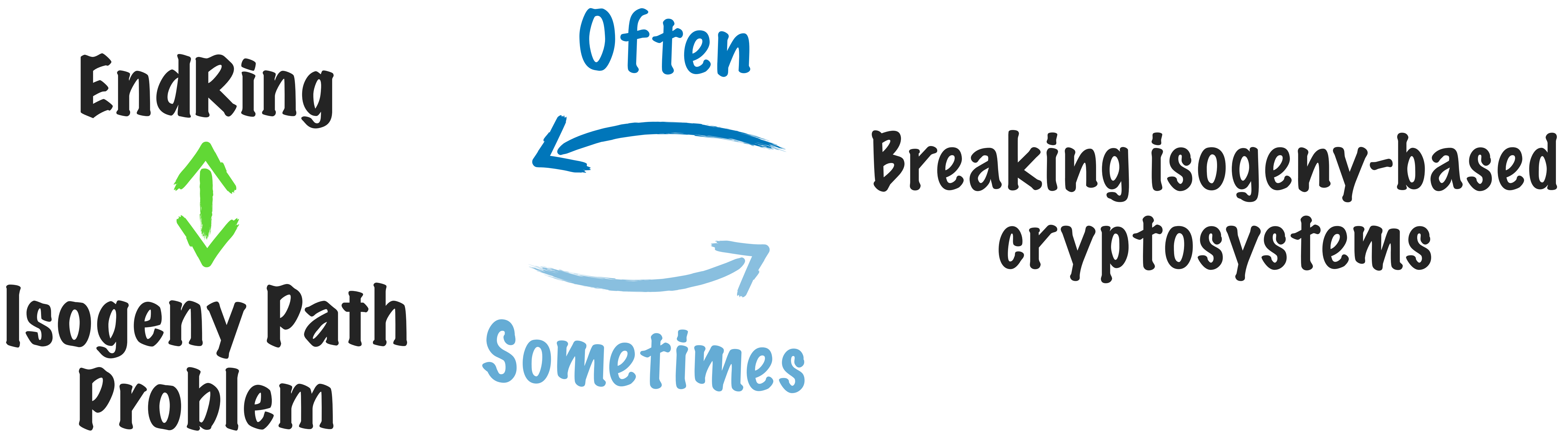

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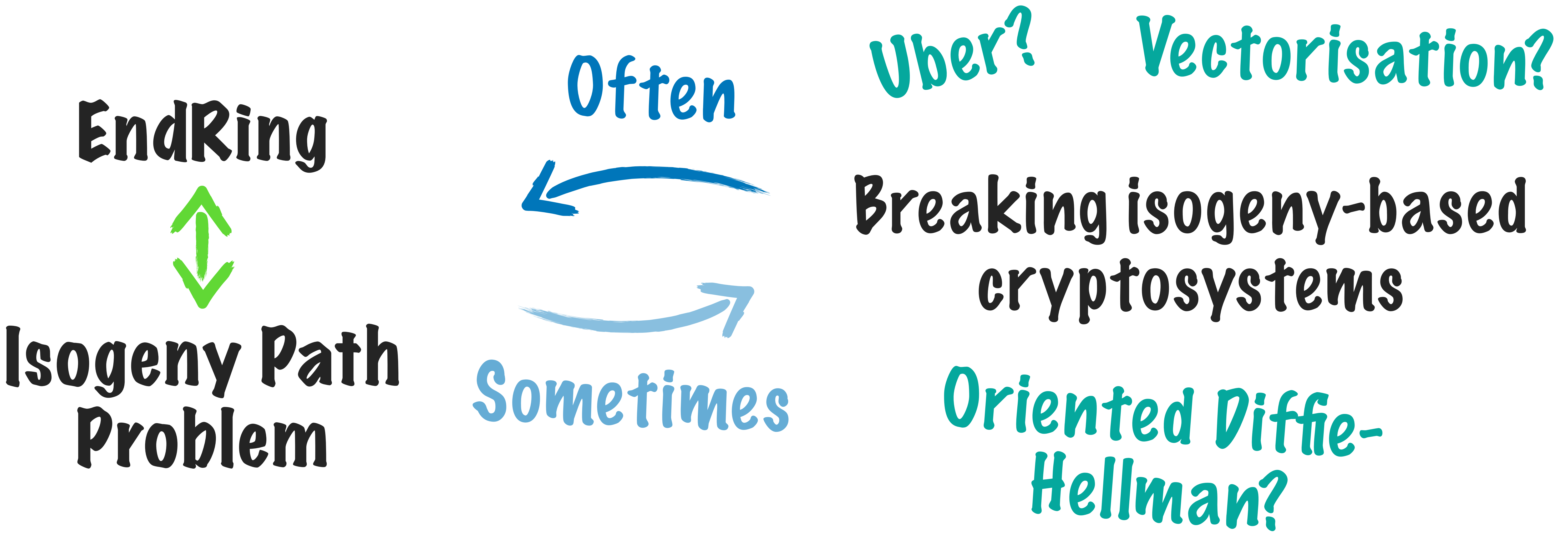

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EndRing

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Isogeny Path Problem

Often

Sometimes

Uber?

Vectorisation?

Breaking isogeny-based cryptosystems

Oriented Diffie-Hellman?

Goal: get a sharper picture of the situation

Orientations

Of supersingular elliptic curves

Oriented elliptic curves

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- $\mathbb{Z}[\alpha] \subset \mathrm{End}(E)$ is a subring of dimension 2

Oriented elliptic curves

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Fix a quadratic ring $\mathcal{O}$. An $\mathcal{O}$-orientation is an injective homomorphism

$$\iota : \mathcal{O} \to \mathrm{End}(E)$$

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$\mathrm{Ell}_{\mathcal{O}}(p)$ is the set of (supersingular) $\mathcal{O}$-oriented curves over $\overline{\mathbb{F}}_p$

Action of the class group

Each quadratic ring $\mathcal{O}$ comes with a finite abelian group $\mathrm{Cl}(\mathcal{O})$, the **ideal class group** of $\mathcal{O}$

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There is an action of $\mathrm{Cl}(\mathcal{O})$ on $\mathrm{Ell}_{\mathcal{O}}(p)$

$$\begin{aligned} * : \mathrm{Cl}(\mathcal{O}) \times \mathrm{Ell}_{\mathcal{O}}(p) &\to \mathrm{Ell}_{\mathcal{O}}(p) \\ (\mathfrak{a}, E) &\longmapsto \mathfrak{a} * E \end{aligned}$$

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\end{aligned}
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$$\mathfrak{b} * (\mathfrak{a} * E) = (\mathfrak{b}\mathfrak{a}) * E$$

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$$\mathfrak{e} * E = E$$

Cryptography from orientations

Key exchange on the CSIDH

CSIDH key exchange

Alice

Bob

CSIDH key exchange

Alice

Fix $E_0 \in \mathrm{Ell}_{\mathcal{O}}(p)$, with $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Bob

CSIDH key exchange

Alice

Fix $E_0 \in \mathrm{Ell}_{\mathcal{O}}(p)$, with $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

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Sample secret $\mathfrak{a} \in \mathrm{Cl}(\mathcal{O})$

CSIDH key exchange

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Sample secret $\mathfrak{a} \in \mathrm{Cl}(\mathcal{O})$

Compute $\mathfrak{a} * E_0$

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Sample secret $\mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$

Compute $\mathfrak{b} * E_0$

Compute $\boldsymbol{E}_{\mathbf{AB}} = \mathfrak{a} * (\mathfrak{b} * E_0)$

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Bob

Sample secret $\mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$

Compute $\mathfrak{b} * E_0$

$$\xleftarrow{\mathfrak{b} * E_0}$$

Compute $E_{AB} = \mathfrak{a} * (\mathfrak{b} * E_0)$

Compute $E_{BA} = \mathfrak{b} * (\mathfrak{a} * E_0)$

CSIDH key exchange

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Sample secret $\mathfrak{a} \in \mathrm{Cl}(\mathcal{O})$

Compute $\mathfrak{a} * E_0$

$$\xrightarrow{\ \mathfrak{a} * E_0\ }$$

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Sample secret $\mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$

Compute $\mathfrak{b} * E_0$

Compute $E_{AB} = \mathfrak{a} * (\mathfrak{b} * E_0)$

Compute $E_{BA} = \mathfrak{b} * (\mathfrak{a} * E_0)$

$$E_{AB} = \mathfrak{a} * (\mathfrak{b} * E_0) = \mathfrak{b} * (\mathfrak{a} * E_0) = E_{BA}$$

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$$\boldsymbol{E}_{\mathsf{AB}} = \mathfrak{a} * (\mathfrak{b} * E_0) = \mathfrak{b} * (\mathfrak{a} * E_0) = \boldsymbol{E}_{\mathsf{BA}}$$

$\boldsymbol{E}_{\mathsf{AB}} = \boldsymbol{E}_{\mathsf{BA}}$ is Alice and Bob's shared secret

CSIDH key exchange

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Sample secret $\mathfrak{a} \in \mathrm{Cl}(\mathcal{O})$

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Compute $\boldsymbol{E}_{\mathbf{AB}} = \mathfrak{a} * (\mathfrak{b} * E_0)$

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A spy sees E_0, $\mathfrak{a} * E_0$, and $\mathfrak{b} * E_0$. Can they recover the secret $(\mathfrak{a}\mathfrak{b}) * E_0$?

CSIDH key exchange

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Compute $\mathfrak{a} * E_0$

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Compute $\mathfrak{b} * E_0$

Compute $\boldsymbol{E_{AB}} = \mathfrak{a} * (\mathfrak{b} * E_0)$

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The CSIDH problem

Vectorisation

And the oriented endomorphism ring problem

Vectorisation

$\mathscr{O}$-Diffie-Hellman: Given $\mathscr{O}$-oriented E, $\mathfrak{a} * E$ and $\mathfrak{b} * E$, compute $(\mathfrak{a}\mathfrak{b}) * E$.

Vectorisation

$\mathcal{O}$-Diffie-Hellman: Given $\mathcal{O}$-oriented E, $\mathfrak{a} * E$ and $\mathfrak{b} * E$, compute $(\mathfrak{a}\mathfrak{b}) * E$.

$\mathcal{O}$-Vectorisation: Given $\mathcal{O}$-oriented E and $\mathfrak{a} * E$, find $\mathfrak{a}$.

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Evidently*, $\mathcal{O}$-Diffie-Hellman reduces to $\mathcal{O}$-Vectorisation

* actually not so evident, because applying the action of $\mathfrak{a}$ on $\mathfrak{b} * E$ to get $\mathfrak{a}\mathfrak{b} * E$ may not be efficient. This issue can be fixed

Vectorisation

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Theorem 1: There is a quantum polynomial time reduction from $\mathcal{O}$-Vectorisation to $\mathcal{O}$-Diffie-Hellman (assuming GRH).

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Theorem 1: There is a quantum polynomial time reduction from $\mathcal{O}$-Vectorisation to $\mathcal{O}$-Diffie-Hellman (assuming GRH).

Previous work: **subexponential**, **heuristic**, quantum reduction **[Galbraith, Panny, Smith, Vercauteren 2021]**

$\mathcal{O}$-Vectorisation

$\mathcal{O}$-Diffie-Hellman

$\mathcal{O}$-Vectorisation

$\mathcal{O}$-Diffie-Hellman

$\Updownarrow$ $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Breaking CSIDH

$\mathcal{O}$-Vectorisation

$\mathcal{O}$-Diffie-Hellman

$\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Breaking CSIDH

$\mathcal{O}$-Vectorisation $\rightleftarrows$ $\mathcal{O}$-Diffie-Hellman

Quantum

$\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Breaking CSIDH

How does this relate to EndRing ?

$\mathcal{O}$-Vectorisation $\rightleftarrows$ $\mathcal{O}$-Diffie-Hellman

Quantum

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Breaking CSIDH

Oriented EndRing

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Theorem 2: Given the factorisation of $\mathrm{disc}(\mathcal{O})$, the problems $\mathcal{O}$-Vectorisation and $\mathcal{O}$-EndRing are equivalent (assuming GRH).

Oriented EndRing

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Theorem 2: Given the factorisation of $\mathrm{disc}(\mathcal{O})$, the problems $\mathcal{O}$-Vectorisation and $\mathcal{O}$-EndRing are equivalent (assuming GRH).

Previous work: **subexponential** reduction for $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$ from $\mathcal{O}$-Vectorisation to $\mathcal{O}$-EndRing **[Castryck, Panny, Vercauteren 2020]**

$$
\begin{tikzcd}
\mathcal{O}\text{-Vectorisation} \arrow[r, shift right, "\text{Quantum}"'] & \mathcal{O}\text{-Diffie-Hellman} \arrow[l, shift right] \arrow[d, leftrightarrow, "\mathcal{O} = \mathbb{Z}[\sqrt{-p}]"] \\
& \text{Breaking CSIDH}
\end{tikzcd}
$$

$\mathcal{O}$-EndRing

$\mathcal{O}$-Vectorisation

$\mathcal{O}$-Diffie-Hellman

Quantum

$\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Breaking CSIDH

$$\begin{tikzcd} \mathcal{O}\text{-EndRing} \arrow[r, leftrightarrow] & \mathcal{O}\text{-Vectorisation} \arrow[r, bend right, "\text{Quantum}"'] & \mathcal{O}\text{-Diffie-Hellman} \arrow[l, bend right] \arrow[d, leftrightarrow, "\mathcal{O} = \mathbb{Z}[\sqrt{-p}]"] \\ & & \text{Breaking CSIDH} \end{tikzcd}$$

EndRing

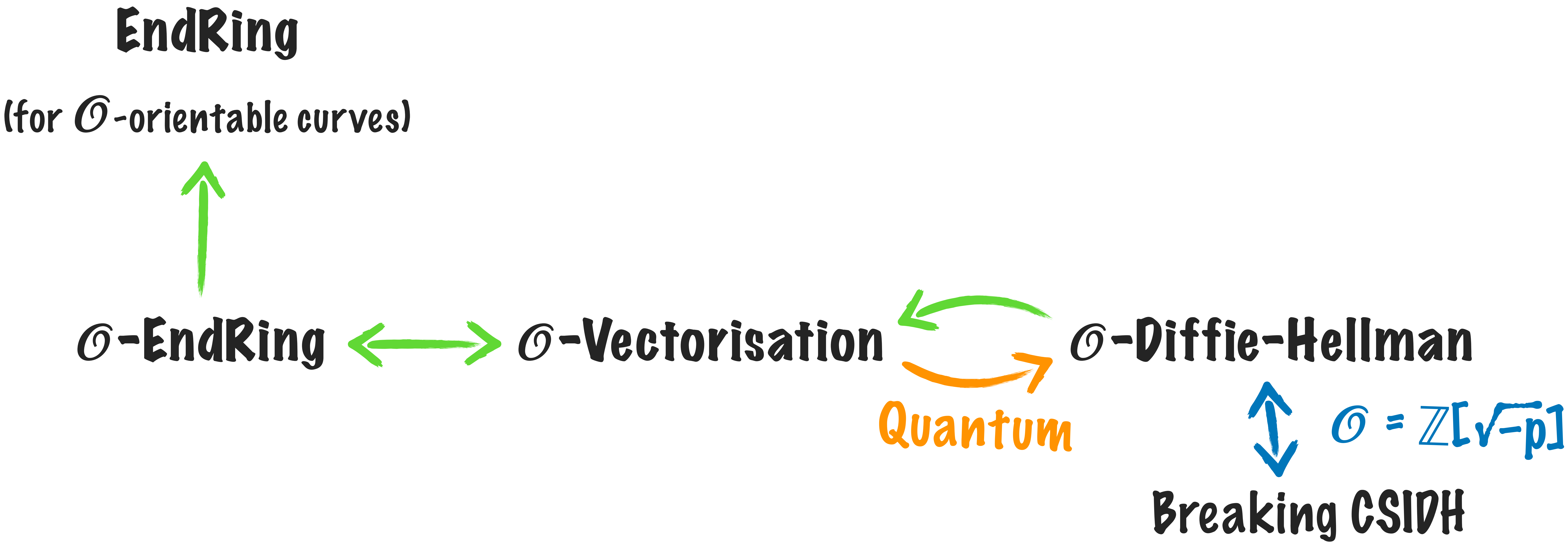
(for $\mathcal{O}$-orientable curves)

$\mathcal{O}$-EndRing $\longleftrightarrow$ $\mathcal{O}$-Vectorisation $\rightleftarrows$ $\mathcal{O}$-Diffie-Hellman

Quantum

$\updownarrow$ $\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Breaking CSIDH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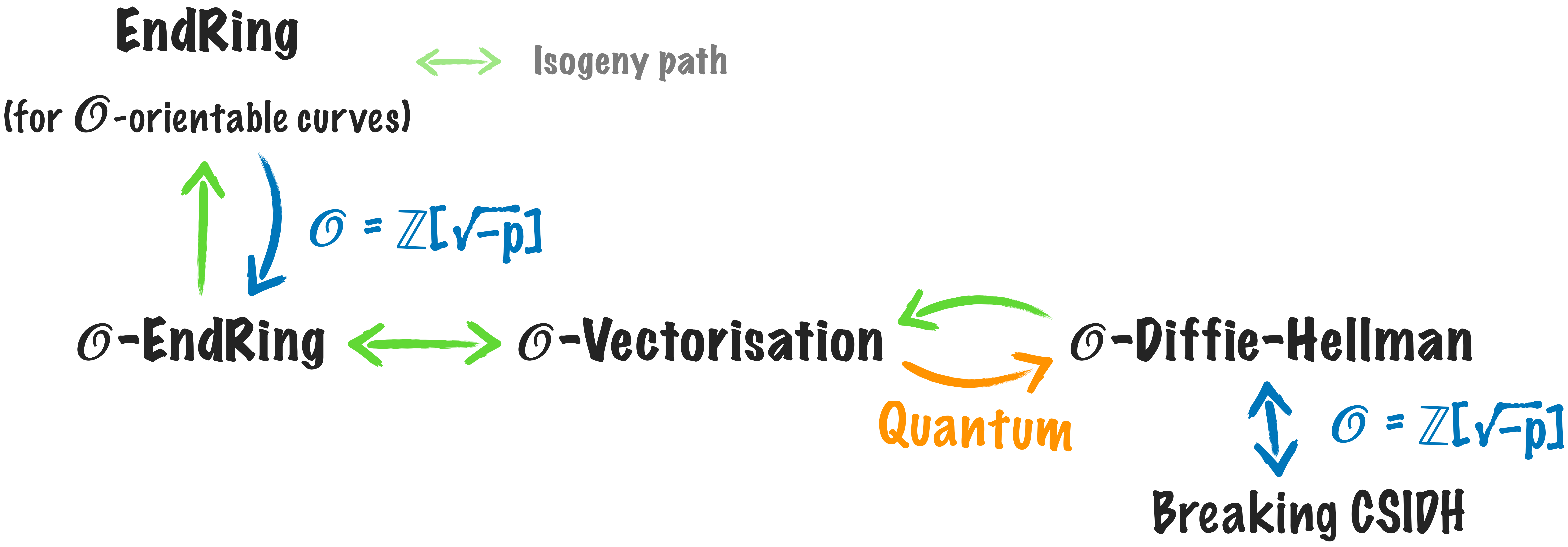
Breaking CSIDH


EndRing
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Isogeny path
$\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$
$\mathcal{O}$-EndRing
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$\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Breaking a few isogeny-based schemes

Sometimes

$\mathcal{O}$-EndRing

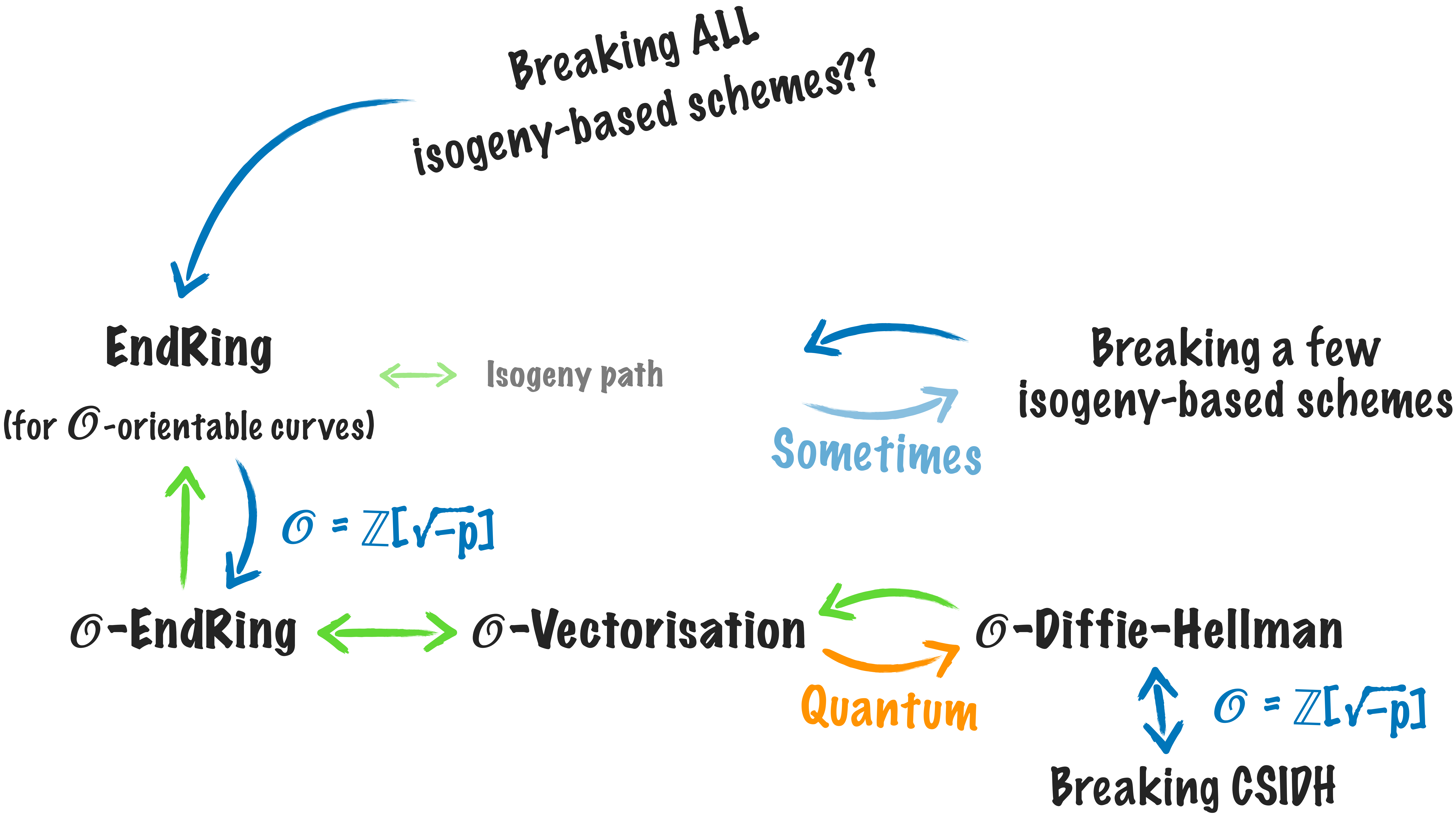
$\mathcal{O}$-Vectorisation

$\mathcal{O}$-Diffie-Hellman

Quantum

$\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$

Breaking CSIDH



Breaking ALL
isogeny-based schemes??
EndRing
(for 𝒪-orientable curves)
Isogeny path
Breaking a few
isogeny-based schemes
Sometimes
𝒪 = ℤ[√−p]
𝒪-EndRing
𝒪-Vectorisation
𝒪-Diffie-Hellman
Quantum
𝒪 = ℤ[√−p]
Breaking CSIDH

The Uber isogeny problem

And another oriented endomorphism ring problem

Uber

Many isogeny-based cryptosystems reduces to:

$\mathcal{O}$-Uber: Given $\mathcal{O}$-orient**ed** (E,ι), and an $\mathcal{O}$-orient**able** curve F, find $\mathfrak{a}$ such that $\mathfrak{a} * E = F$.

Uber

Many isogeny-based cryptosystems reduces to:

$\mathcal{O}$-Uber: Given $\mathcal{O}$-orient**ed** (E,ι), and an $\mathcal{O}$-orient**able** curve F, find $\mathfrak{a}$ such that $\mathfrak{a} * E = F$.

[De Feo, Delpech de Saint Guilhem, Fouotsa, Kutas, Leroux, Petit, Silva, W. 2021] SIDH, CSIDH, OSIDH, Séta reduce to $\mathcal{O}$-Uber

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$\mathcal{O}$-EndRing*: Given $\mathcal{O}$-orient**able** E, compute $\mathrm{End}(E)$ and an $\mathcal{O}$-orientation ι of E.

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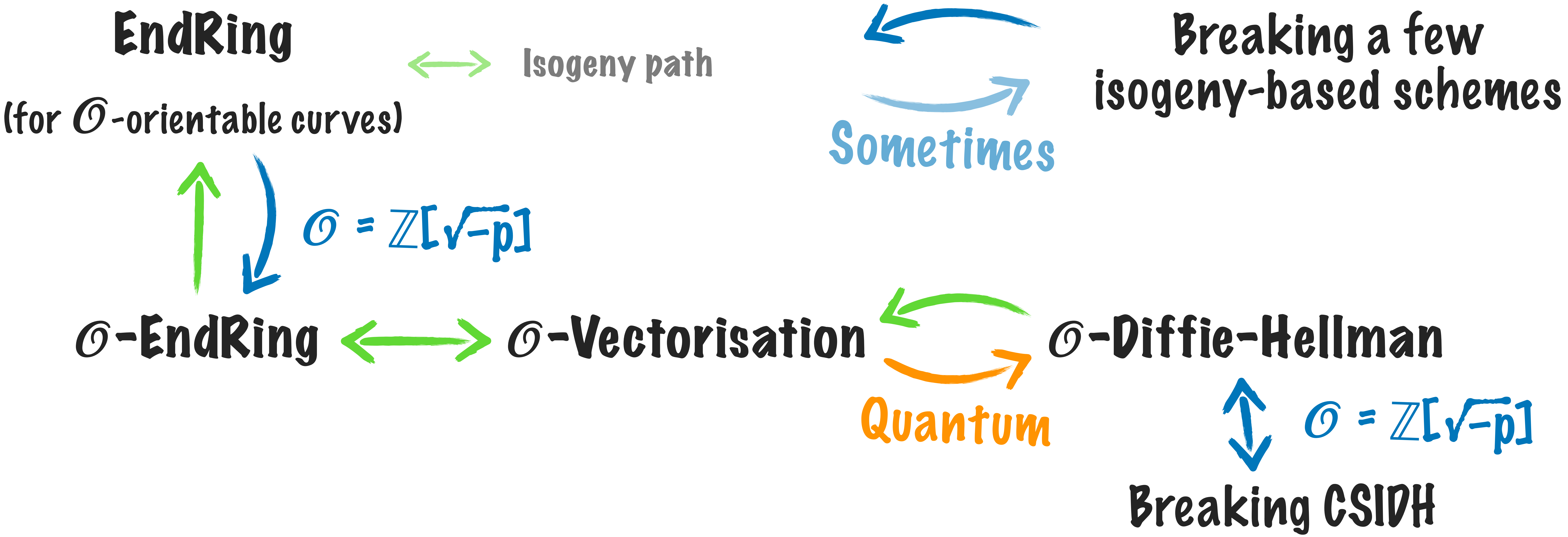
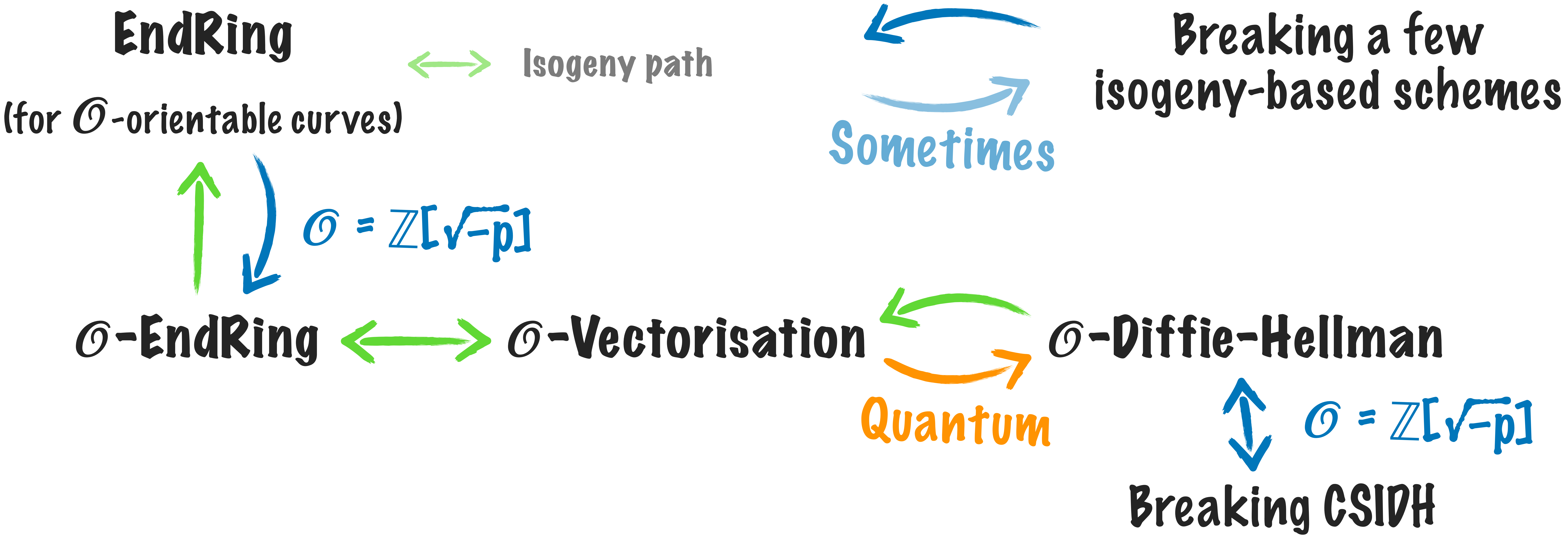
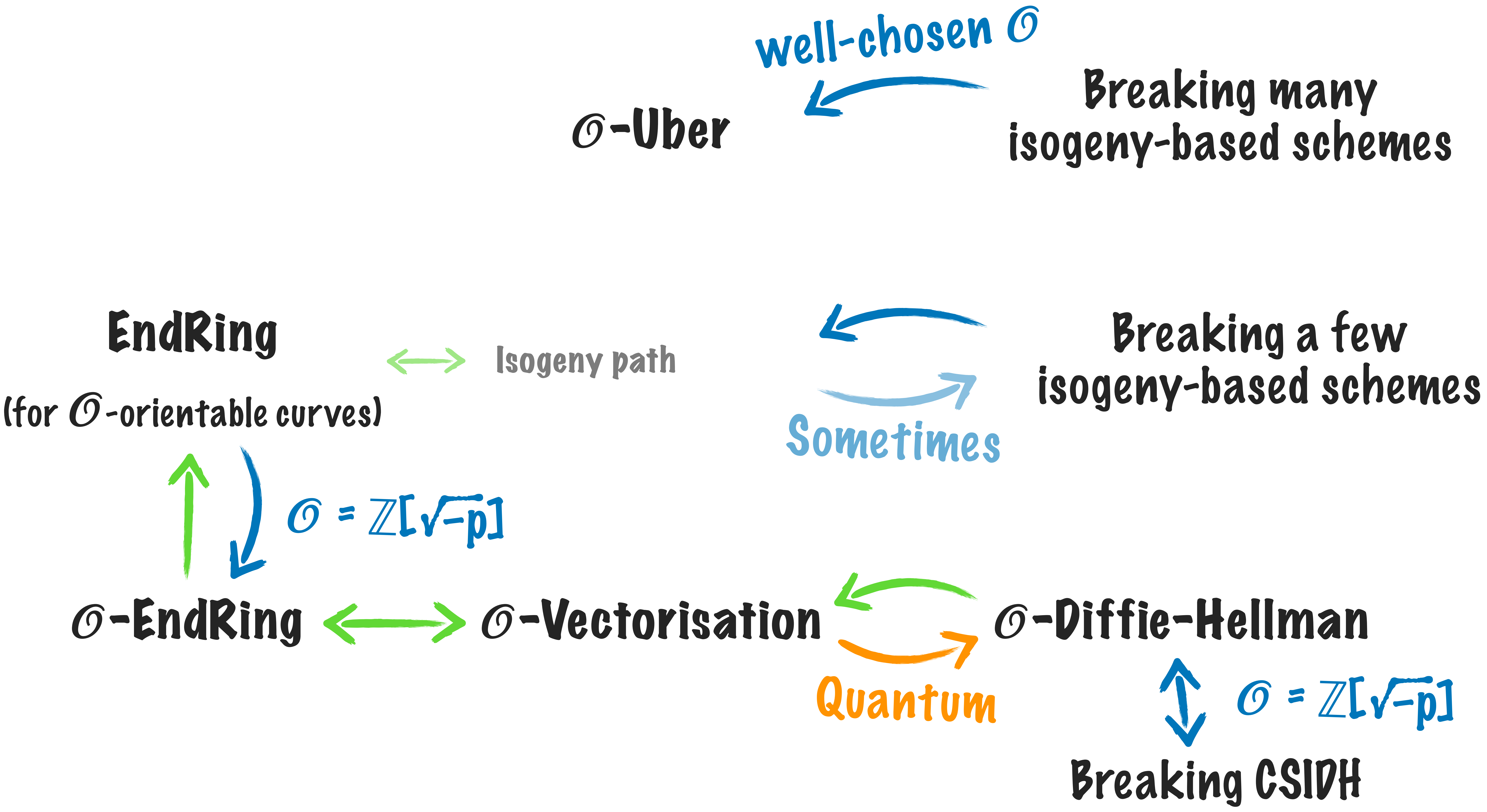
Theorem 3: Given the factorisation of $\mathrm{disc}(\mathcal{O})$, the problems $\mathcal{O}$-Uber and $\mathcal{O}$-EndRing* are equivalent (assuming GRH).


EndRing
(for 𝒪-orientable curves)
Isogeny path
Breaking a few isogeny-based schemes
Sometimes
𝒪 = ℤ[√−p]
𝒪-EndRing
𝒪-Vectorisation
𝒪-Diffie-Hellman
Quantum
𝒪 = ℤ[√−p]
Breaking CSIDH

$\mathcal{O}$-Uber




well-chosen $\mathcal{O}$
$\mathcal{O}$-Uber
Breaking many isogeny-based schemes
EndRing
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$\mathcal{O} = \mathbb{Z}[\sqrt{-p}]$
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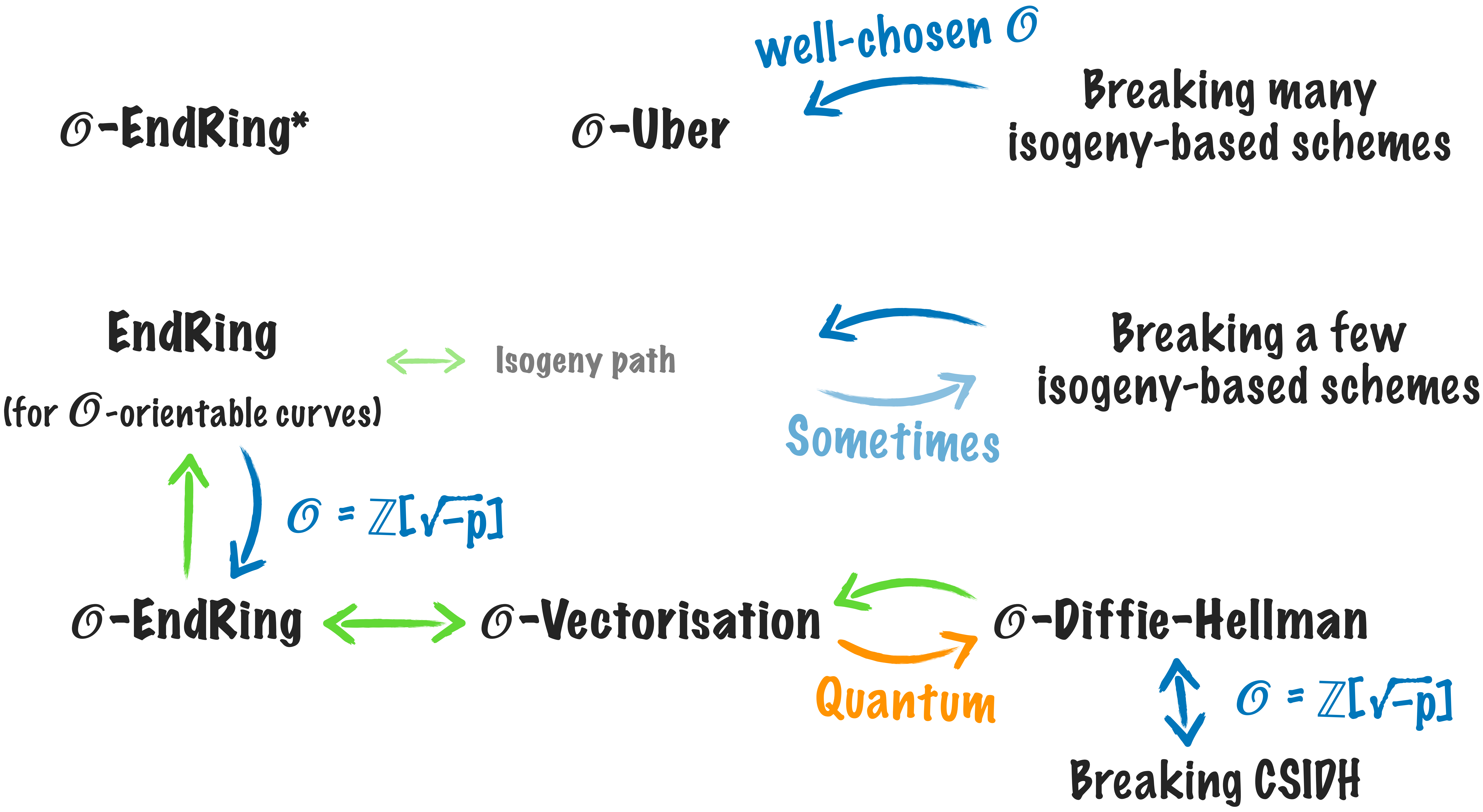

$\mathcal{O}$-EndRing*
$\mathcal{O}$-Uber
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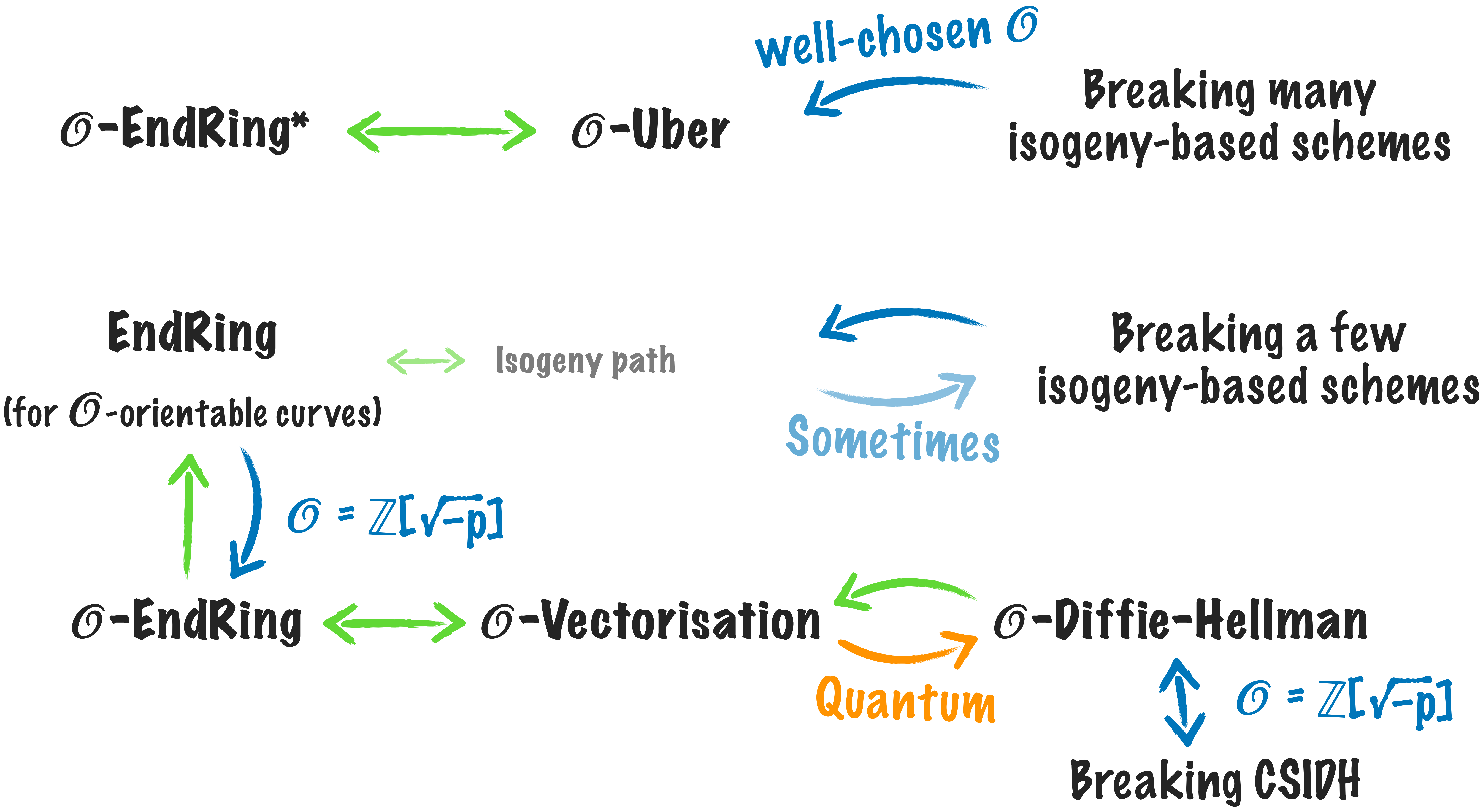

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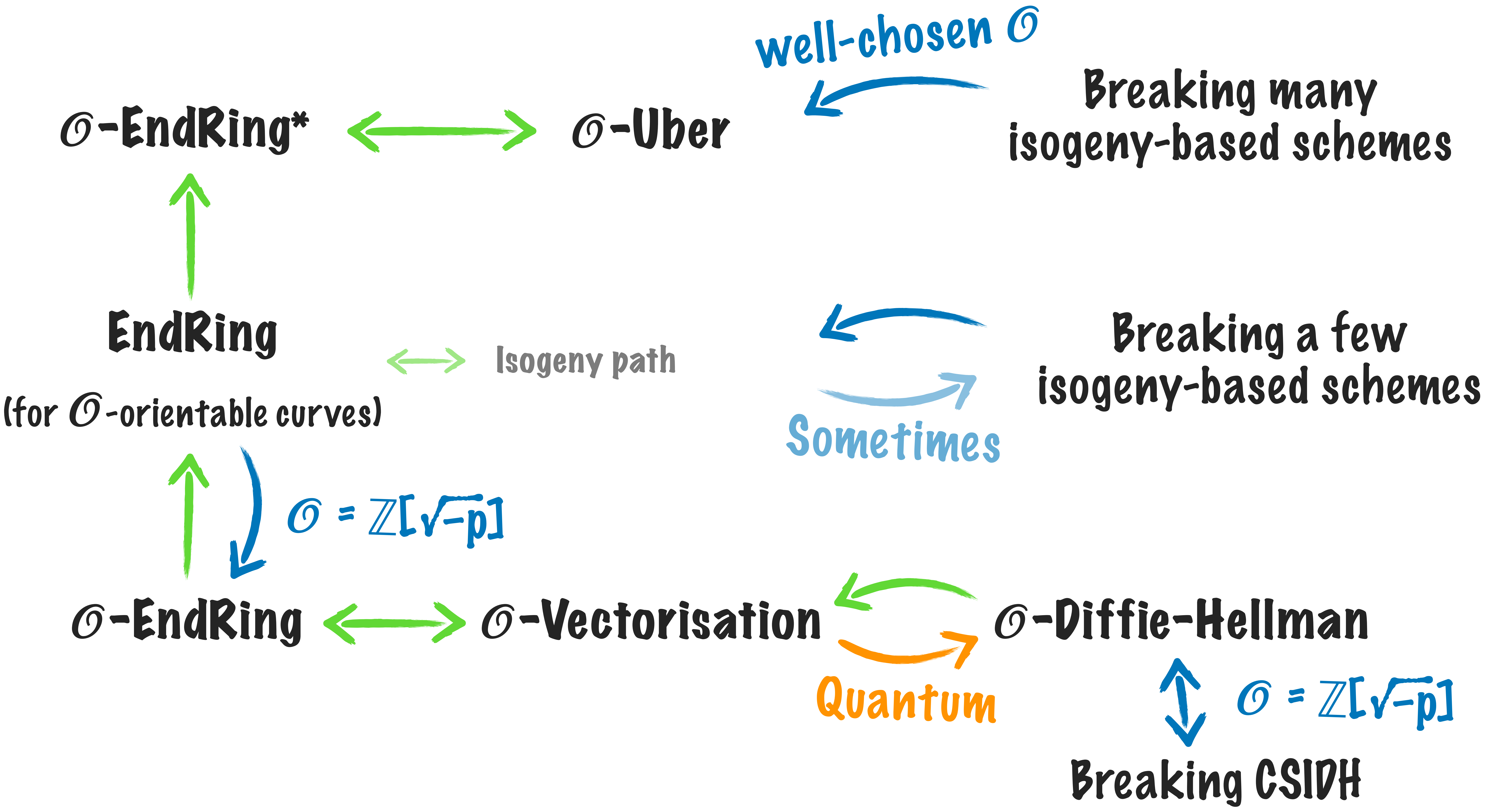

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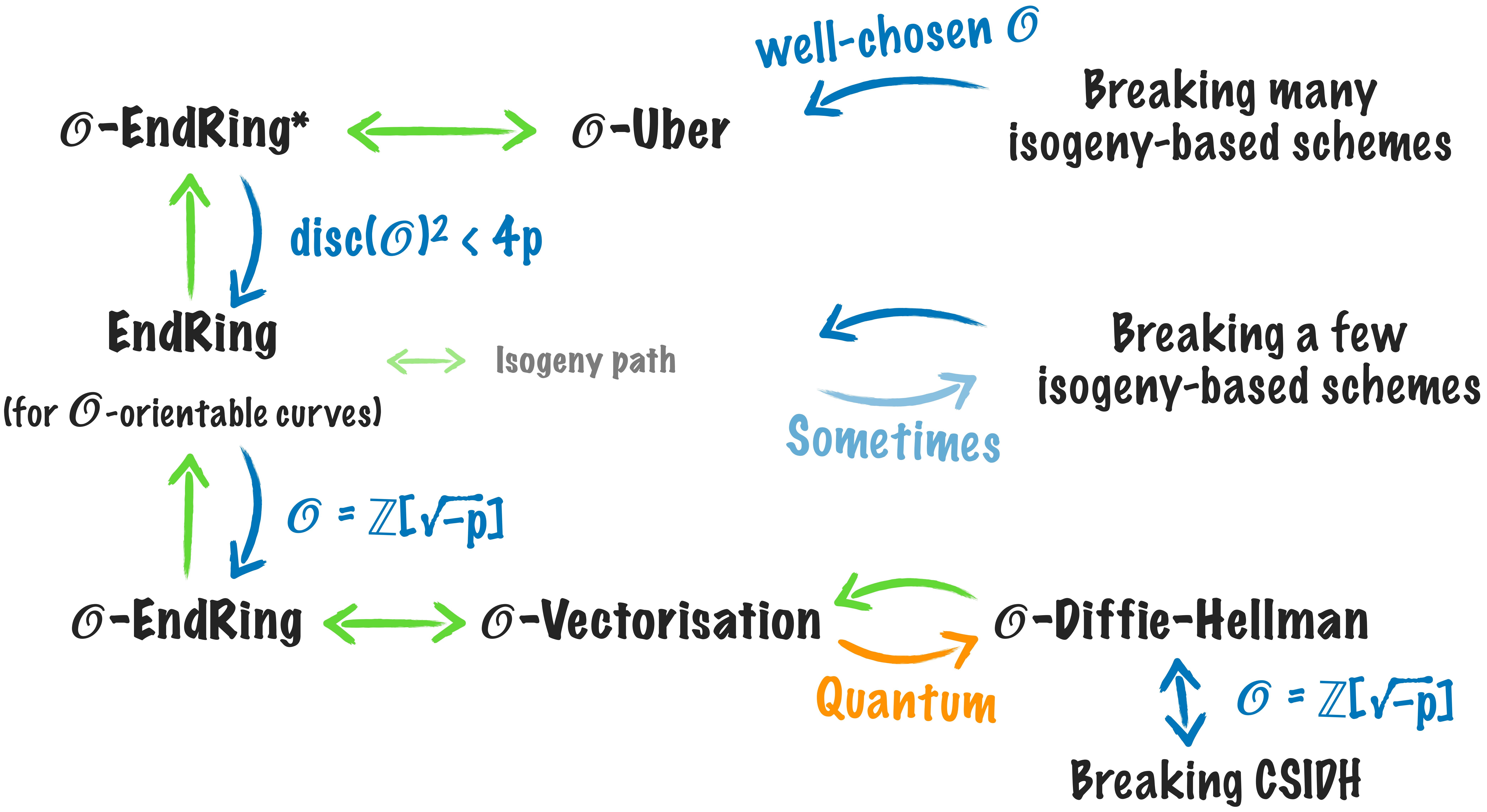

$\mathcal{O}$-EndRing*
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well-chosen $\mathcal{O}$
Breaking many isogeny-based schemes
$\mathrm{disc}(\mathcal{O})^2 < 4p$
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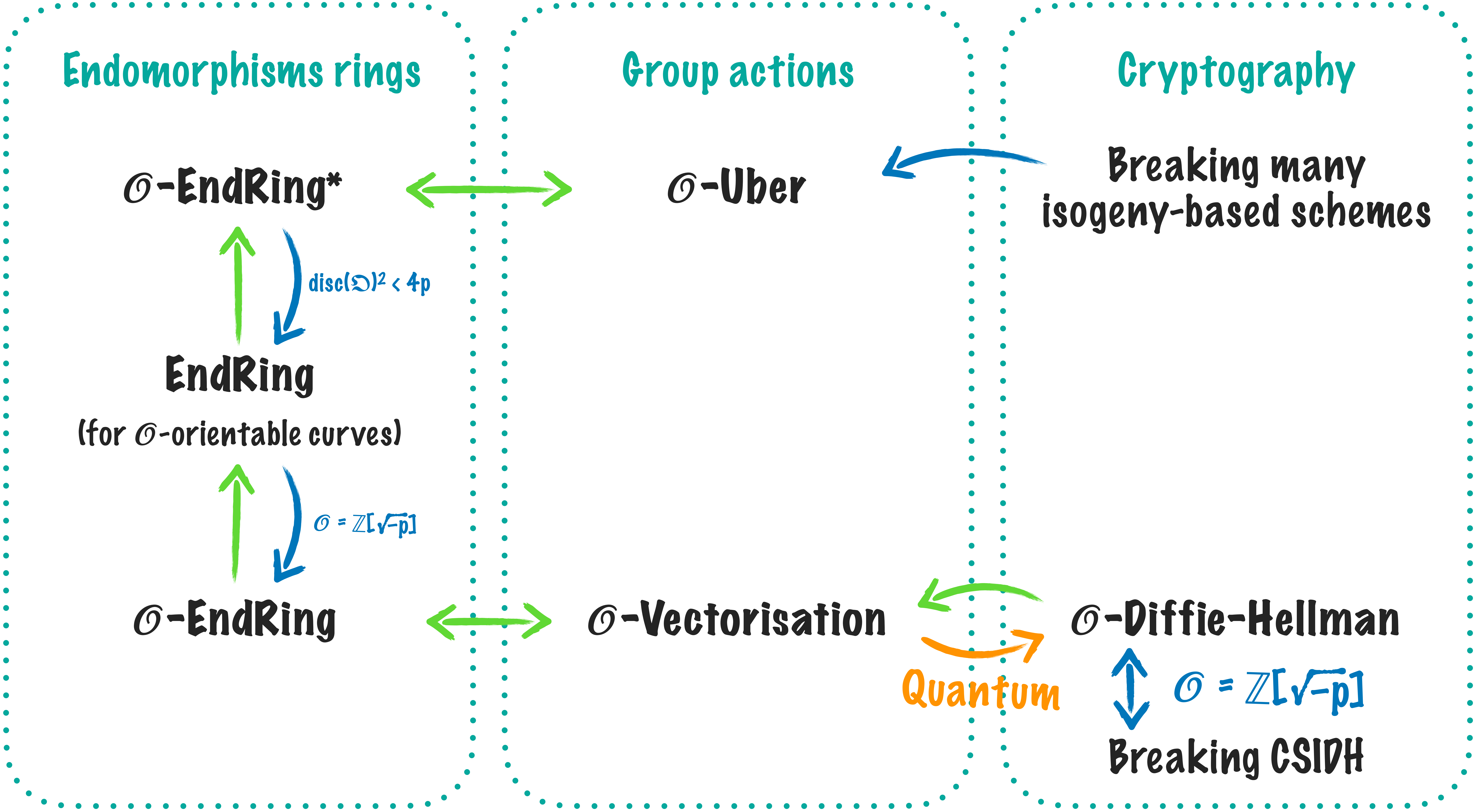
Endomorphisms rings
Group actions
Cryptography
$\mathcal{O}$-EndRing*
$\mathcal{O}$-Uber
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Breaking CSIDH

Orientations and the supersingular endomorphism ring problem

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