Time-Memory tradeoffs for large-weight syndrome decoding in ternary codes

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Definitions, motivation

The fixed-weight syndrome decoding problem (SDP)

- We consider $[n, k]$-ternary linear codes
- $n =$ length, $k =$ dimension

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$\boldsymbol{s} \in \mathbb{F}_3^{n-k}$ (syndrome)
$w \in [\![0, n]\!]$ (weight)

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In our case:

- $\boldsymbol{H} \twoheadleftarrow \left\{ \boldsymbol{M} \in \mathbb{F}_3^{(n-k)\times n} \mid \boldsymbol{M} \text{ full-rank} \right\}$; $\boldsymbol{s} \twoheadleftarrow \mathbb{F}_3^{n-k}$
- $k/n, w/n$ fixed
- Number of solutions exponential in n

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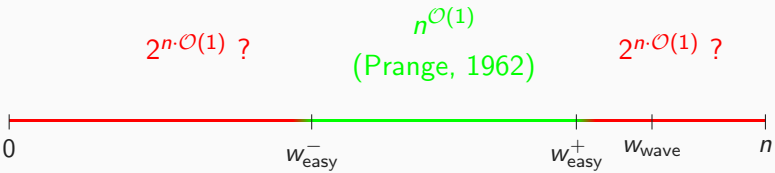
- Decoding is easier with access to $\boldsymbol{H}$ $\implies$ trapdoor
- One security requirement: SDP with a random parity-check matrix and syndrome must be hard

The SDP in the Wave regime

- With random parity-check matrix and syndrome, k/n fixed:



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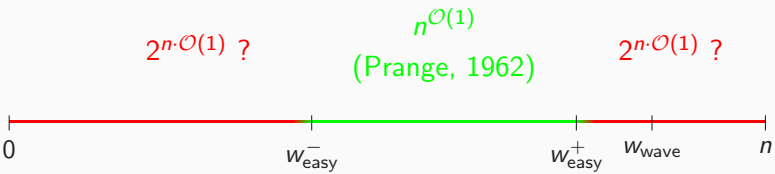

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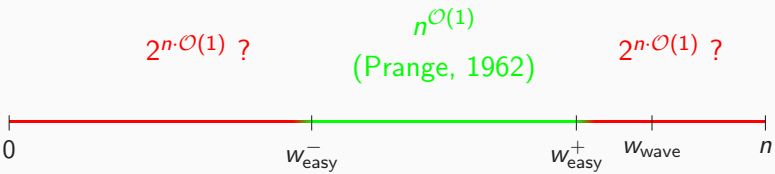

- Best known attack (Bricout et al., 2019): based on the PGE+SS framework with k-tree+representations: $\mathcal{O}(2^{0.0176n})$ in time **and** memory
- In this work, we investigate tradeoffs for this problem

The PGE+SS framework

$SDP(\boldsymbol{H} \in \mathbb{F}_3^{(n-k)\times n}, \boldsymbol{s} \in \mathbb{F}_3^{n-k}, w \in [\![0, n]\!])$:

$\boldsymbol{e} \in \mathbb{F}_3^n$ st., $\boldsymbol{He} = \boldsymbol{s}$, $\mathrm{wt}(e) = w$

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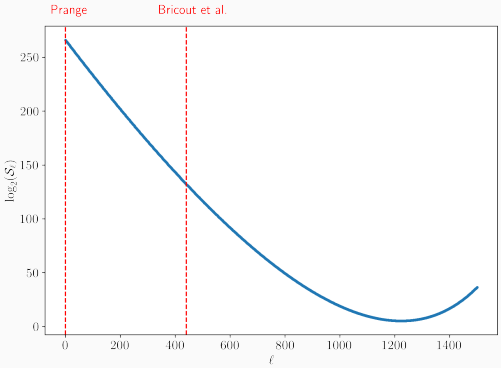
$\Longrightarrow$ Need to solve efficiently $\boldsymbol{H}_2\boldsymbol{e}_2 = \boldsymbol{s}_2$

- Let $\mathcal{S}_\ell$ be the number of required $\boldsymbol{e}_2$ full weight solutions to $\boldsymbol{H}_2\boldsymbol{e}_2 = \boldsymbol{s}_2$
- Large $\ell \implies$ hard sub-problem, but smaller $\mathcal{S}_\ell$
- One ℓ parameter can provide one tradeoff



r-list sum problem

Transform the SDP into a r-list problem

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Input: r lists $L_1, L_2, \ldots, L_r$
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- $\boldsymbol{He} = \boldsymbol{s} \iff \underbrace{\boldsymbol{H}_1\boldsymbol{e}_1}_{L_1} + \underbrace{\boldsymbol{H}_2\boldsymbol{e}_2}_{L_2} \cdots + \underbrace{\boldsymbol{H}_r\boldsymbol{e}_r - \boldsymbol{s}}_{L_r} = \boldsymbol{0}$

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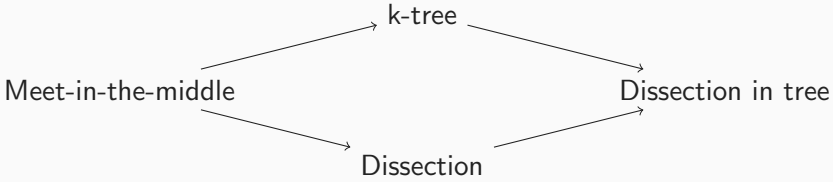
- Do not need to add **all** possible e_i's in L_i
- ⚠ Constraint: $\forall i, \#L_i \leq 2^{\frac{k+\ell}{r}}$

The algorithms used

- Freedom on r, ℓ, $\#L_i$ parameter
- Generalizations of the meet-in-the-middle algorithm:



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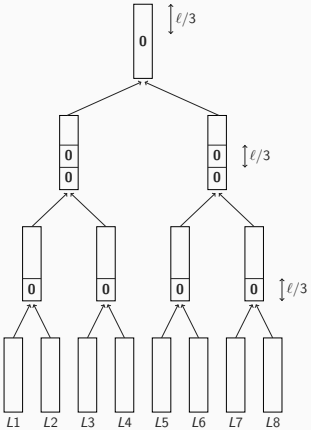
The **granularity** of an algorithm is the minimal number of solutions that can be returned by one iteration without changing its amortized cost

- Some algorithms are very memory friendly, but have a granularity too coarse

The k-tree algorithm

The k-tree algorithm (Wagner, 2002)

- Need $r = 2^a$
- Key idea: apply the meet-in-the-middle pairwise on small constraints



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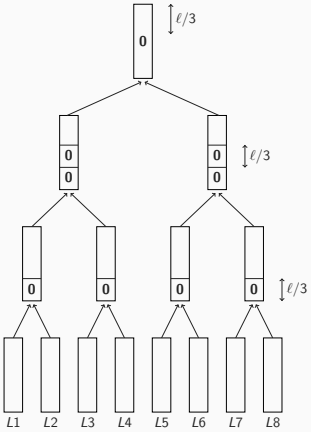

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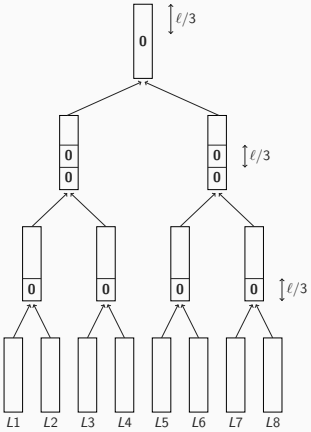

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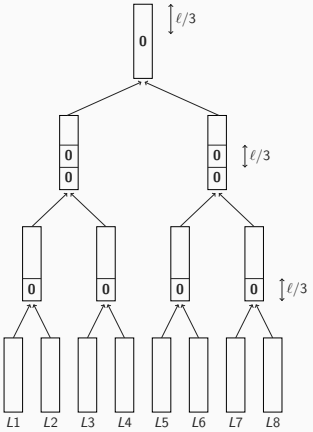

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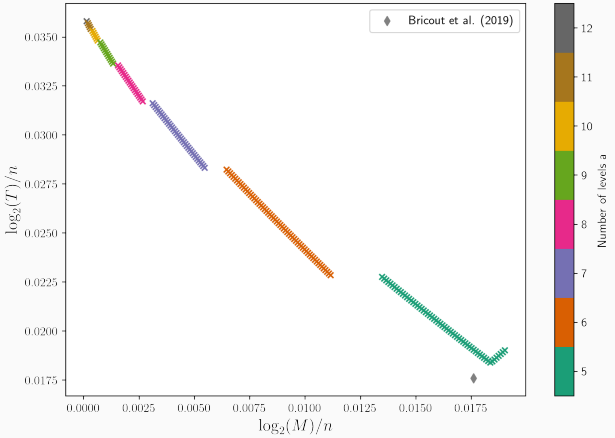
  $\Longrightarrow$ Want a as large as possible
- ⚠ Constraint:

$$3^{\ell/a} < 2^{\frac{k+\ell}{2^a}}$$

Time-Memory tradeoffs using the k-tree algorithm



The smoothing technique

- The extended k-tree algorithm (Minder and Sinclair, 2011), adapted in ternary high-weight SDP as smoothing technique (Bricout et al., 2019)
- Relax the constraint: enables to add one level

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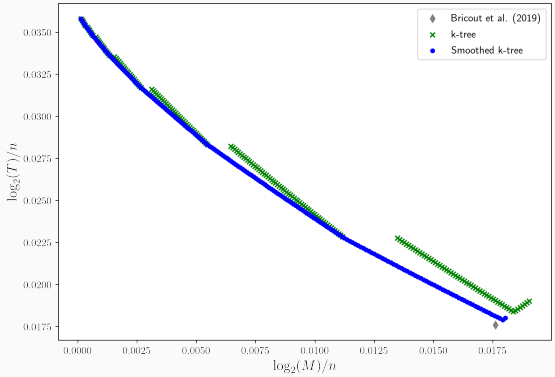


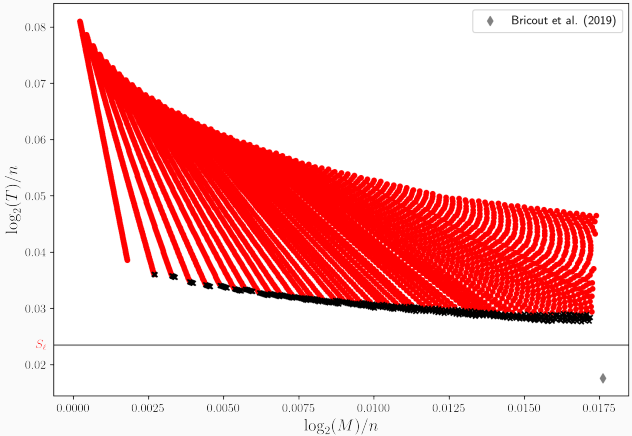
The dissection framework

Dissection features

- The dissection framework (Dinur et al., 2012) is a generalization of the meet-in-the-middle
- Memory-friendly family of exhaustive algorithms
- Proposes Time-Memory tradeoffs with initial list size 3^m
- Splits the r-list problem in an asymmetric way

r-dissection TM tradeoffs, $r < 400$

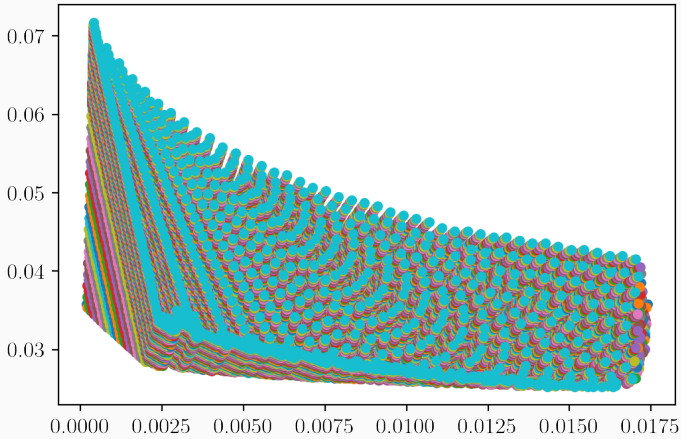
Fixed ℓ parameter ($\ell = 0.04n$)



Black dot = too many returned solutions $\implies$ granularity problems

Dissection: obtain the best tradeoffs

The best tradeoffs are provided by a range of ℓ parameters



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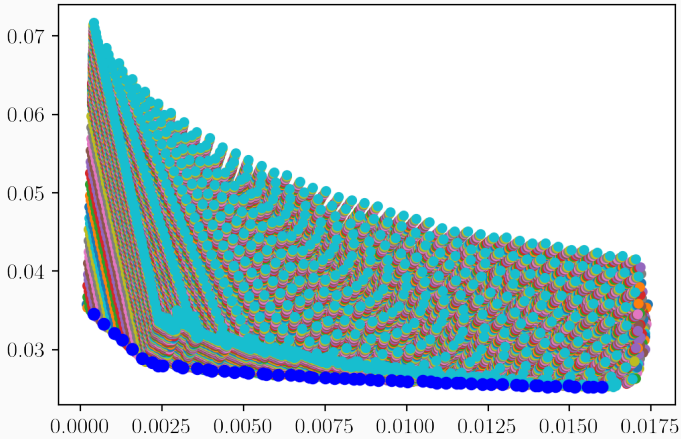

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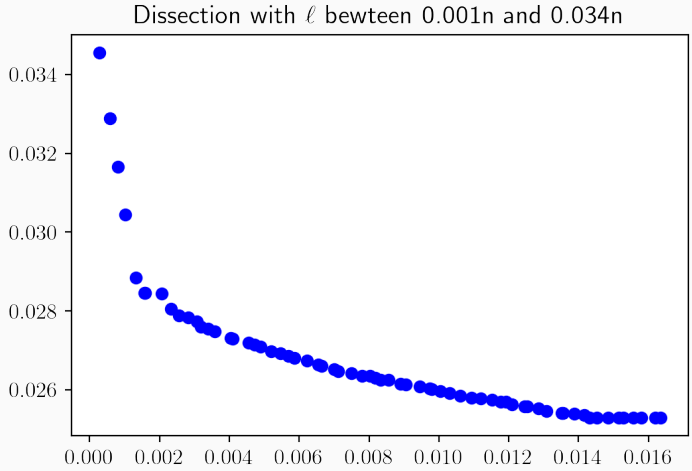
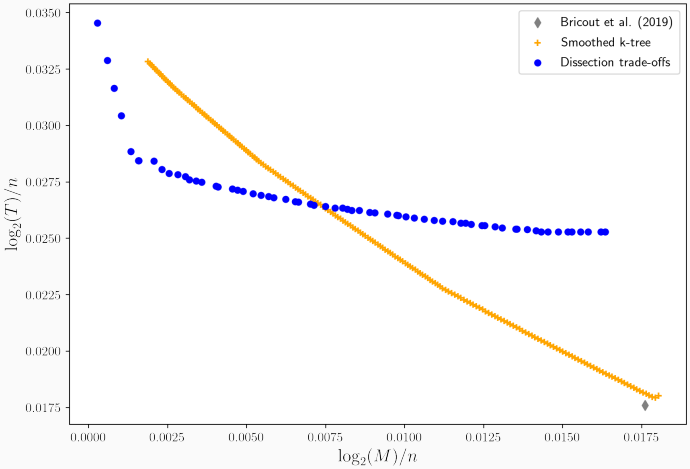


The dissection is better than the k-tree for small memory

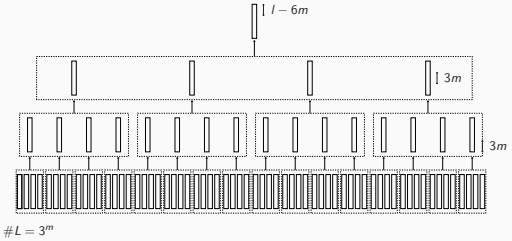


Dissection in tree

Principle

Combine dissection in a tree structure (Dinur, 2019)

Example: 4-dissection with 3 levels ($X = 4, h = 3$)



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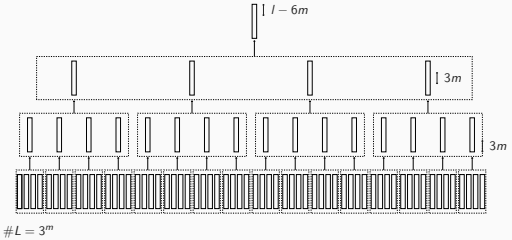
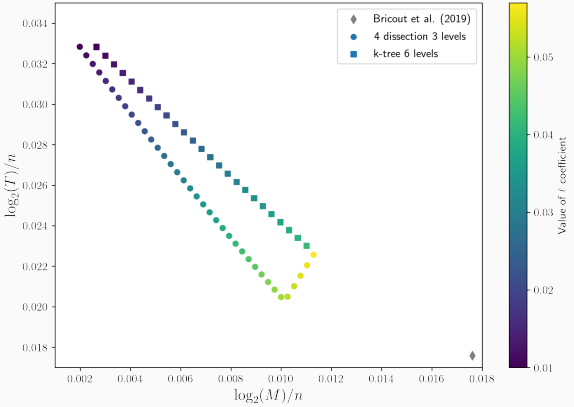


- Each dissection done is exhaustive
- Solutions are returned in constant amortized time
- ⚠ Again same constraint:

$$3^m < 2^{\frac{k+\ell}{x^h}}$$

4-dissection with three levels versus k-tree

Both have the same number of lists, but different merging strategies



- Dissection in tree needs less memory $\implies$ it is applicable for more ℓ parameters

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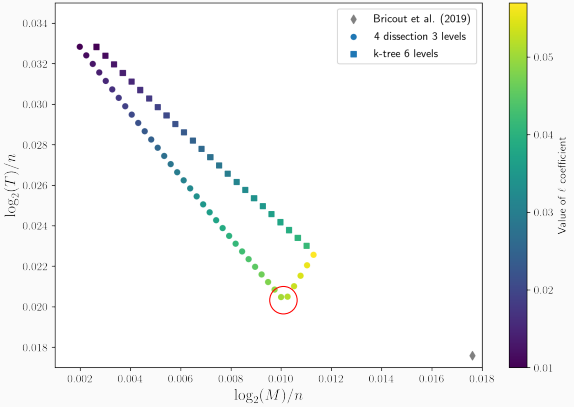
Both have the same number of lists, but different merging strategies



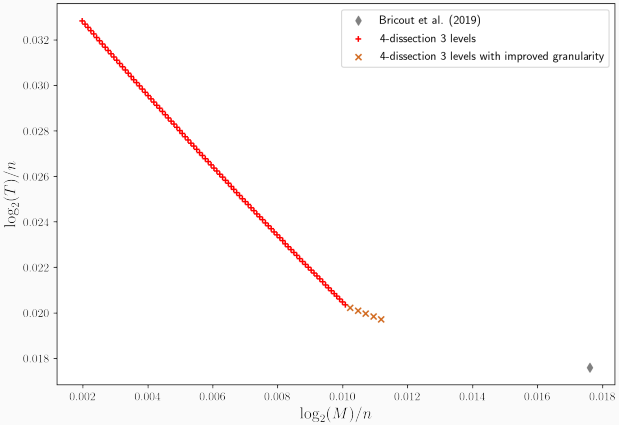
- Dissection in tree needs less memory $\implies$ it is applicable for more ℓ parameters
- The curve rebounds $\Rightarrow$ granularity problems

Granularity improvement

Idea: be not exhaustive in the dissection

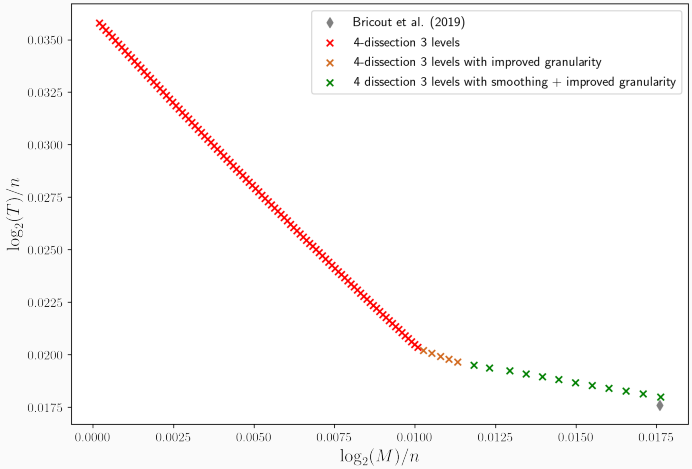
$\Longrightarrow$ solutions are more quickly decimated

$\Longrightarrow$ memory efficiency decreases



Combination of smoothing and granularity improvement

When the previous improvement can not be applied because of the constraint

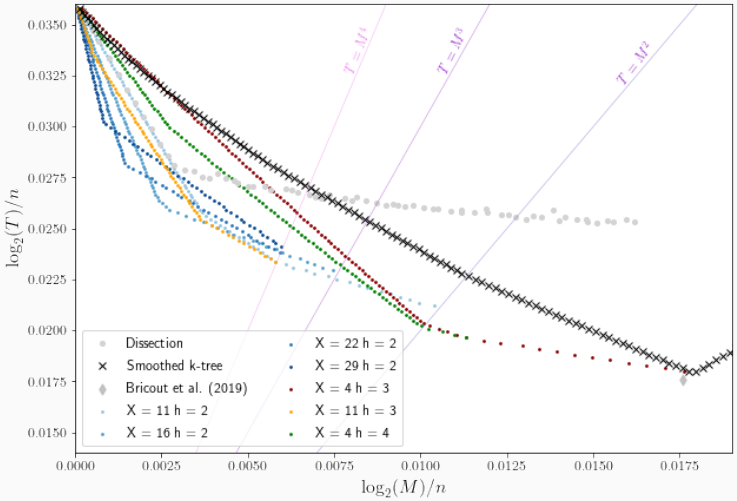


Results

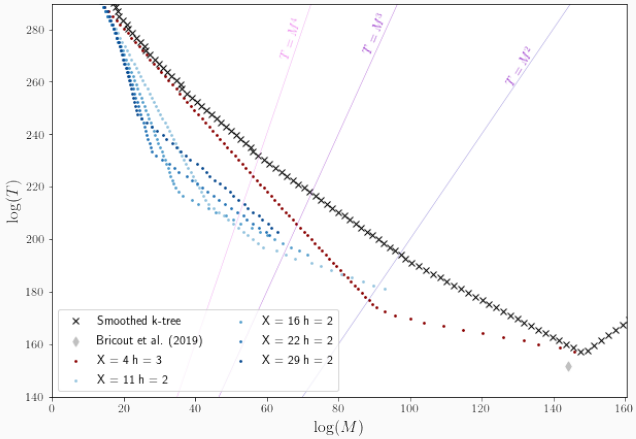
Numerical results

Time	Memory	Time $\times$ Memory	Tradeoff	Algorithm
$2^{0.0176n}$	$2^{0.0176n}$	$2^{0.0352n}$	$T = M$	k-tree + representations (Bricout et al.)
$2^{0.02014n}$	$2^{0.01007n}$	$2^{0.03021n}$	$T = M^2$	4,4-dissection
$2^{0.02256n}$	$2^{0.007521n}$	$2^{0.03008n}$	$T = M^3$	2,11-dissection
$2^{0.02335n}$	$2^{0.005838n}$	$2^{0.02919n}$	$T = M^4$	3,11-dissection

Time/Memory plots



Time/Memory plot with Wave parameters



$n = 7236, k = 4892, w = 6862$

Conclusion

- We investigated Time-Memory tradeoffs for the ternary syndrome decoding problem in the Wave regime
- The studied algorithms are k-tree, dissection and dissection in tree, embedded within the PGE+SS framework
- Some tweaks were necessary to decrease the *granularity* of the building blocks

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Thank you for your attention!