Oblivious-Transfer Complexity of Noisy Coin-Toss via Secure Zero Communication Reductions

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[1]Stanford University, [2]Technion, [3]IIT Bombay

November 2022

Outline

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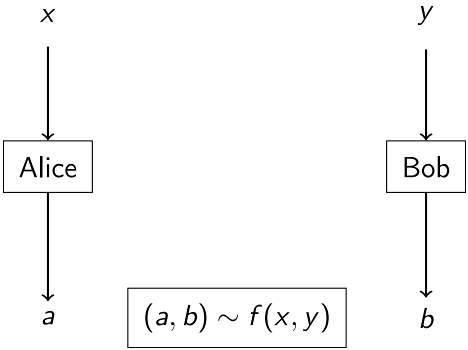
OT complexity



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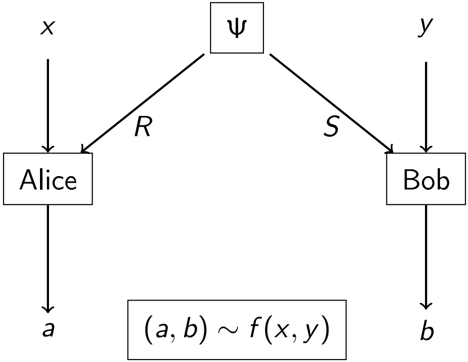

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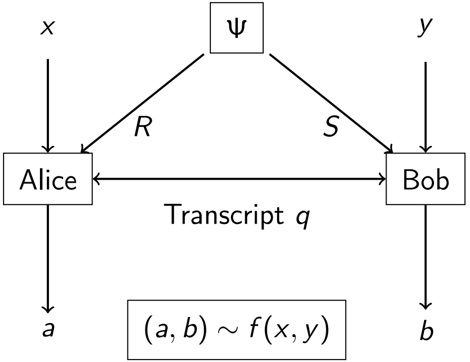

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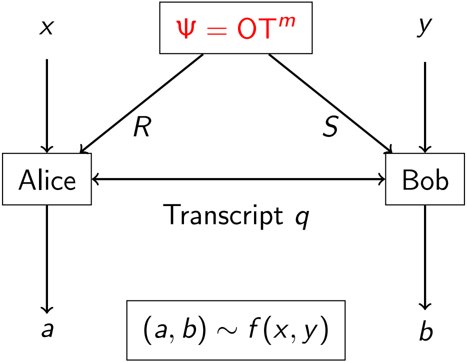

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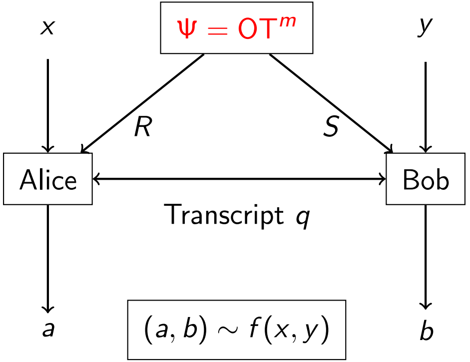

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  - For deterministic functions

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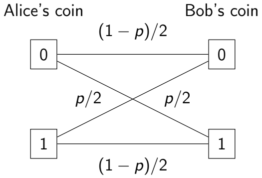
Yes!

Noisy coin-tossing

p-noisy coin toss is defined as follows:

Noisy coin-tossing



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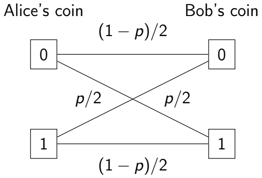

p-noisy coin toss is defined as follows:

- Alice and Bob get a uniform bit each
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- Circuit complexity of sampling $f_{p\text{-coin}}$ is $\Theta(\log(1/p))$ (need that many bits as randomness)
- By basic GMW protocol [GMW87] $|f_{p\text{-coin}}|_{\mathsf{OT}} = O(\log 1/p)$
- Can we securely compute $f_{p\text{-coin}}$ with fewer OTs?

Contributions

- Main Result: $|f_{p\text{-coin}}|_{\mathsf{OT}} = \Theta(\log 1/p)$

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$$|f|_{\mathsf{emb}} \leq |f|_{\mathsf{szcr}} \lesssim |f|_{\mathsf{OT}}$$

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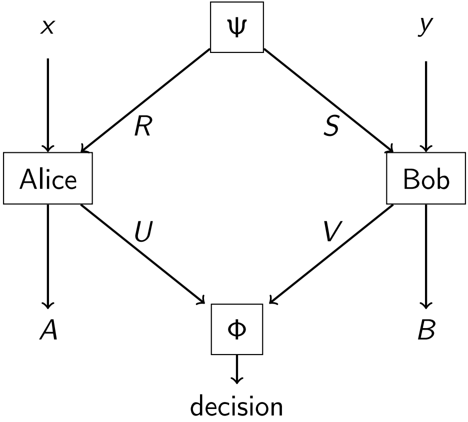
SZCR



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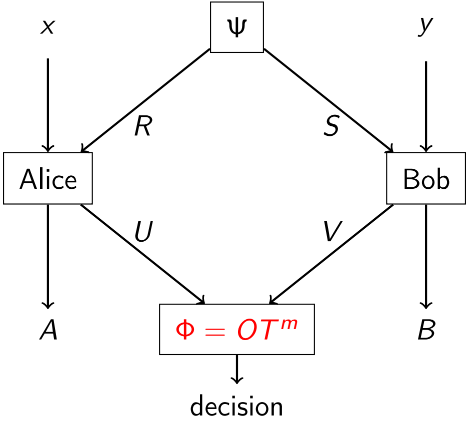

$|f|_{\text{szcr}}$ = minimum number of OTs needed by an SZCR of f

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SZCR and OT complexity

Using our construction for SZCR given an MPC protocol, we will show:

$$|f|_{\text{szcr}} \lesssim |f|_{\text{OT}}$$

Same result was shown in [NPP20] for deterministic f

Deterministic functions [NPP20]

Given an MPC protocol using m OTs for computing f, an SZCR for f using OT^m:

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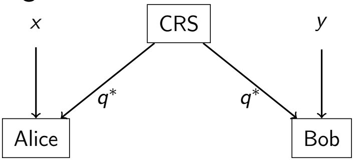

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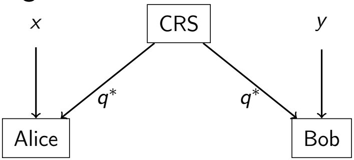

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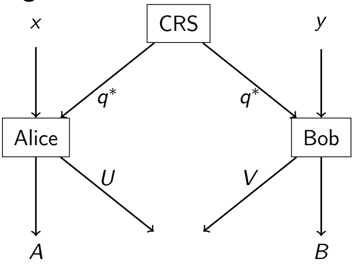

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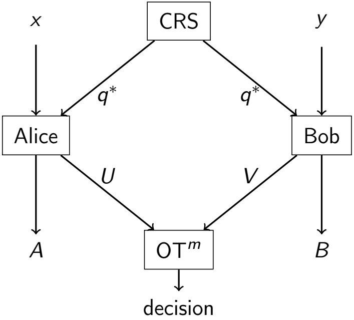

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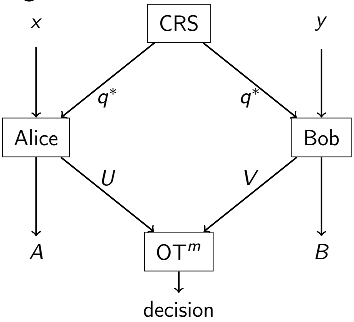

1. Sample output
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Remarks on the construction:

- Comes with a lower bound on the acceptance probability
- Generalizes to correlations beyond OT
- Can avoid the need for common randomness

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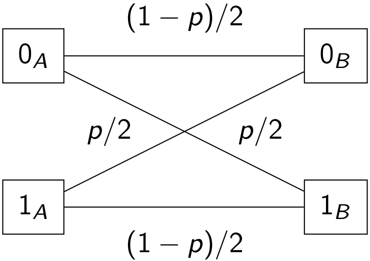
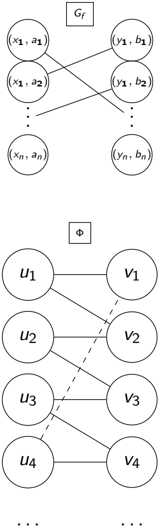
Evaluation Graph

For a function $f : X \times Y \to A \times B$, we define the evaluation graph G_f as:

- Bipartite graph on $(X \times A), (Y \times B)$
- Weight of edge $((x, a), (y, b)) = Pr[f(x, y) = (a, b)]$



SZCR as a graph embedding



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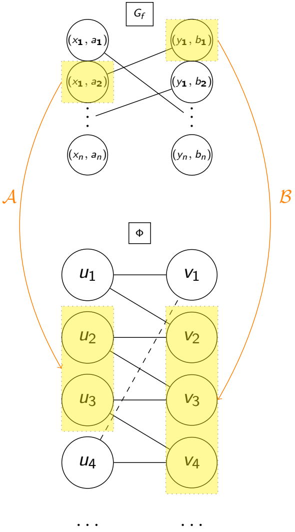

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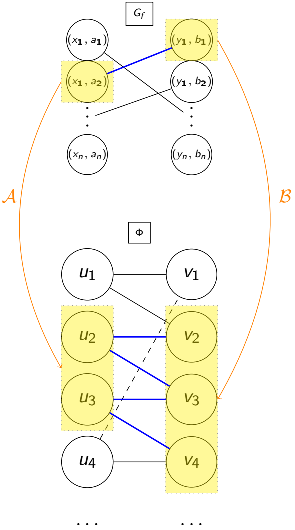

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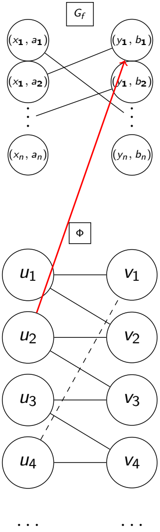

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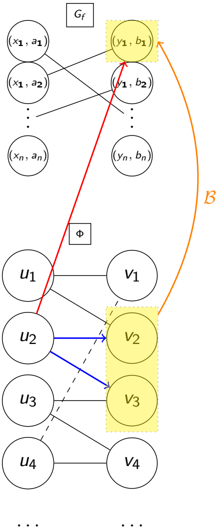

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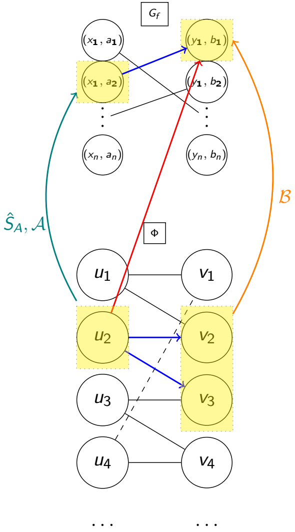

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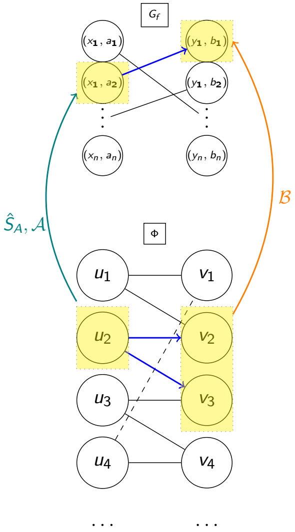

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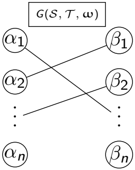
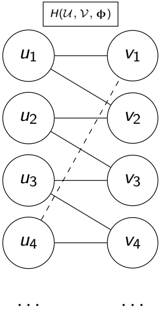
For security, we need both to be equivalent

Simplifying to a Balanced Embedding





Balanced embedding of weighted bipartite graph $G(\mathcal{S}, \mathcal{T}, \omega)$ into $H(\mathcal{U}, \mathcal{V}, \phi)$ where weight function ω is non-negative:

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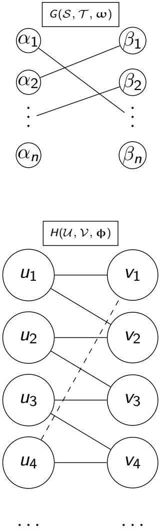

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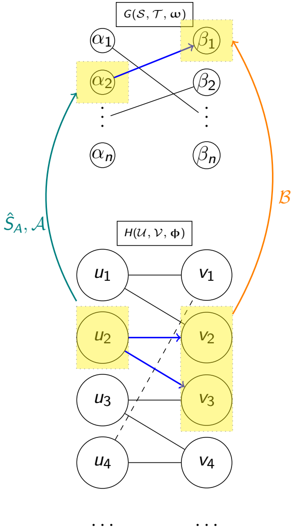

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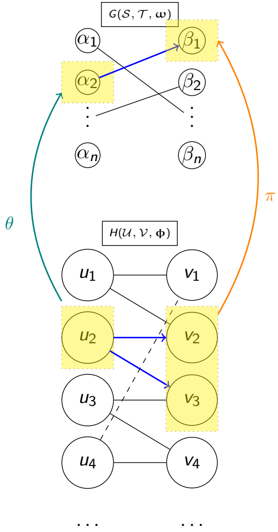

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Balancing (similar to security):

$$\sum_{v \in \mathcal{V}} \pi(v, \beta) \cdot \phi(u, v) = \theta(u, \alpha) \cdot \omega(\alpha, \beta) \forall u \in \mathcal{U}$$

Balanced Embedding complexity

Balanced Embedding

For $\pi, \theta : (\mathcal{U} \times \mathcal{S}) \cup (\mathcal{V} \times \mathcal{T}) \to \mathbb{R}_{\geq 0}$, (π, θ) is a balanced embedding of $G(\mathcal{S}, \mathcal{T}, \boldsymbol{\omega})$ into $H(\mathcal{U}, \mathcal{V}, \boldsymbol{\phi})$ if for all $(\alpha, \beta) \in \mathcal{S} \times \mathcal{T}$:

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$$\sum_{u \in \mathcal{U}} \pi(u, \alpha) \cdot \theta(u, \alpha) = 1 \qquad \sum_{v \in \mathcal{V}} \pi(v, \beta) \cdot \theta(v, \beta) = 1 \qquad \text{if } \boldsymbol{\omega}(\alpha, \beta) > 0$$

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Balanced Embedding Complexity

The *balanced embedding complexity* of f, $|f|_{\mathrm{emb}}$ is the smallest m such that G_f has a balanced embedding into $H_{\boldsymbol{\phi}^m_{\mathrm{OT}}}$. By construction, we conclude: $|f|_{\mathrm{emb}} \leq |f|_{\mathrm{szcr}}$

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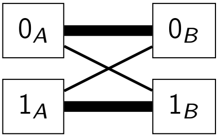
Bounding Embedding Complexity



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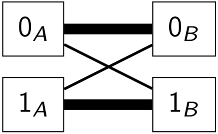

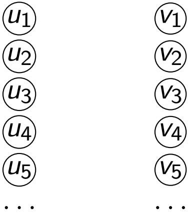

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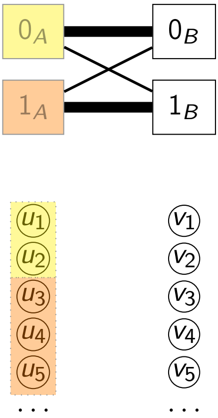

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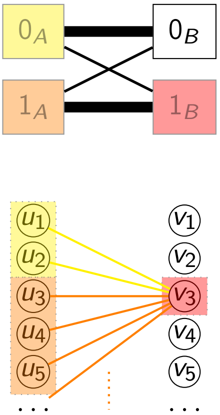

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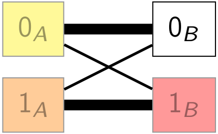

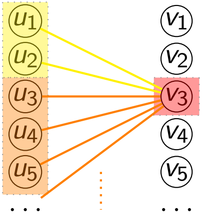

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- We find that $2^m \geq \frac{1-p}{p}$

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Conclusion

- We showed:
  - $|f|_{\mathsf{emb}} \leq |f|_{\mathsf{szcr}} \lesssim |f|_{\mathsf{OT}}$ for all (possibly randomised) f
  - $|f_{p\text{-coin}}|_{\mathsf{emb}} = \omega(\log 1/p)$

Hence, $|f_{p\text{-coin}}|_{\mathsf{OT}} = \omega(\log 1/p)$

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Hence, $|f_{p\text{-coin}}|_{\mathsf{OT}} = \omega(\log 1/p)$

Open: Super-linear balanced-embedding complexity for deterministic functions

- For explicit functions, this faces circuit complexity barriers
- Immediate question: Do such functions **exist**?