Radical Isogenies on Montgomery Curves

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Overview

Isogeny-based cryptography

- a candidate for **post-quantum cryptography**,
- **small** keys and ciphertext,
- **slow** because of isogeny computation.

Radical isogenies

- formulas for computing **repeating isongenies of the same degree**,
- proposed by [CDV2020] (Castryck, Decru, and Vercauteren @Asiacrypt 2020),
- The original formulas are constructed on **Tate normal forms**.

This work

- constructs radical isogenies of degree 3, 4 on **Montgomery curves**,
- **reduce the cost** of transforms between curves in some protocols,
- **prove a conjecture** left open by [CDV2020].

Elliptic curves

Definition 1

An **elliptic curve** is a smooth algebraic curve of genus one.

- An elliptic curve E has an **abelian group structure**, i.e., we can define $P+Q$ for $P, Q \in E$.
- There are **many forms** of elliptic curves.
- In isogeny-based cryptography, we often use **Montgomery curves**

$$y^2 = x^3 + Ax^2 + x,$$

because of efficient scalar multiplications and isogeny formulas.

Isogenies (1/2)

Definition 2

An **isogeny** is a nonzero rational homomorphism between elliptic curves.

Let $\varphi : E \to E'$ be an isogeny.

- We can define the **degree** of φ, denoted by $\deg \varphi$.
- There is the **dual isogeny** $\hat{\varphi} : E' \to E$ $(\deg \hat{\varphi} = \deg \varphi)$.

Example (degree 2)

$$E_1 : y^2 = x^3 + 6x^2 + x,\ \ E_2 : y^2 = x^3 - 12x^2 + 32x,$$

$$E_1 \to E_2,\ \ (x, y) \mapsto \left(\frac{y^2}{x^2}, \frac{y(x^2 - 1)}{x^2} \right).$$

Isogenies (2/2)

E: an elliptic curve over K, N: an integer coprime to $\mathrm{char}(K)$

one to one correspondence

$$\begin{array}{ccc} \{\text{subgroups of } E \text{ of order } N\} & \overset{1:1}{\longleftrightarrow} & \{\text{isogenies of degree } N \text{ from } E\} \\ G & \longmapsto & \varphi_G : E \to E/G \\ & & \text{s.t. } \ker \varphi_G = G \end{array}$$

- This work considers the case that G is **cyclic**.
  I.e., we consider a subgroup of the form $\langle P \rangle$.
- $(E; P, Q \in E) \mapsto (\varphi_{\langle P \rangle}(Q), E/\langle P \rangle$ can be **efficiently computed**.
  (**Vélu's formula**)
- $(E, E/\langle P \rangle) \mapsto P$ is consider to be **hard**.
  (the security of isogeny-based cryptography)

CSIDH and CSURF

CSIDH

- isogeny-based key-exchange,
- by Castryck, Lange, Martindale, Panny, and Renes @Asiacrypt 2018,
- uses elliptic curves $/\mathbb{F}_p$ and isogenies $/\mathbb{F}_p$ with $p \equiv 3 \pmod 8$,
- uses only isogenies of **odd degrees**.

CSURF

- variant of CSIDH by Castryck and Decru @PQCrypto 2020,
- uses $p \equiv 7 \mod 8$,
- also uses isogenies of **degree 2 and 4**.

E: an elliptic curve over K,
N: an integer coprime to $\mathrm{char}(K)$,
P: a point on E of order N.

A **radical isogeny** is a formula of a map

$$\begin{array}{ccc} \text{(elliptic curve, order-}N\text{ point)} & & \text{(elliptic curve, order-}N\text{ point)} \\ (E, P \in E) & \longmapsto & (E/\langle P\rangle, P' \in E/\langle P\rangle), \end{array}$$

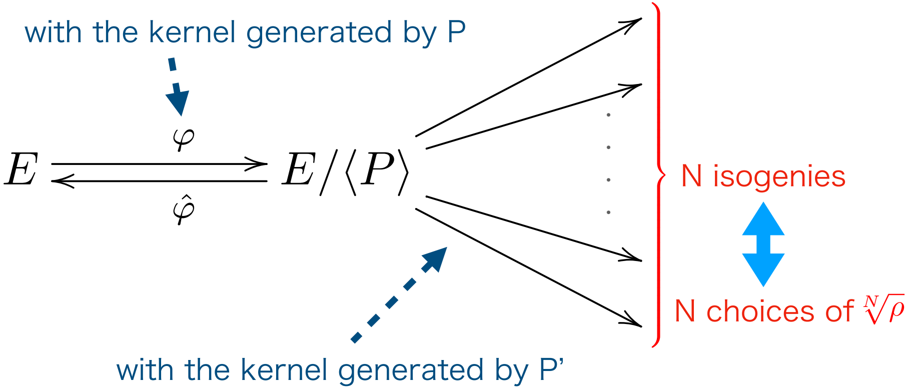
where $\langle \hat{\varphi}(P')\rangle = \langle P\rangle$.

Thoery of radical isogenies

One can chose a form of $E/\langle P\rangle$ such that

$$E/\langle P\rangle \text{ and } P' \text{ are defined over } K(\sqrt[N]{\rho}),$$

where ρ is the Tate pairing $\tau_N(P, -P)$.


with the kernel generated by P
$E \underset{\hat{\varphi}}{\overset{\varphi}{\rightleftarrows}} E/\langle P \rangle$
N isogenies
N choices of $\sqrt[N]{\rho}$
with the kernel generated by P'

Radical Isongenies (3/6)

[CDV2020] uses forms of elliptic curves such that P and P' are $(0, 0)$.

$N = 3$

$$E : y^2 + a_1 xy + a_3 y = x^3.$$

$N \geq 4$

Tate normal form

$$E : y^2 + (1 - c)xy - by = x^3 - bx^2$$

(b, c satisfy a relation depending on N).

Radical Isongenies (4/6)

$\mathbf{N = 3}$

$E : y^2 + a_1xy + a_3y = x^3$, $P = (0,0)$,

$E/\langle P\rangle : y^2 + a_1'xy + a_3'y = x^3$, $P' = (0,0)$,

$$a_1' = -6\alpha + a_1,\ a_3' = 3a_1\alpha^2 - a_1^2\alpha + 9a_3,$$

α is a cube root of $-a_3$.

$\mathbf{N = 4}$

$E : y^2 + xy - by = x^3 - bx^2$, $P = (0,0)$,

$E/\langle P\rangle : y^2 + xy - b'y = x^3 - b'x^2$, $P' = (0,0)$,

$$b' = \frac{\alpha(4\alpha^2 + 1)}{(2\alpha + 1)^4},$$

α is a fourth root of $-b$.

Iteration of radical isogenies of degree $N = 3$:

$$E \xrightarrow{\langle(0,0)\rangle} E' \xrightarrow{\langle(0,0)\rangle} E'' \xrightarrow{\langle(0,0)\rangle}$$

$$(a_1, a_3) \qquad (a_1', a_3') \qquad (a_1'', a_3'')$$

$$a_1' = -6\alpha + a_1 \qquad a_1'' = -6\alpha' + a_1'$$

$$a_3' = 3a_1\alpha^2 - a_1^2\alpha + 9a_3 \qquad a_3'' = 3a_1'{\alpha'}^2 - {a_1'}^2\alpha' + 9a_3'$$

No computation for the kernels of intermediate isogenies.
$\Rightarrow$ **accelerating isogenies of small degrees** in CSIDH and CSURF.
(especially in **CSURF**. $\because$ one can use $N = 4$.)

Q. How to choose a radical $\alpha = \sqrt[3]{-a_3}$?

A. Choose $\alpha \in \mathbb{F}_p$ in CSIDH and CSURF.

$\because$ There is the unique N-th root in $\mathbb{F}_p$ if $\#\mathbb{F}_p^\times = p - 1$ is coprime to N.

Radical Isogenies (6/6)

Radical isogenies in CSURF

- One needs to **transform to a Montgomery curve**
  $\because$ generating the first kernel and computing higher degree isogenies.

- In the case $N = 4$, there are two fourth roots in $\mathbb{F}_p$.
  $\Rightarrow$ The choice is conjectured but **not proven**.

This work

- constructs radical isogenies of degree 3 and 4 on Montgomery curves.

- proves the conjecture on $N = 4$.

Montgomery Curves

Montgomery Curves

A **Montgomery curve** is an elliptic curve defined by

$$y^3 = x^3 + Ax^2 + x, \quad A^2 \neq 4.$$

We call A the **Montgomery coefficient**.

- The order of the point $(0, 0)$ is 2.
- $[2](1, -) = [2](-1, -) = (0, 0)$.
- $C_E^{(4)} \coloneqq \langle (1, -) \rangle$.
- If $(t, -)$ is a point of order 3 then

$$A = \frac{-3t^4 - 6t^2 + 1}{4t^3}.$$

I.e., t determines the Montgomery coefficient.

Our Contribution 1-1: A Formula of Degree 3

A pair $(E, (t, -))$ of a Montgomery curve and a point of order 3 is represented by t.

$\Rightarrow$ There exists a radical isogeny: $t \mapsto t'$.

Theorem 1

$(E, (t, -))$: a pair of a Montgomery curve and a point order 3,

$\varphi : E \to E/\langle (t, -) \rangle$: an isogeny with kernel $\langle (t, -) \rangle$.

$(t', -)$: a point on $E/\langle (t, -) \rangle$ of order 3 such that $\langle \hat{\varphi}((t', -)) \rangle = \langle (t, -) \rangle$.

Then

$$t' = 3t\alpha^2 + (3t^2 - 1)\alpha + 3t^3 - 2t,$$

where α is a **cube root** of $t(t^2 - 1)$.

Our Contribution 1-2: A Formula of Degree 4

A pair $(E, C_E^{(4)})$ of a Montgomery curve and a point of order 4 is represented by A.

$\Rightarrow$ There exists a radical isogeny: $A \mapsto A'$.

Theorem 2

E : a Montgomery curve with coefficient A,

$\varphi : E \to E'$: an isogeny of kernel $C_E^{(4)}$ such that $\hat{\varphi}(C_{E'}^{(4)}) = C_E^{(4)}$,

A' : the Montgomery coefficient of E', $a := 4(A+2)$, $a' := 4(A'+2)$.
Then

$$a' = \frac{(\alpha+2)^4}{\alpha(\alpha^2+4)}.$$

where α is a **fourth root** of a.

Comparison (1/2)

[CDV2020]:

$$
\begin{tikzcd}
E_T \arrow[r] \arrow[rrrr, phantom, bend left, "\overbrace{\hspace{6cm}}^{\text{radical isogenies}}"] & E_T^{(1)} \arrow[r] & \cdots \arrow[r] & & E_T^{(n)} \arrow[d, "\text{transform}"] \\
E_M \arrow[u, "\text{transform}"] & & & & E_M^{(n)}
\end{tikzcd}
$$

This work:

$$
\begin{tikzcd}
E_M \arrow[loop left, "\text{to radical representation}"'] \arrow[r] \arrow[rrrr, phantom, bend left, "\overbrace{\hspace{6cm}}^{\text{radical isogenies}}"] & E_M^{(1)} \arrow[r] & \cdots \arrow[r] & & E_M^{(n)} \arrow[loop right, "\text{recover coefficient}"]
\end{tikzcd}
$$

Comparison (2/2)

Number of operations in $\mathbb{F}_p$ in CSURF

	Degree 3		Degree 4	
	[CDV2020]	**Our formula**	[CDV2020]	**Our formula**
Isogeny	$\mathbf{E} + 3\mathbf{M} + 12\mathbf{A}$	$\mathbf{E} + 5\mathbf{M} + 12\mathbf{A}$	$\mathbf{E} + 3\mathbf{M} + 5\mathbf{A} + \mathbf{I}$	$\mathbf{E} + 3\mathbf{M} + 4\mathbf{A} + \mathbf{I}$
To radial form	$> \mathbf{E}$	0	$2\mathbf{A} + \mathbf{I}$	$3\mathbf{A}$
From radical form	$> 3\mathbf{E}$	$3\mathbf{M} + 9\mathbf{A} + \mathbf{I}$	$2\mathbf{A} + \mathbf{I}$	$\mathbf{M} + 3\mathbf{A}$

E: exponentiation, **M**: multiplication, **A**: addition, **I**: inversion.

Radicals are computed by exponentiation. $\mathbf{E} \approx 1.5 \log p \mathbf{M}$.

Our Contribution 2: Fourth Root in $\mathbb{F}_p$ (1/2)

Theorem 3

Let p be a prime satisfying $p \equiv 7 \pmod 8$. Consider a radical isogeny of degree 4 on Montgomery curves:

$$a \mapsto a' = \frac{(\alpha + 2)^4}{\alpha(\alpha^2 + 4)}.$$

We can compute **the isogeny used in CSURF** by taking

$$\alpha = (-a)^{(p+1)/8}.$$

Our Contribution 2: Fourth Root in $\mathbb{F}_p$ (2/2)

Corollary 1 (Conjecture by [CDV2020])

Let p be a prime satisfying $p \equiv 7 \pmod 8$. Consider a radical isogeny of degree 4 on Tate normal forms:

$$b \mapsto b' = \frac{\alpha(4\alpha^2+1)}{(2\alpha+1)^4}.$$

We can compute **the isogeny used in CSURF** by taking

$$\alpha = b^{(p+1)/8}.$$

Conclusion

Our contribution:

- We constructed radical isogenies of degree 3 and 4 on Montgomery curves.
- Our formulas slightly improve the efficiency of CSURF using radical isogenies.
- We proved a conjecture left as open by [CDV2020].

Future work:

- Other applications; e.g., random walks in isogeny graphs