ON SECRET SHARING, RANDOMNESS, AND RANDOM-LESS REDUCTIONS

Divesh Aggarwal Eldon Chung Maciej Obremski Joao Ribeiro

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Lots of applications

- BPP algorithms
- Cryptographic constructions
- Distributed and Network Protocols
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But can we generate uniform randomness?

IDEAL WORLD

$Pr[X_i = 0] = Pr[X_i = 1] = ½$

$Pr[X = x] = 1/2^n$

MINIMAL REALISTIC ASSUMPTION

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- "Hard" to guess X
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How can we use such a source for crypto applications?

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Can we still build crypto primitives from min-entropy sources?

BUILDING CRYPTO PRIMITIVES FROM WEAK SOURCES

[DOPS04] showed that the following crypto tasks are impossible from block sources (e.g. Santha Vazirani) source.

- Encryption
- Secret Sharing
- Zero knowledge
- Secure two party computation
- Bit commitment

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Are there sources in the middle good for crypto?

Crypto primitives can be built!

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ENCRYPTION $\Longrightarrow$ EXTRACTION [BD07]

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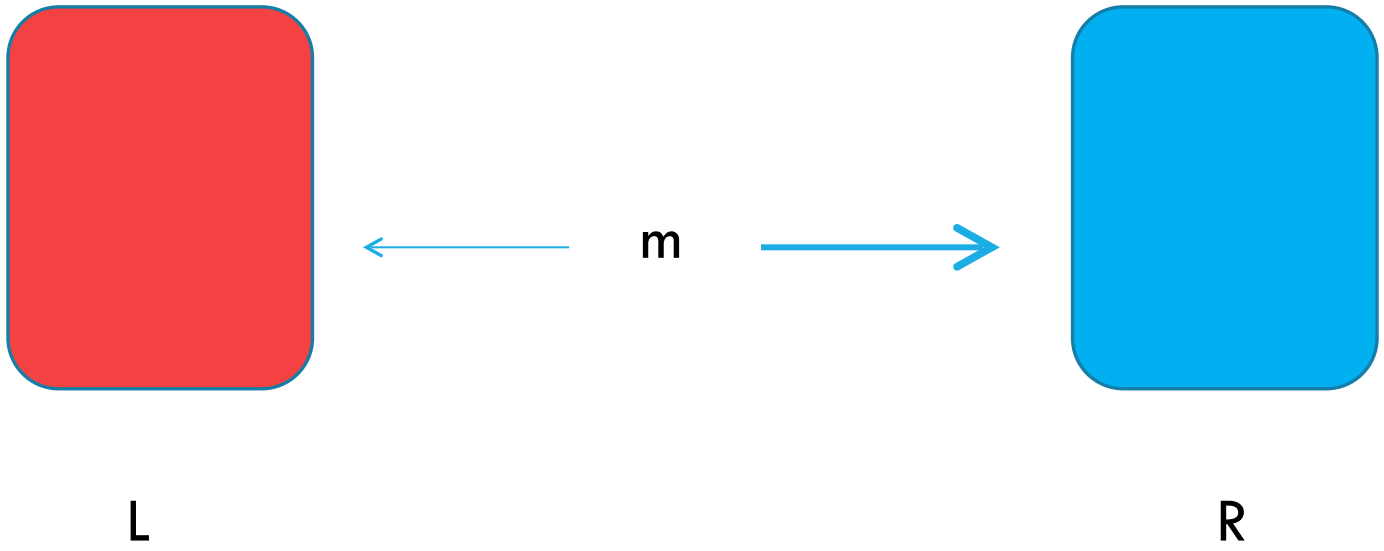
2-OUT-OF-2 SECRET SHARING



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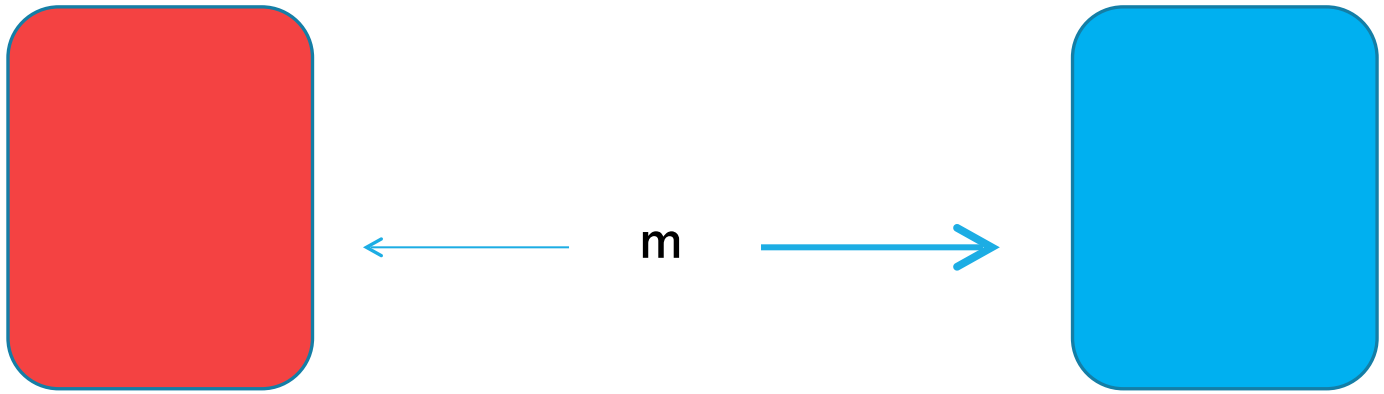
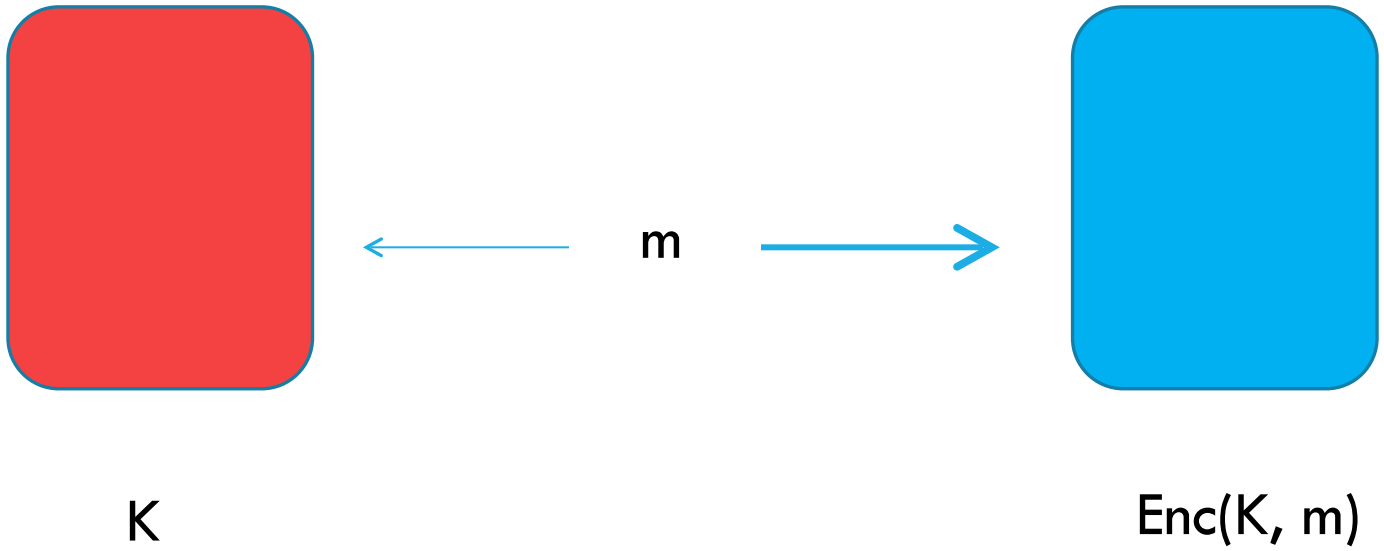


One of L, R does not reveal any information about m. Together, they can reconstruct m.

ENCRYPTION IS ALSO 2-OUT-OF-2 SECRET SHARING



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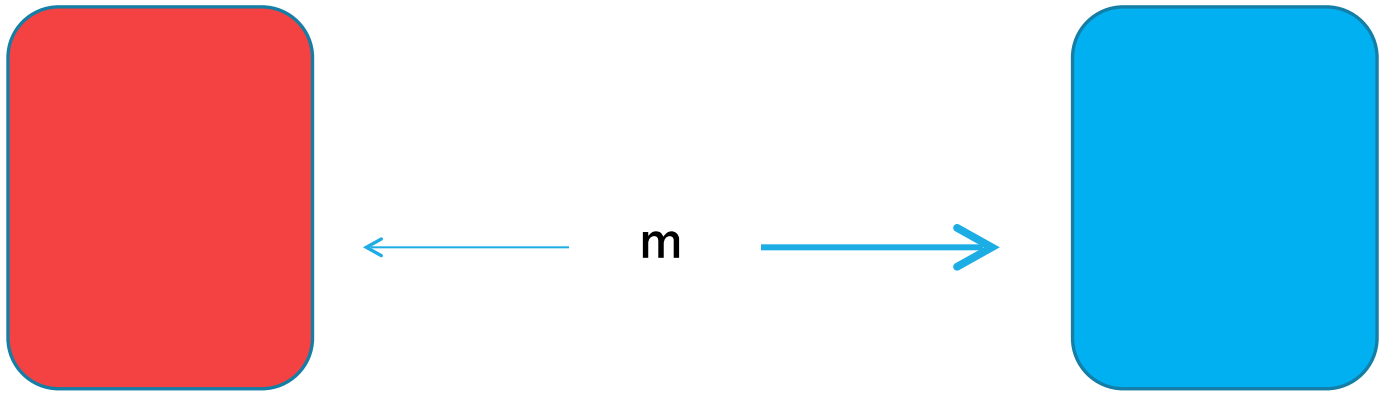
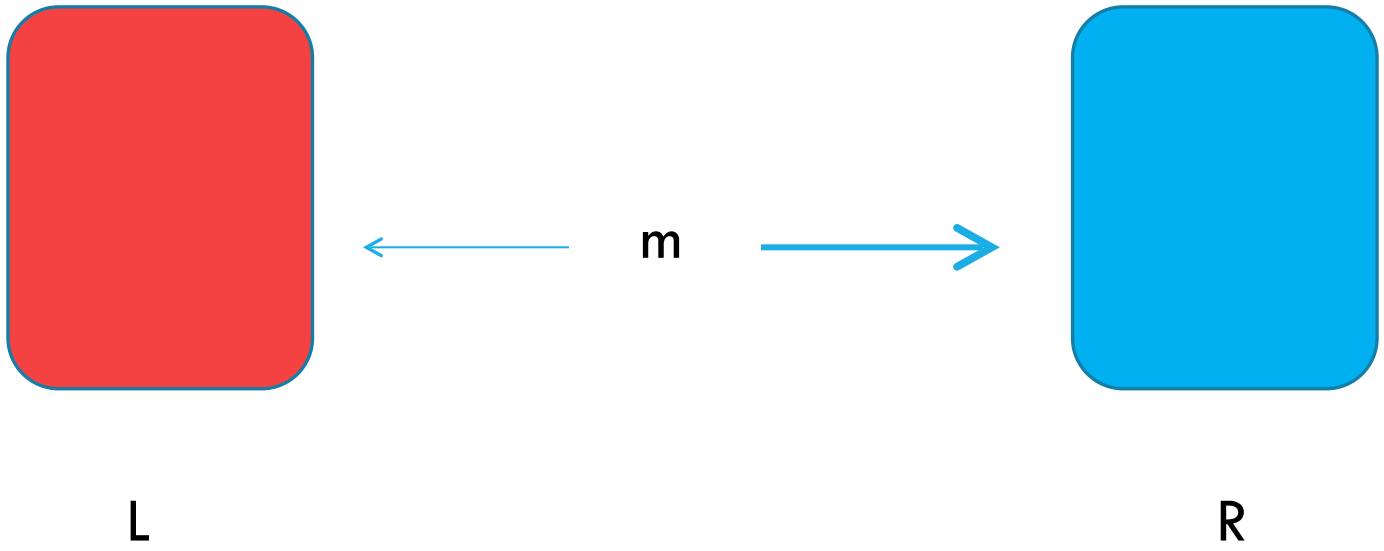


Cannot learn anything about m from any one of K, or Enc(K,m). Together, they determine m.

DOES 2-2 SECRET SHARING IMPLY EXTRACTION



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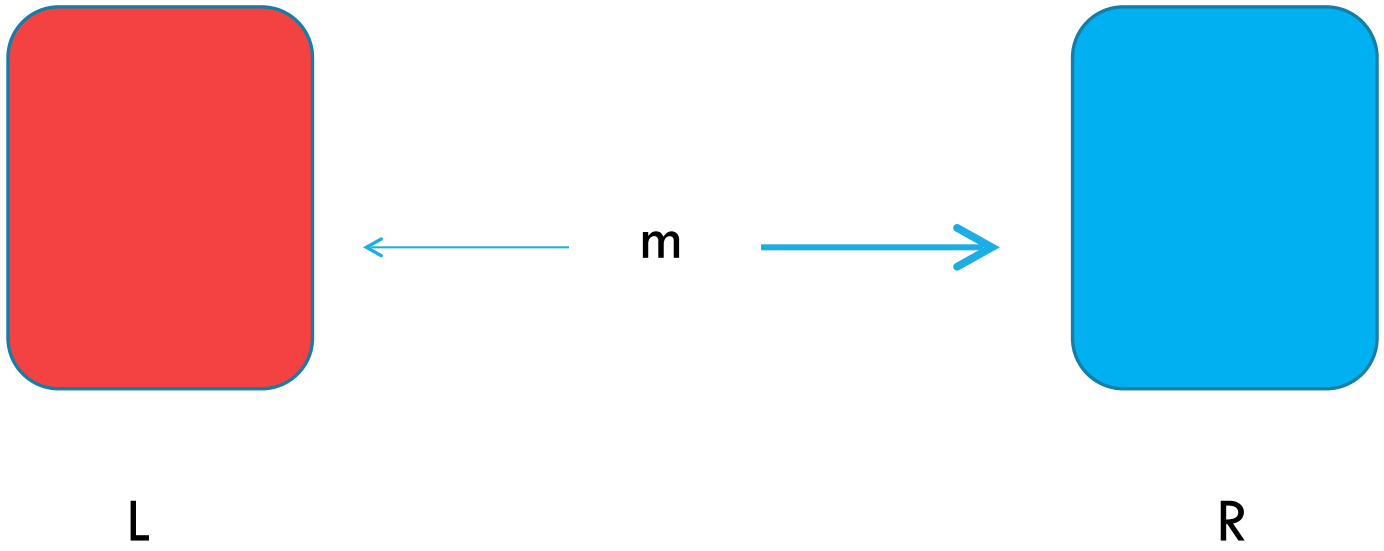

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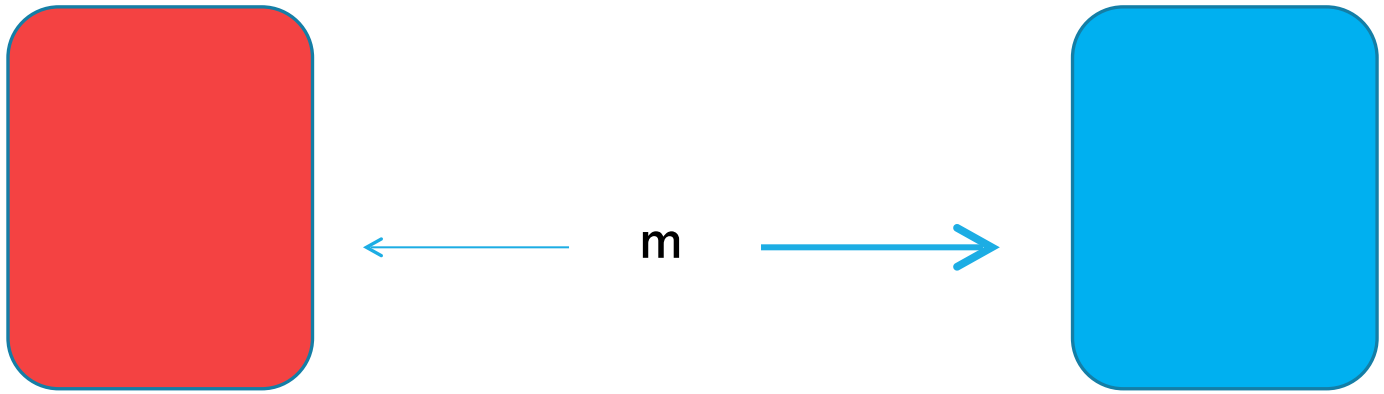

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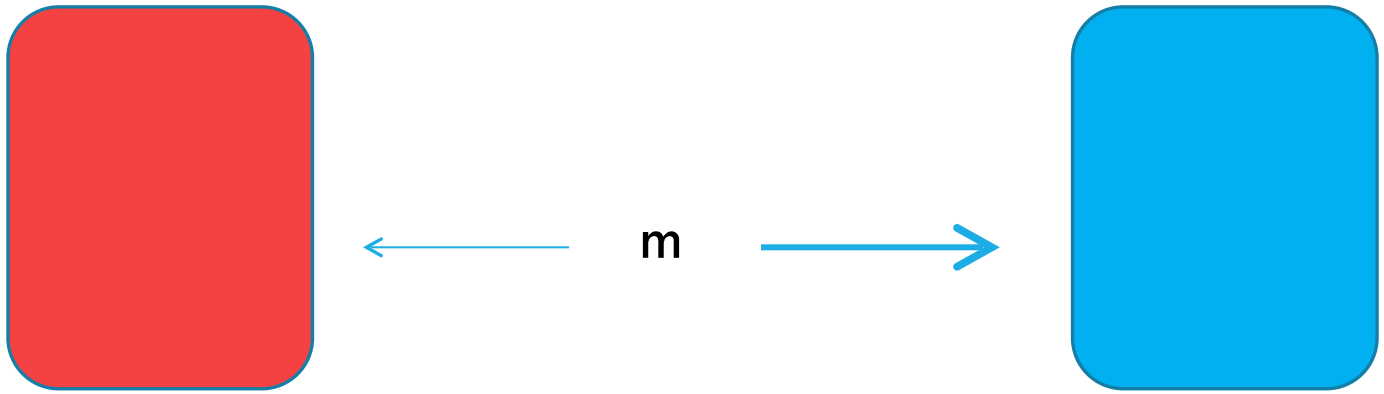

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Quite Hard!! [Me2022]

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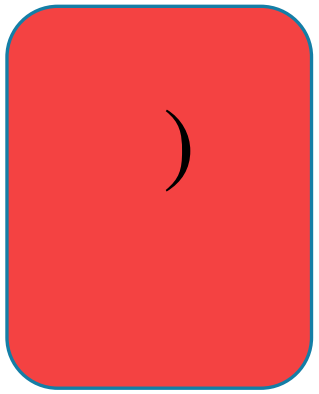
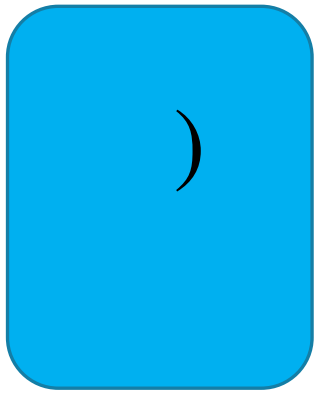
- LeftShare(K, U) is a convex combination of LeftShare(k, U), where key k is any fixed key in the key space.
- Unfortunately, LeftShare(k, U) might not have any entropy.

WE CONSIDER THIS QUESTION FOR LEAKAGE RESILIENT SECRET SHARING

LEAKAGES:

Think about a model where information from two parts may be obtained, but separately.

$f($  $)$ $g($  $)$

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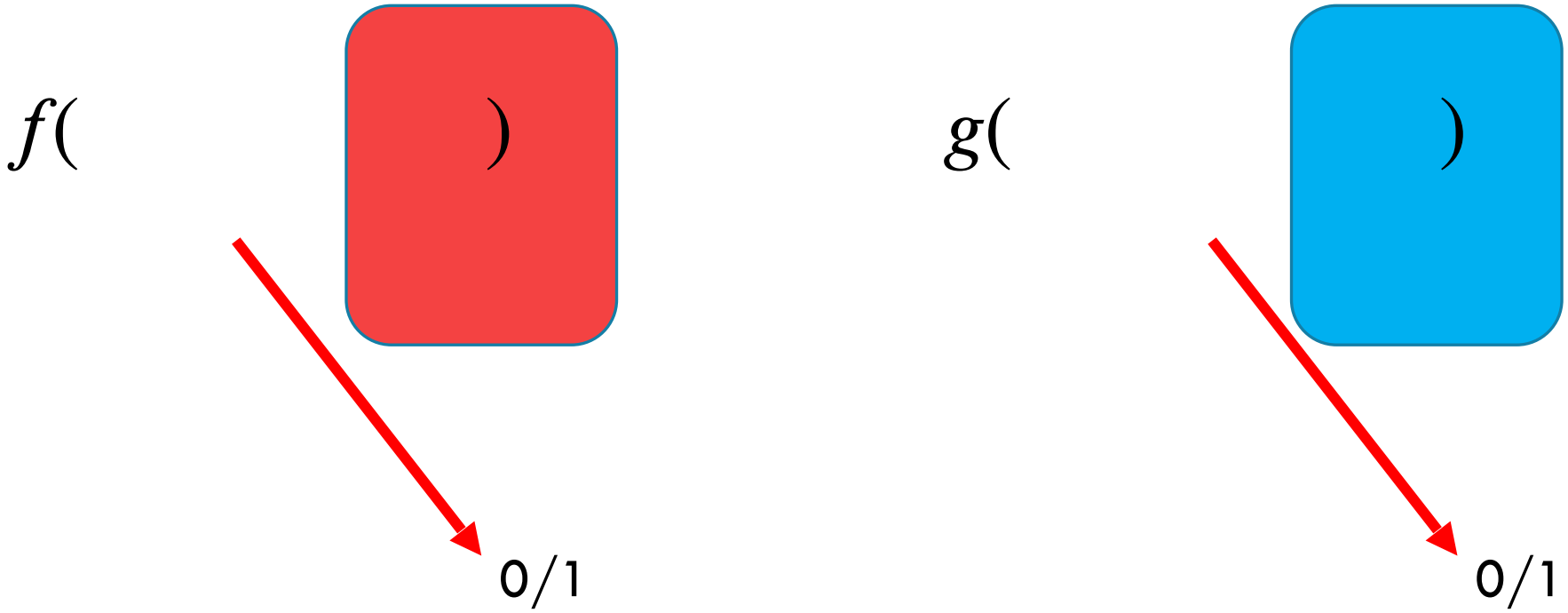

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- An NMC is also a leakage-resilient secret sharing scheme.
- Non-malleable codes in the split-state model imply extraction
- Contrast to the fact that AMD codes can be constructed from an entropy source.

PART 2: RANDOM-LESS REDUCTIONS

- Reducing crypto primitive A to B without additional randomness
    - Given a construction for A, we give a construction for B.

- If B requires an extractable source, we have that A also requires an extractable source.

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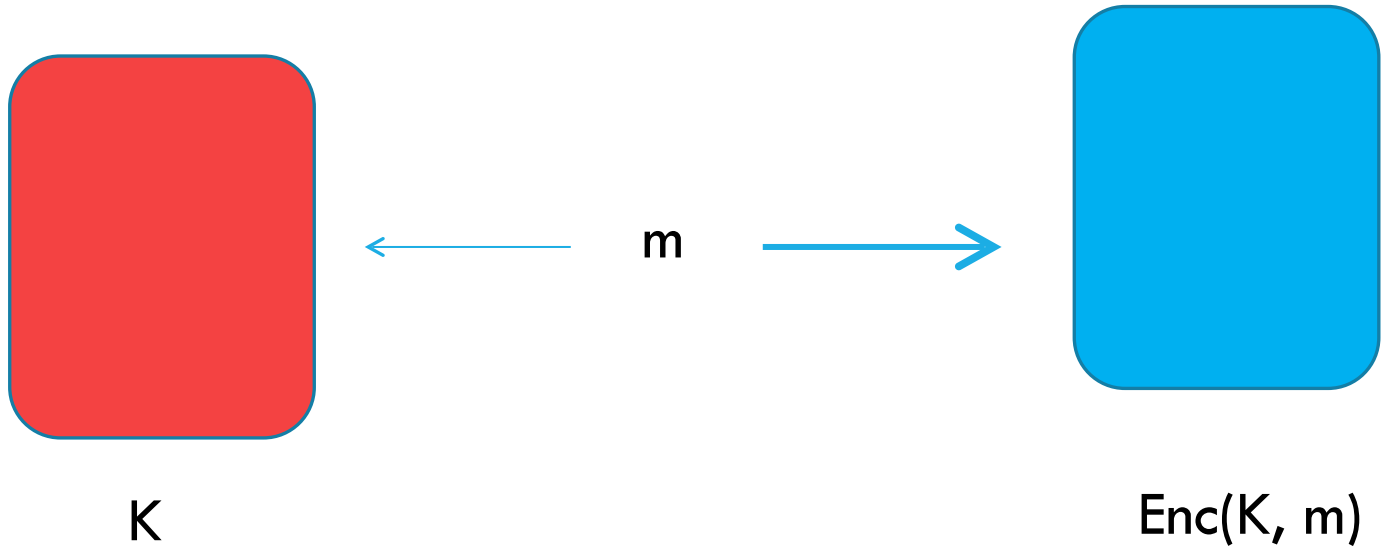

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Motivates distribution designs: Distribute n shares into n' sets such that any t sets contain t' shares, and less than t sets contain less than t' shares.

DISTRIBUTION DESIGNS

- We find distribution designs for several different choices of $n,\ t,\ n',\ t'$
- In some cases, these designs are tight and cannot be improved.

CONCLUSIONS

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Hopefully, our work motivates our main open question: Can we extract randomness from a t out of n secret sharing scheme for any t, n ?

THANK YOU!

Questions?