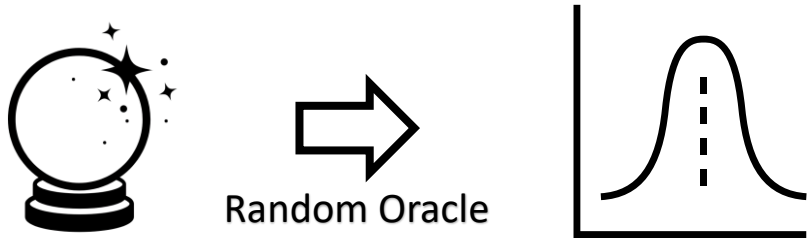
How to Sample a Discrete Gaussian (and more) from a Random Oracle

George Lu Brent Waters

Random Oracles to Structured Distributions



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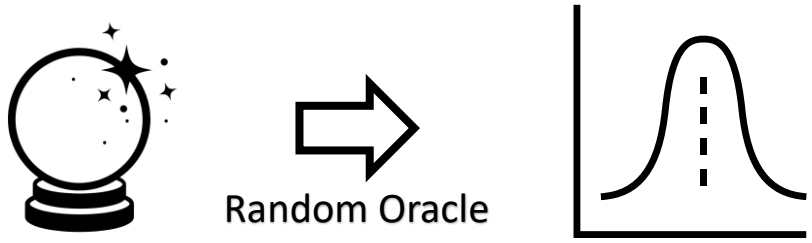

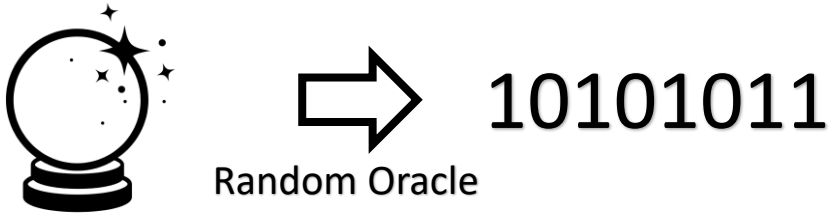

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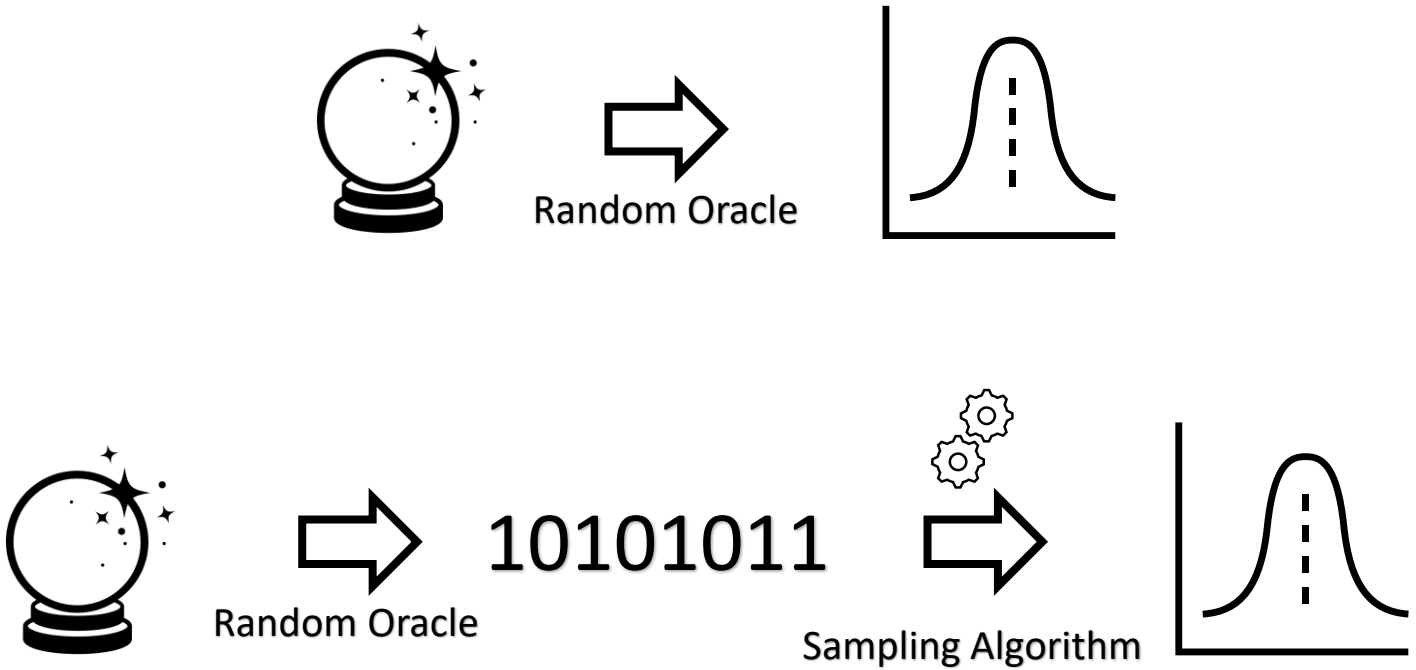

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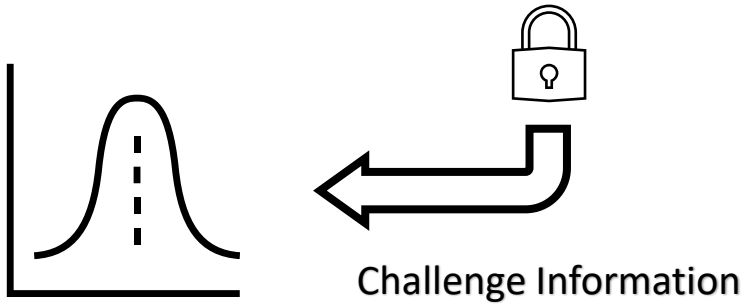

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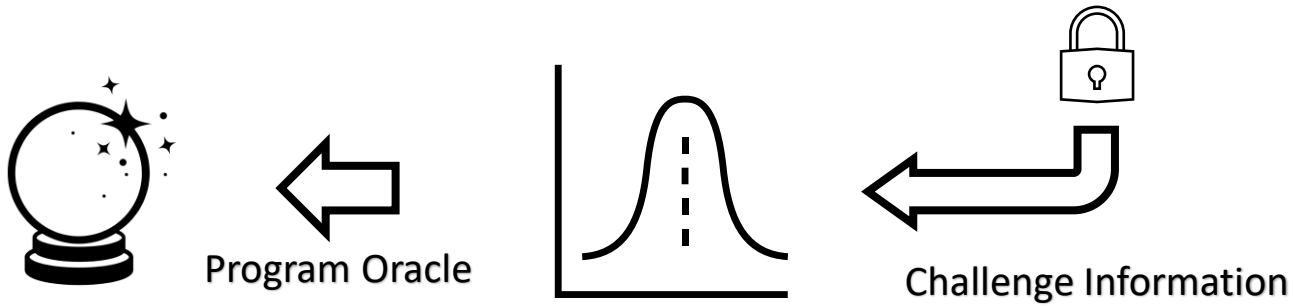

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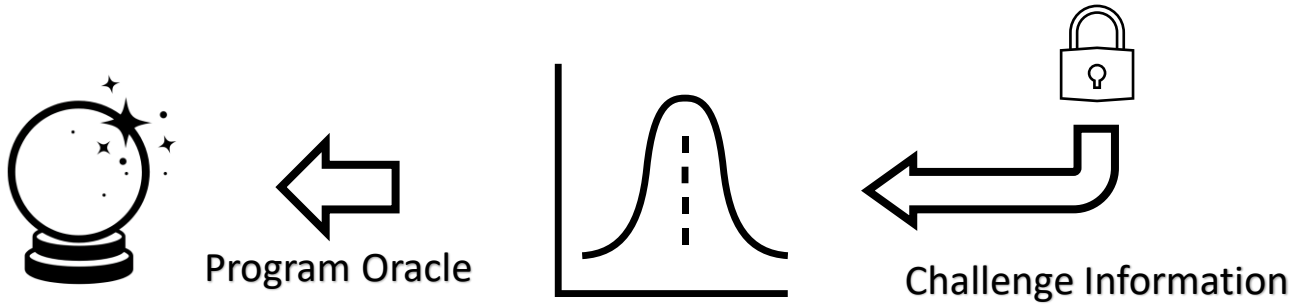

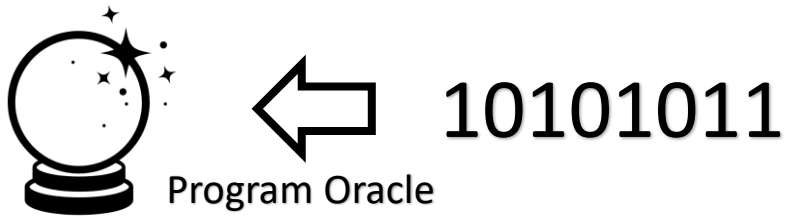

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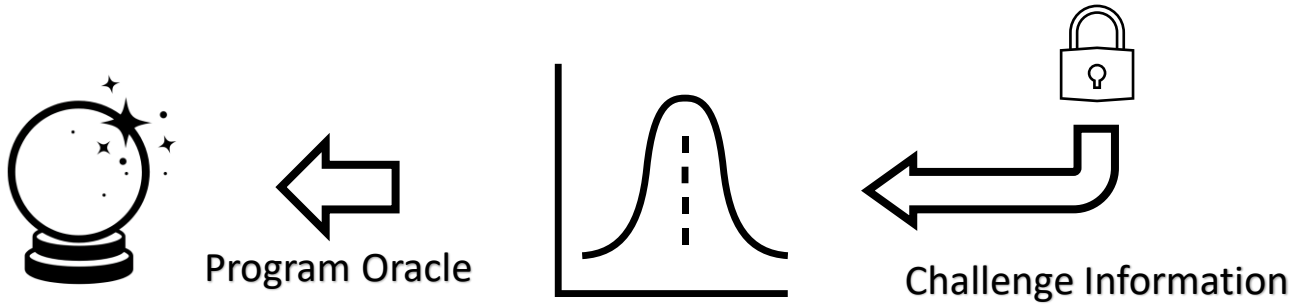

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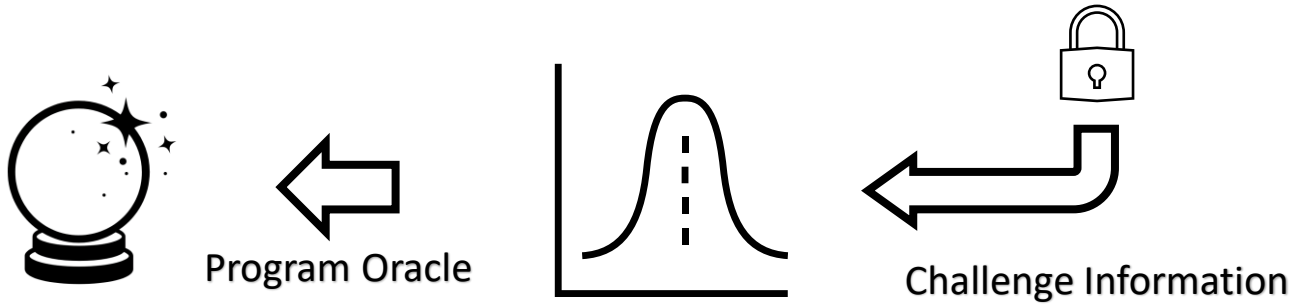

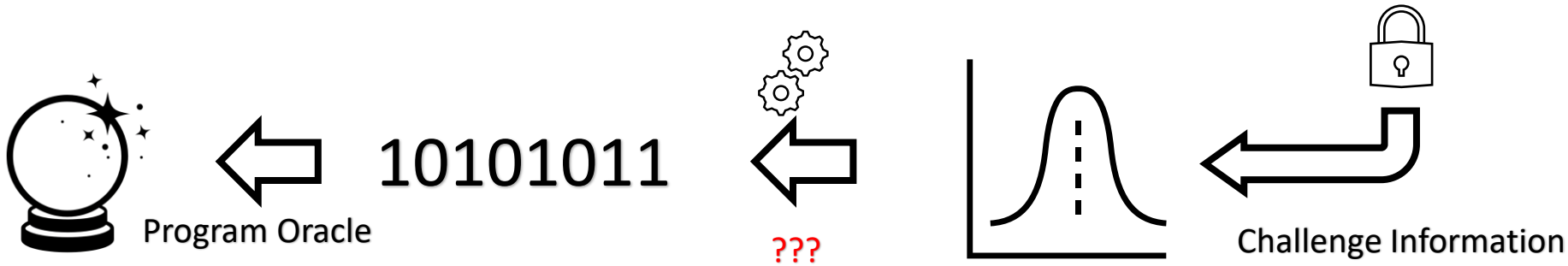

Prior Works

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- Random Oracles to Groups
  - RSA Full domain hash
  - Hash-to-Point on Elliptic Curves

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- Universal Samplers [HJKSWZ16]

Lattice-Based Random Oracle Schemes

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- Homomorphic ABE [BCTW16]
- Multi-Authority ABE [DKW21]
- Broadcast Encryption* [AWY20]

*uniform random public parameters

Lattice-Based Random Oracle Schemes

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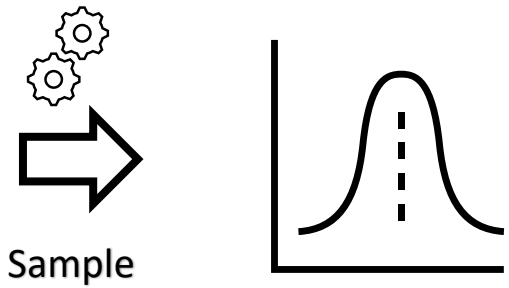
- Ad-hoc workarounds!

*uniform random public parameters

Explainability - Reverse Sampling

10101011



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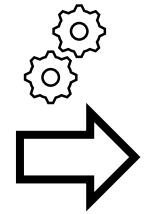
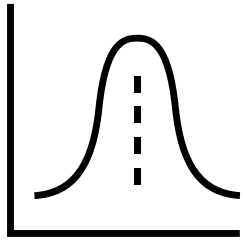


Sample



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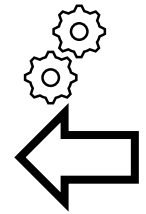
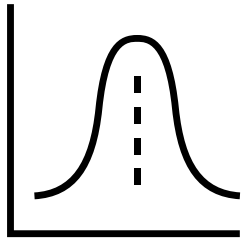


Explain



Explainability - Reverse Sampling

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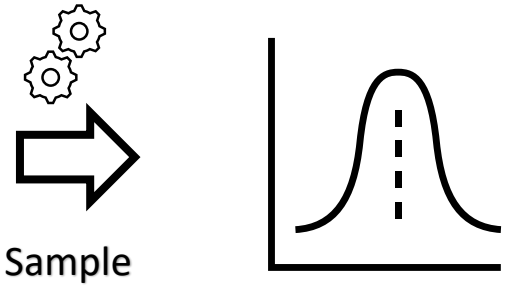

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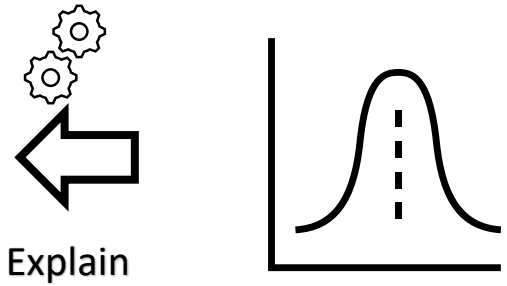
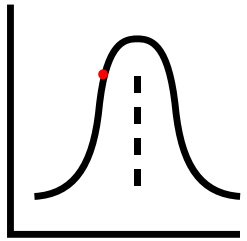


$$\text{Sample}(\text{Explain}(\text{y})) = y$$

Explainability - Reverse Sampling

10101011



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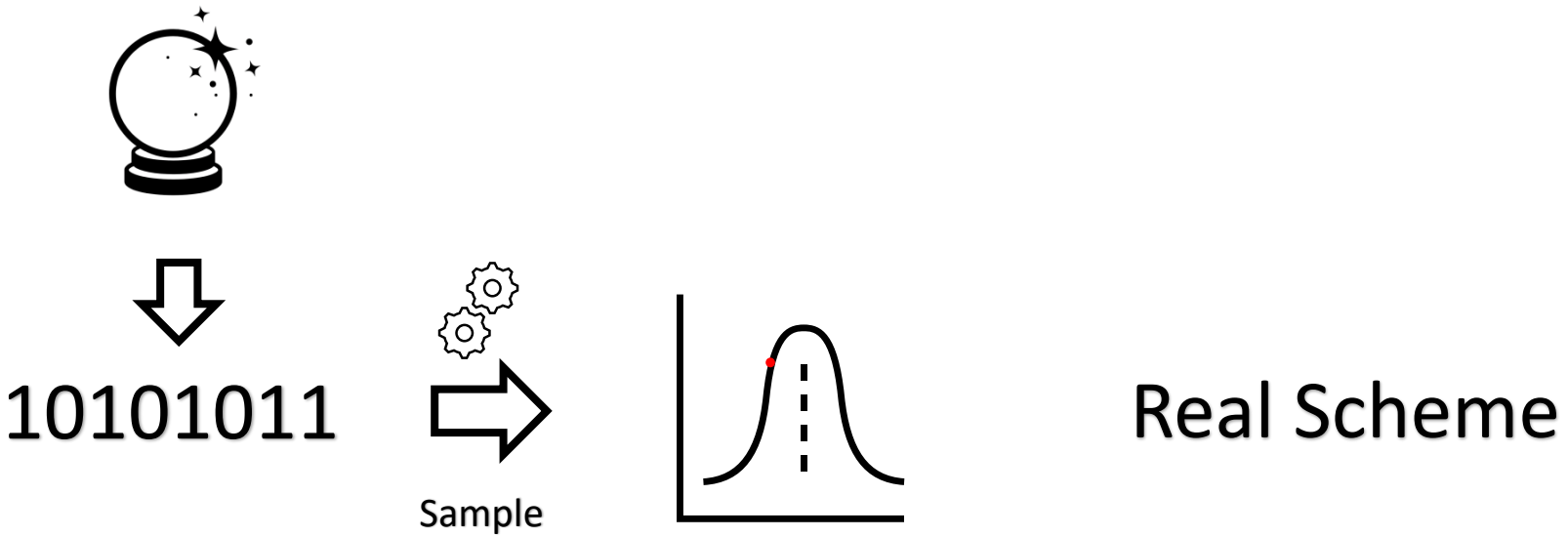
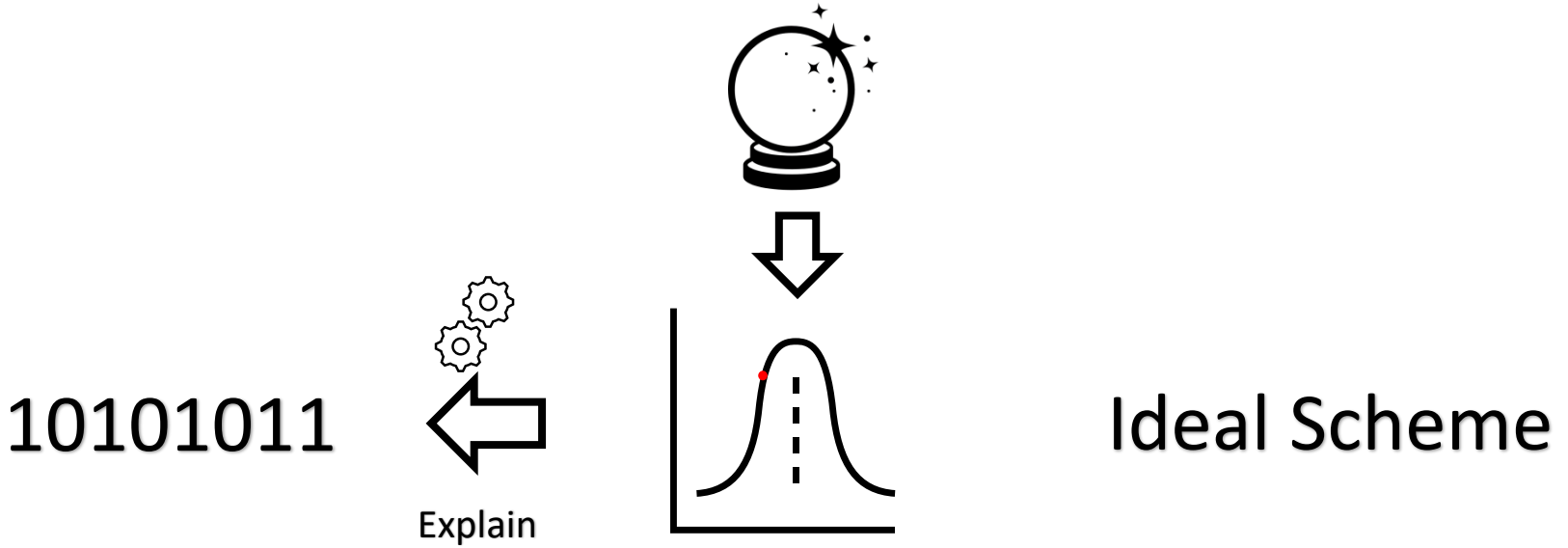


Explainability - Reverse Sampling



Explainability

Distribution 1

$$r \leftarrow \{0,1\}^n$$
$$x \leftarrow \mathsf{Sample}(r)$$

Distribution 2

$$r' \leftarrow \{0,1\}^n$$
$$x \leftarrow \mathsf{Sample}(r')$$
$$r \leftarrow \mathsf{Explain}(1^\kappa, x)$$

Explainability

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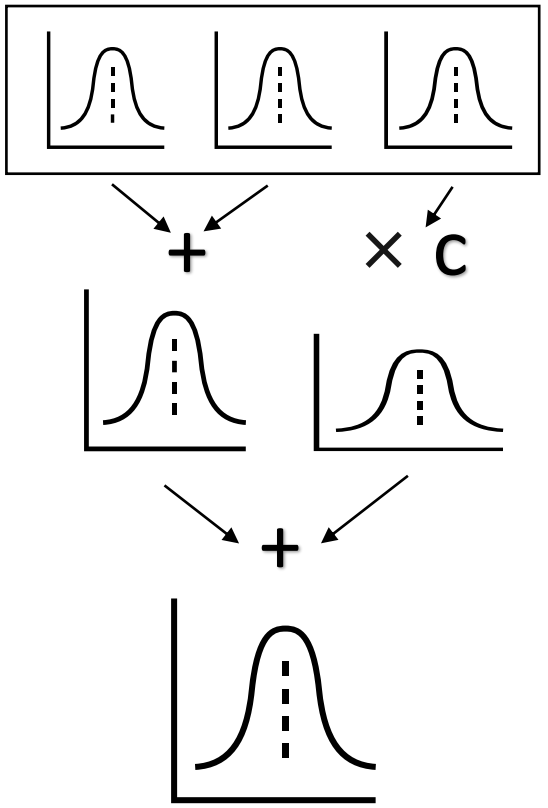
$$(r, x) \approx_{\frac{1}{\kappa}} (r, x)$$

Statistical Distance

Result 1

Explaining the Miccancio-Walter ’17 Discrete Gaussian Sampler

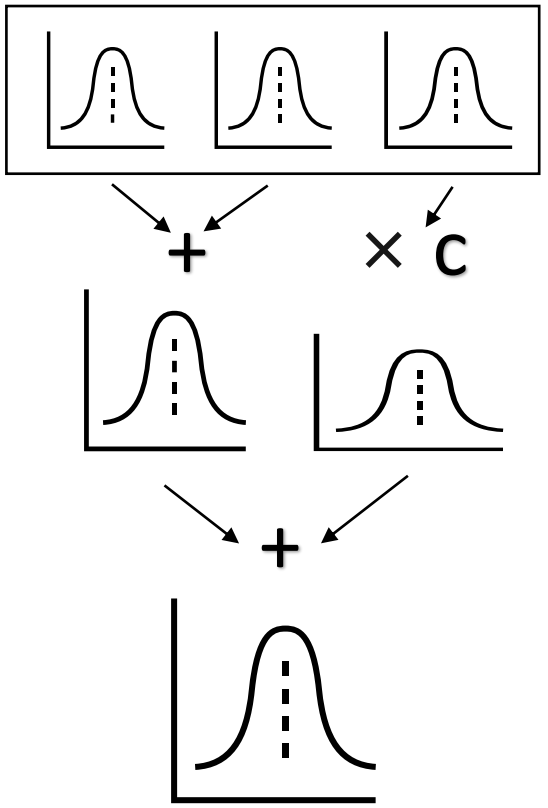
Miccancio-Walter ‘17 Sampler



Offline Phase – Generate ‘Small’ Discrete Gaussian Sample

Online Phase – Combine linear combinations of Small Samples based on μ, σ

Miccancio-Walter ‘17 Sampler



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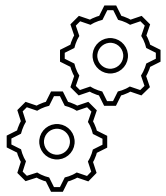
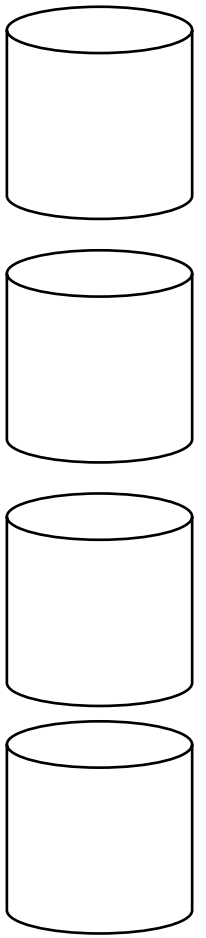
Explaining Polynomial Range Samplers

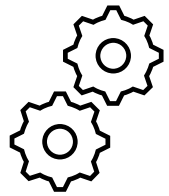
$$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(r)$$

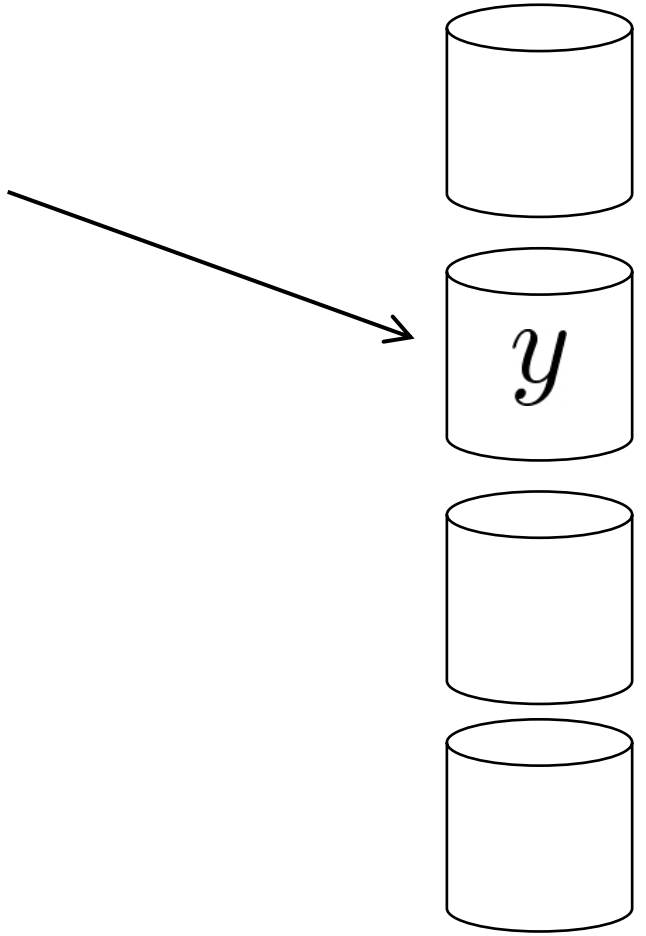




Explaining Polynomial Range Samplers

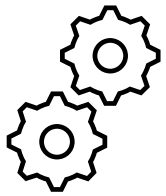
$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(r)$

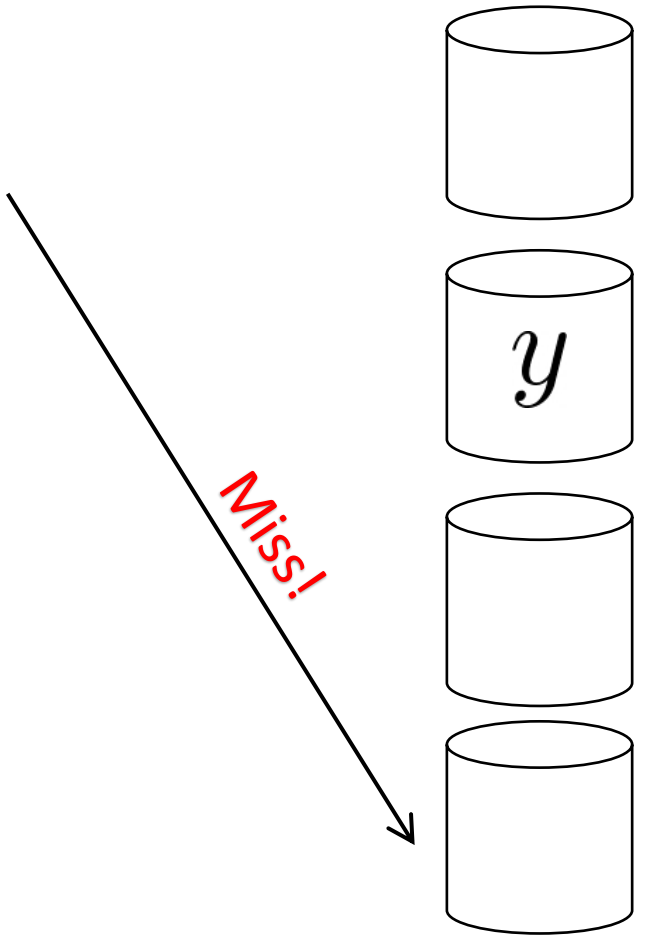




Explaining Polynomial Range Samplers

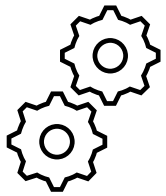
$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(r_0)$

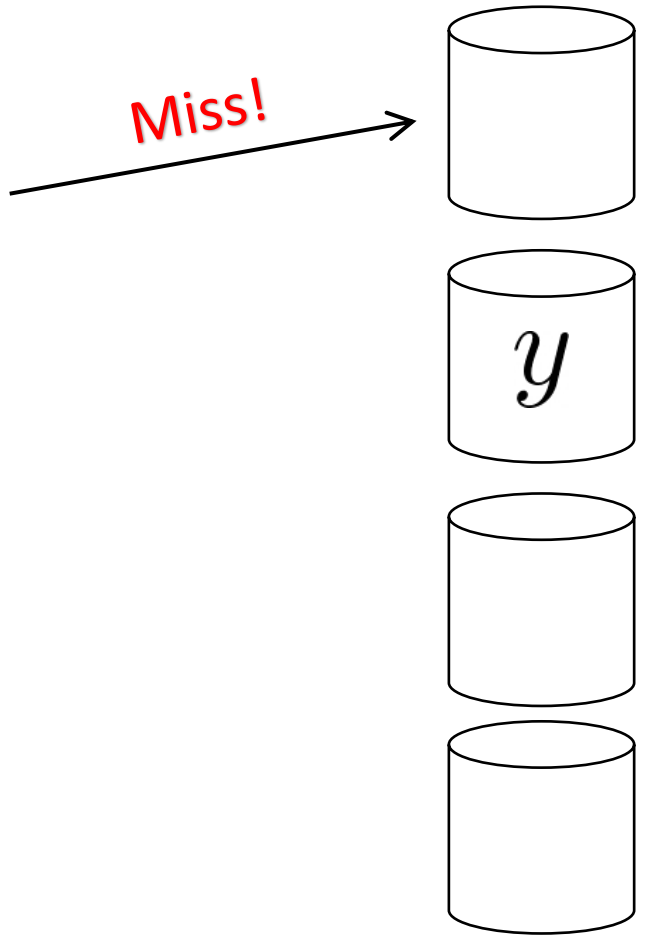




Explaining Polynomial Range Samplers

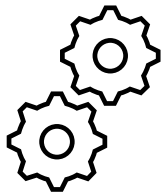
$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(r_1)$

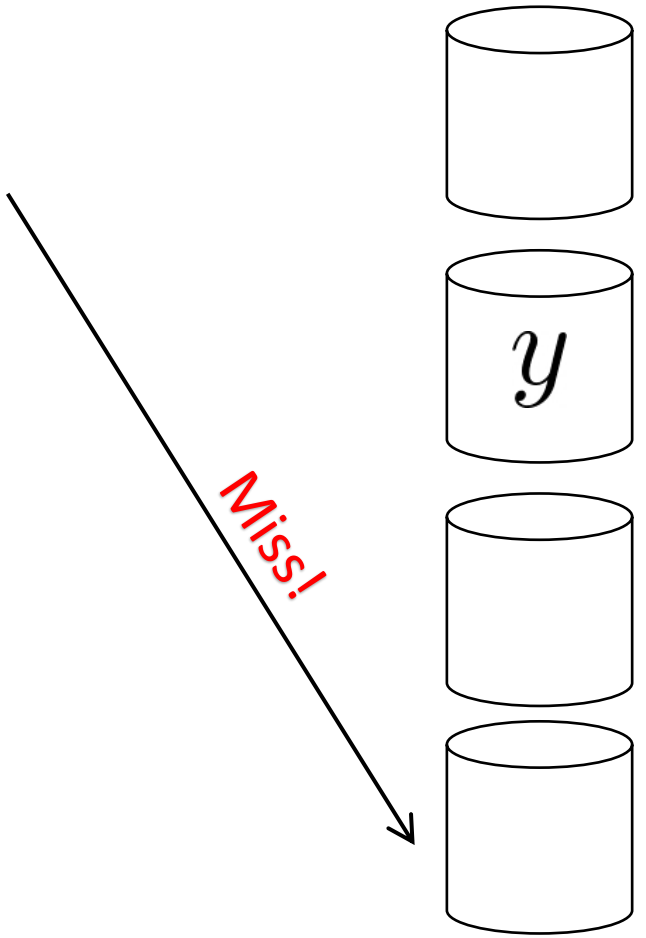




Explaining Polynomial Range Samplers

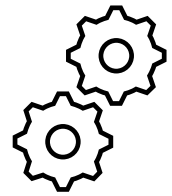
$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(r_2)$

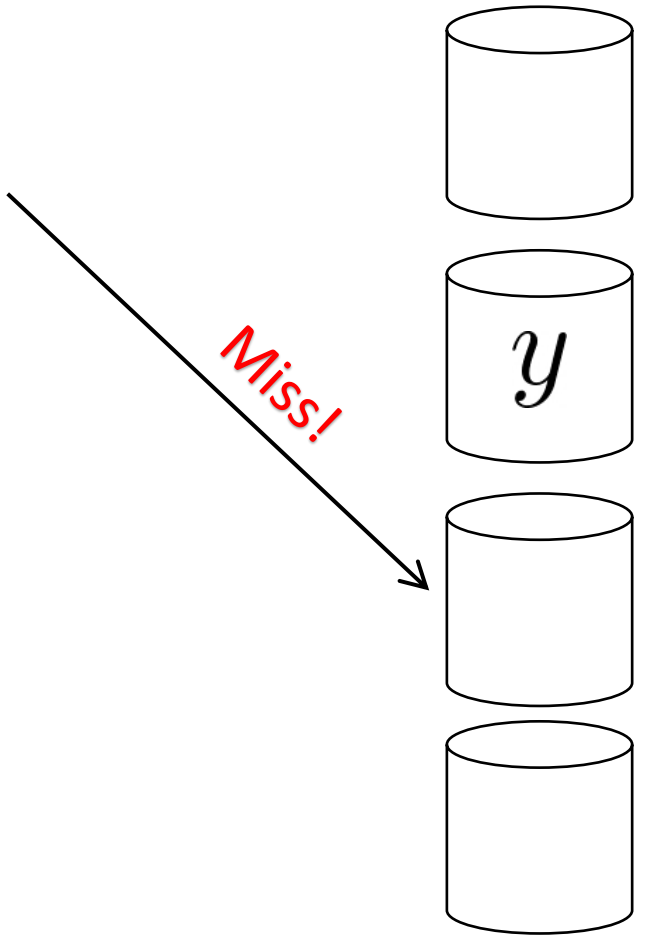




Explaining Polynomial Range Samplers

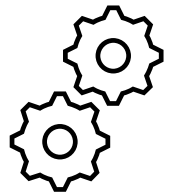
$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(r_3)$

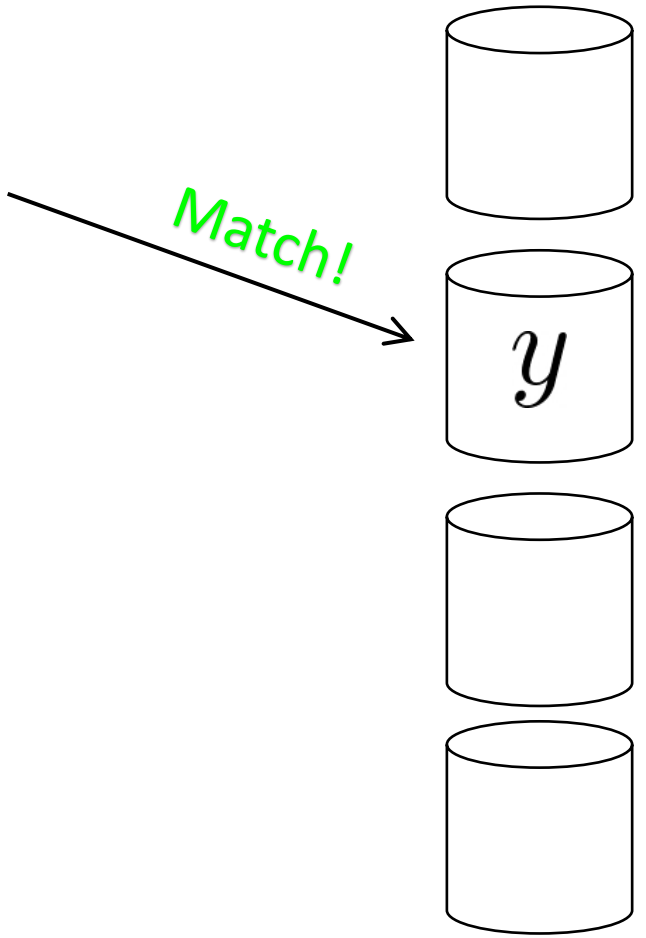




Explaining Polynomial Range Samplers

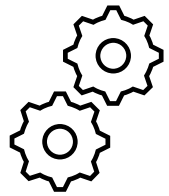
$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(r_4)$

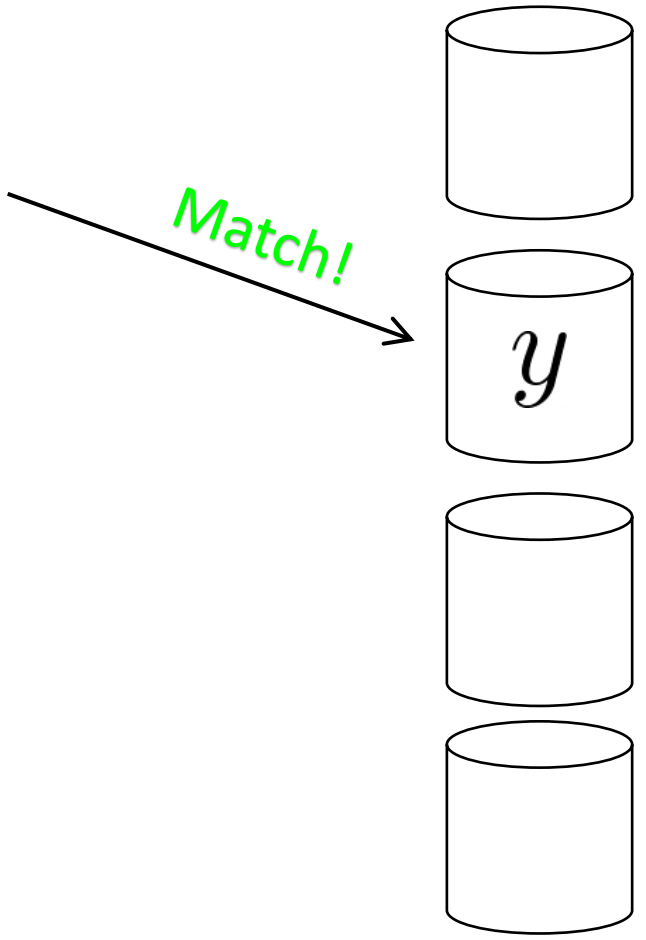




Explaining Polynomial Range Samplers

$\mathsf{Explain}(y) \rightarrow \mathsf{Sample}(\underline{r_4})$





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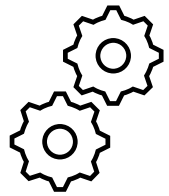

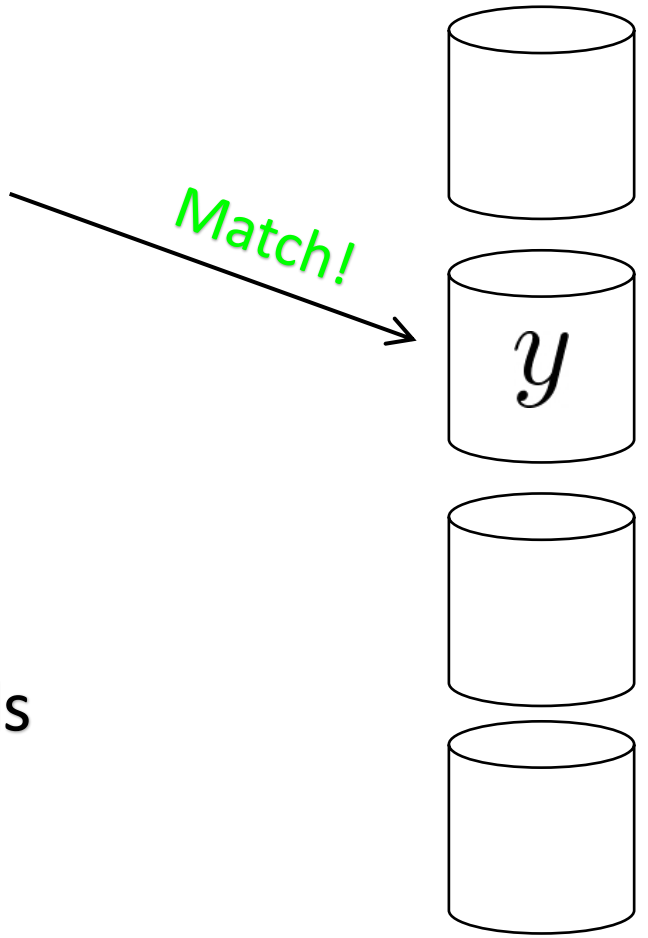

$\kappa \cdot (|range(\mathsf{Sample}(\cdot))| - 1)$ Trials

Result 2

Generic Explainability through Heavy Element Samplers

Result 2

- Encompasses a wide range of mathematical distributions
  - Gaussian
  - Geometric
  - Gamma
  - Poisson
  - Any distribution with monotonic[+] probability density

    …
- Converts non-explainable sampler to explainable one

[+]or decomposable into polynomially many monotonic intervals

Explaining Larger Range Distributions

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- Sampling Function

  SampleOld(r)

Explaining Larger Range Distributions

- Sampling Function
  SampleOld(r)

Any DG Sampler – ex: MW17

Explaining Larger Range Distributions

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- Probability Density function

  PDF(y)

  - Output *proportional* to probability of occurring

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Any DG Sampler – ex: MW17

$$e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Explaining Larger Range Distributions

- Sampling Function

  SampleOld(r)

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  PDF(y)

  - Output *proportional* to probability of occurring

- Heavy Element Sampler

  $\{\text{SampleUniform}(p, r), \text{ExplainUniform}(1^\kappa, p, r)\}$

  - Samples uniform element above probability density threshold p
  - Explainable

Any DG Sampler – ex: MW17

$$e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Explaining Larger Range Distributions

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  - Output *proportional* to probability of occurring

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  $\{\text{SampleUniform}(p, r), \text{ExplainUniform}(1^\kappa, p, r)\}$

  - Samples uniform element above probability density threshold p
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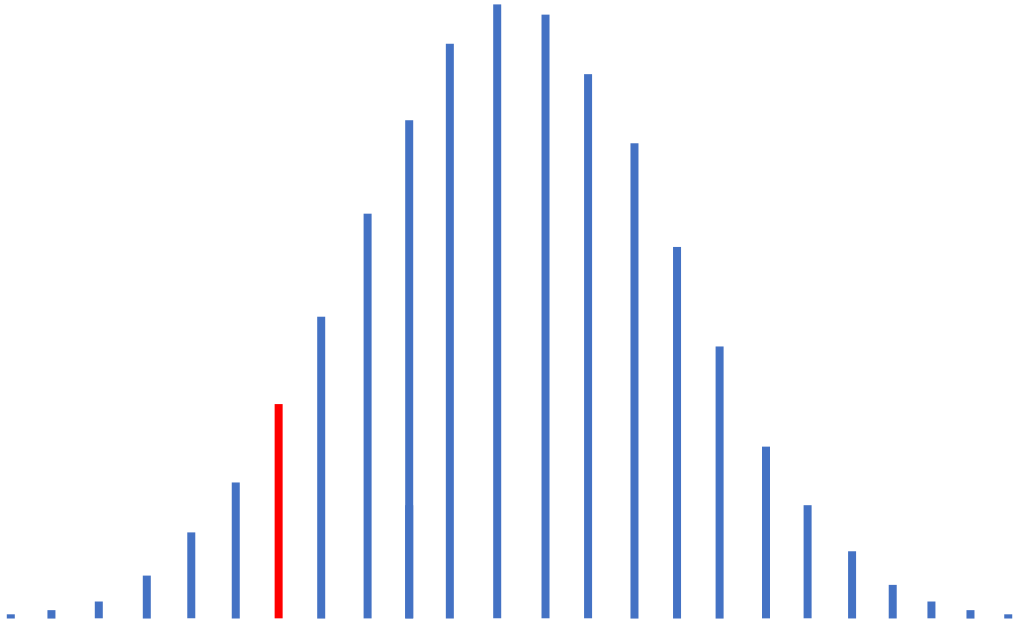
Any DG Sampler – ex: MW17

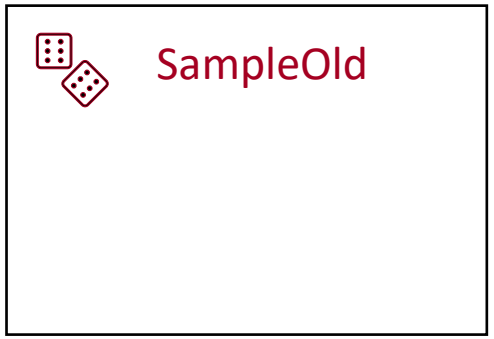
$$e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

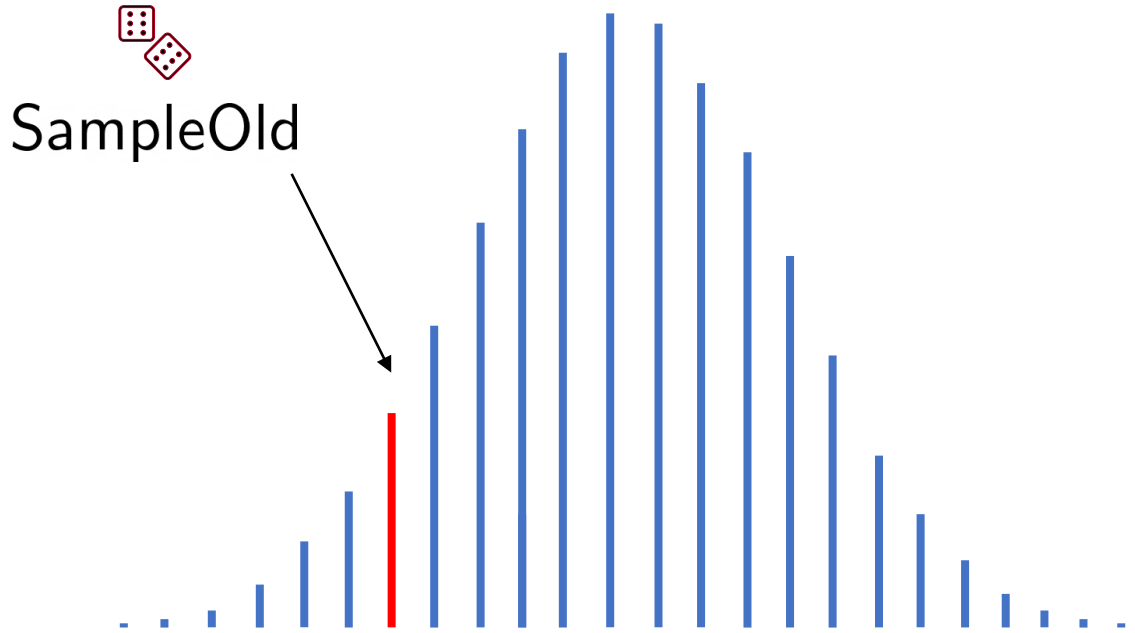
$$\left[\mu - \sqrt{-2\sigma^2 \ln p}, \mu + \sqrt{-2\sigma^2 \ln p}\right]$$

Large Range Sampler



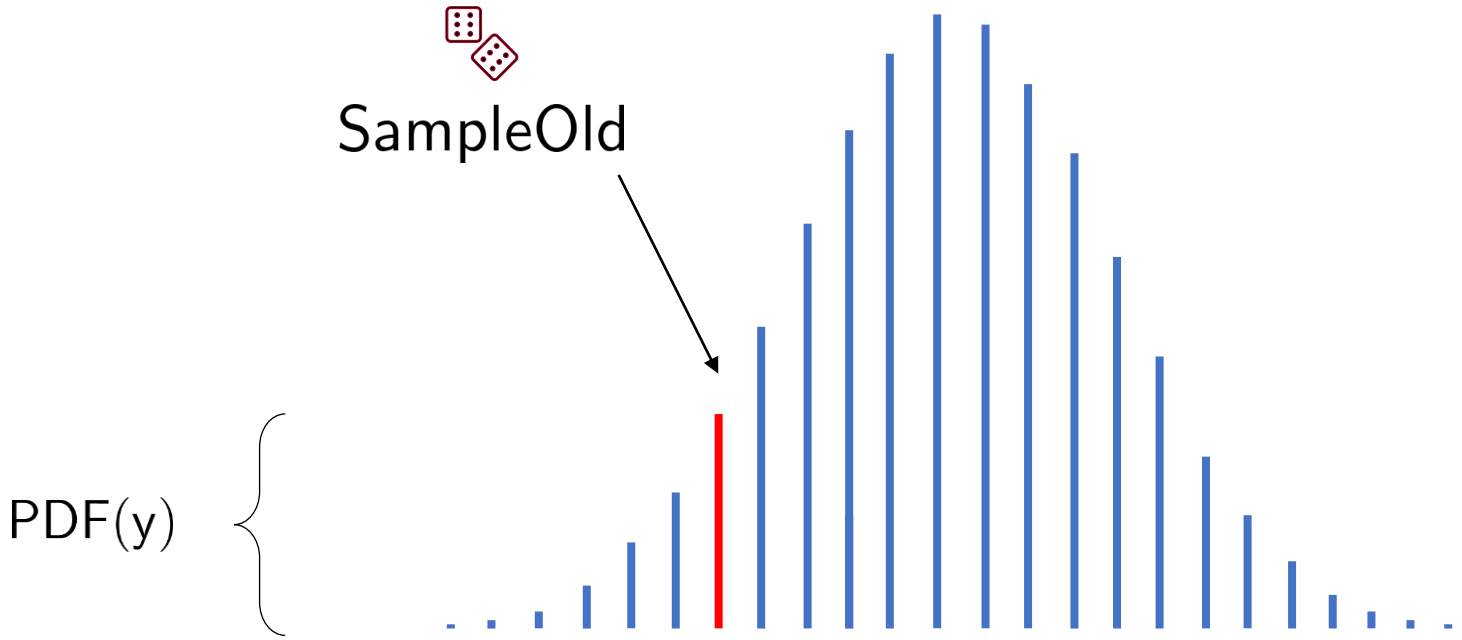
Large Range Sampler





Large Range Sampler

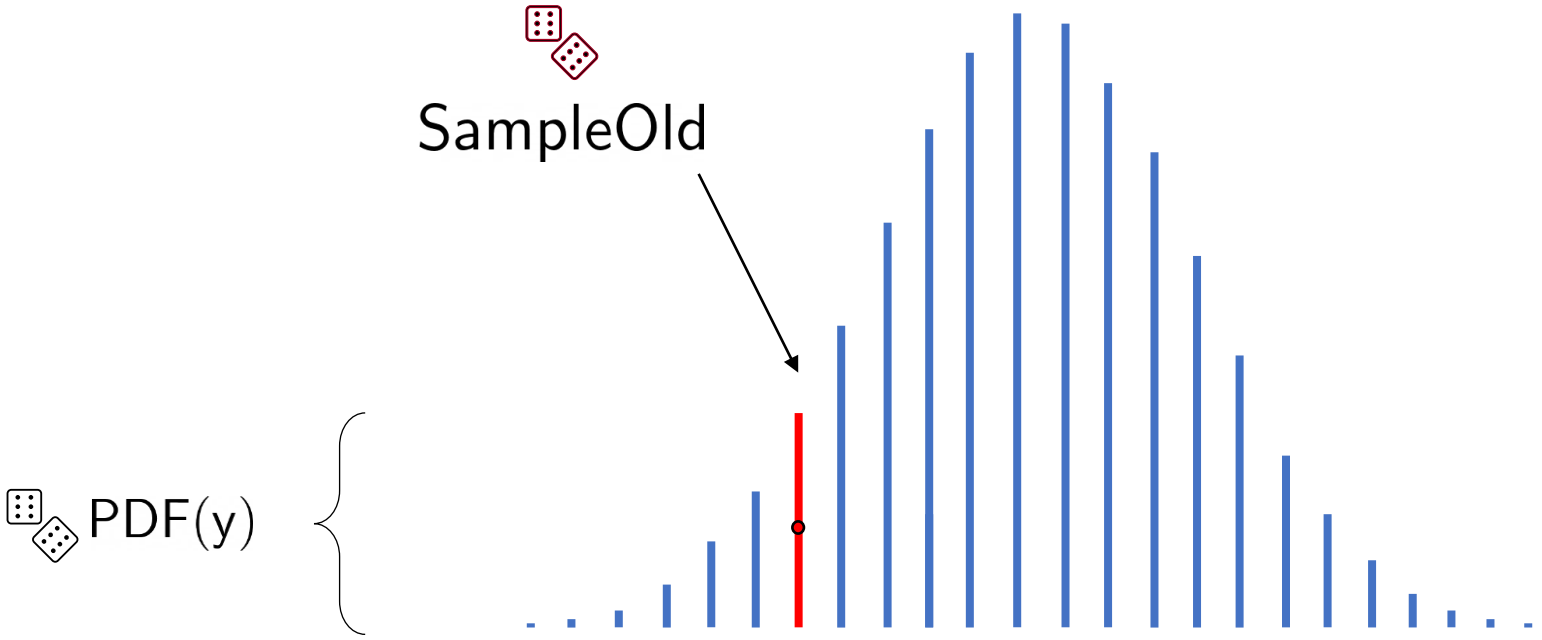
SampleOld



Large Range Sampler

SampleOld

Threshold p



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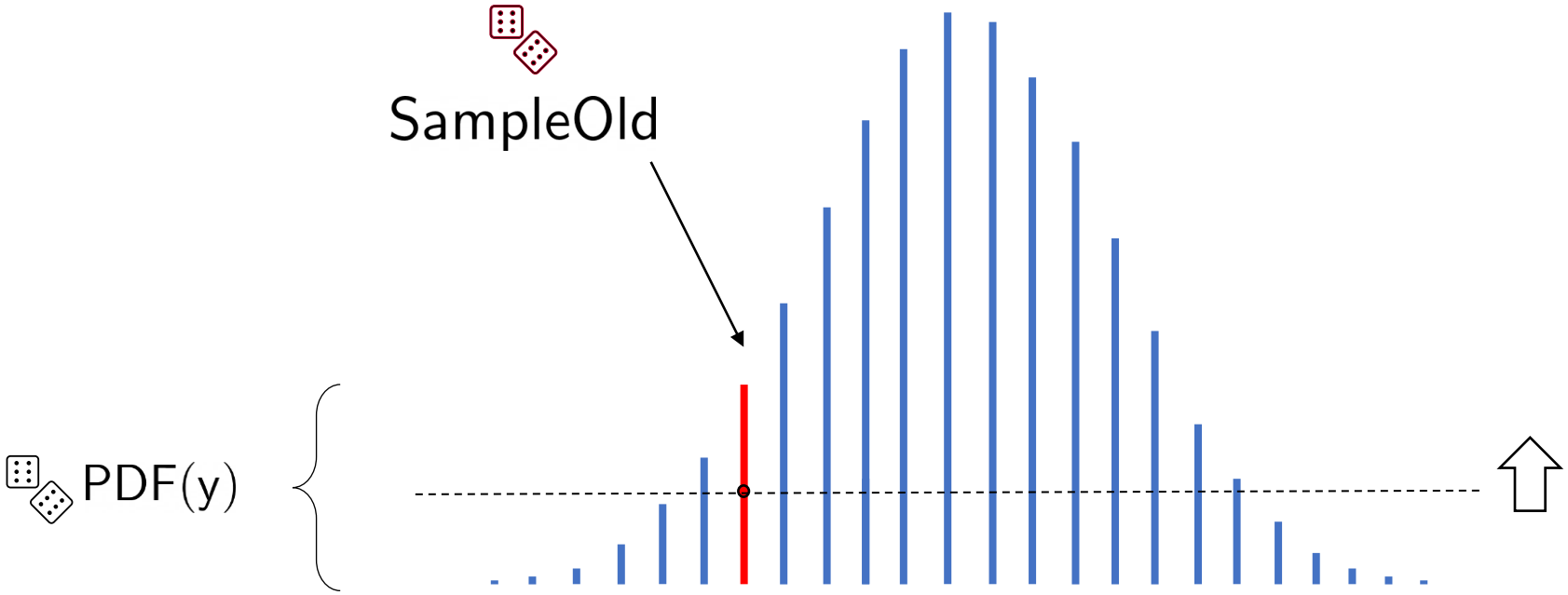
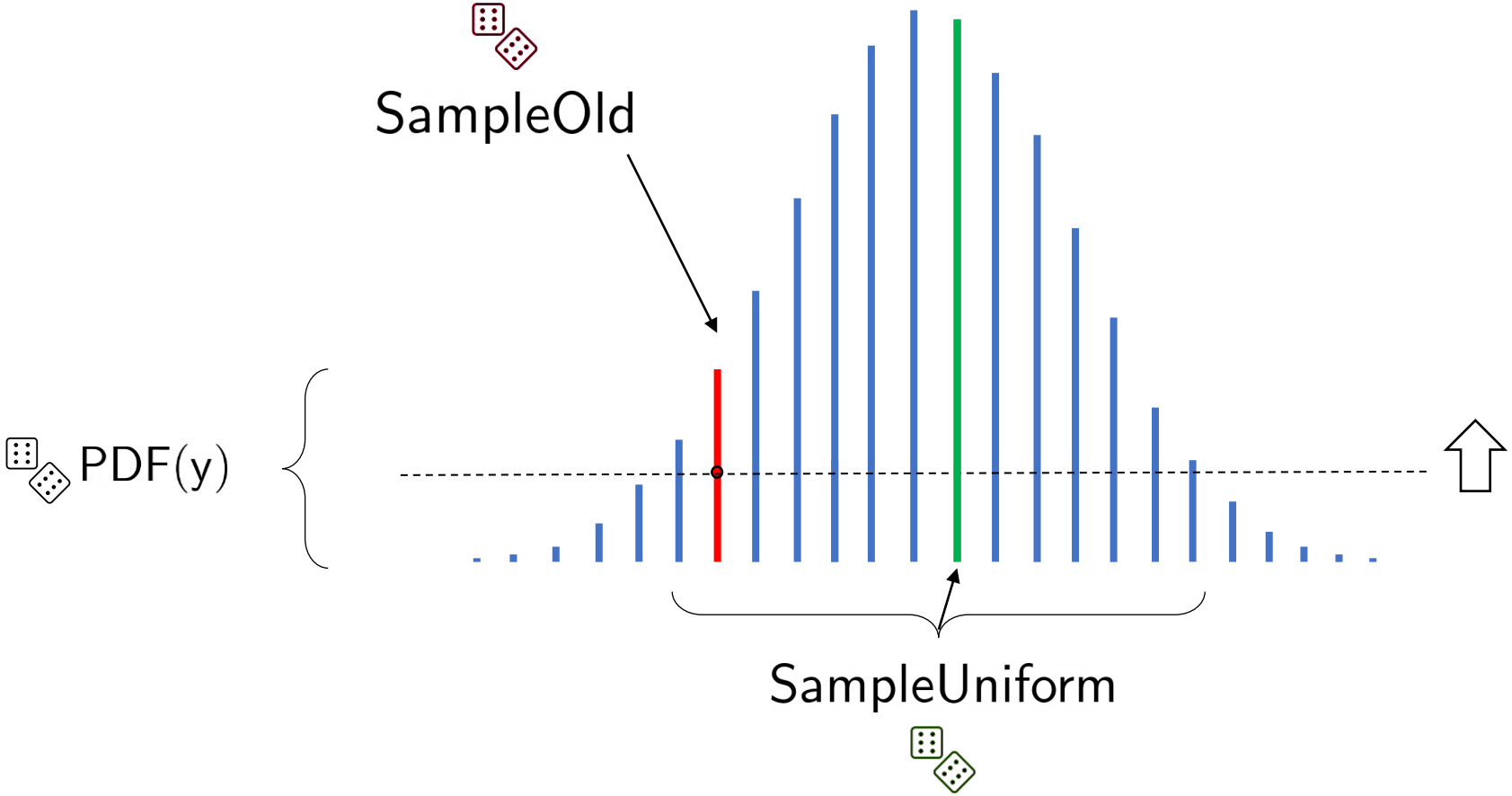


Large Range Sampler

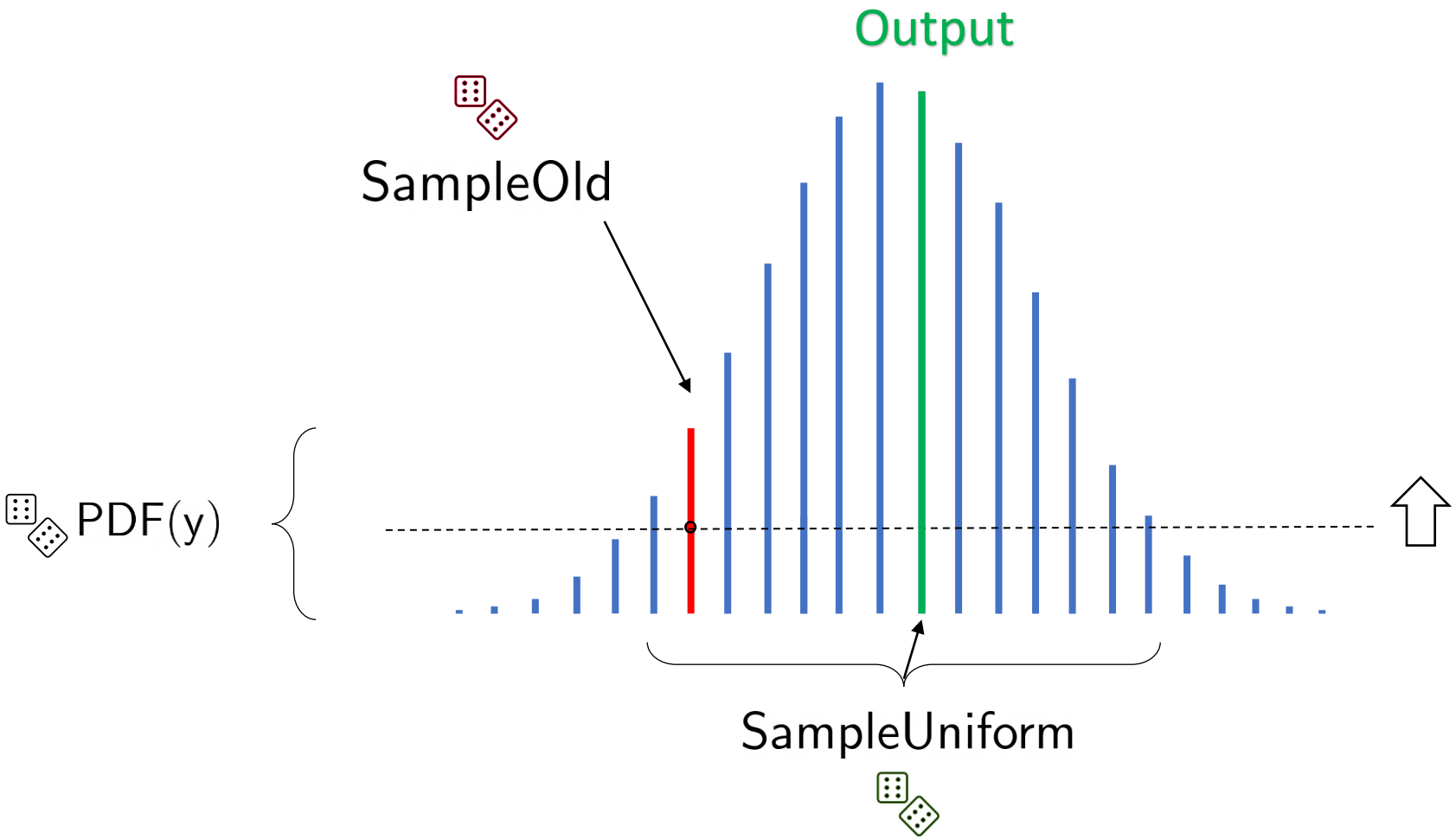


SampleOld

Threshold p

SampleUniform

Large Range Sampler

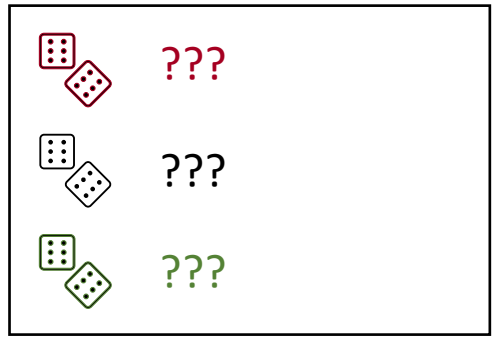


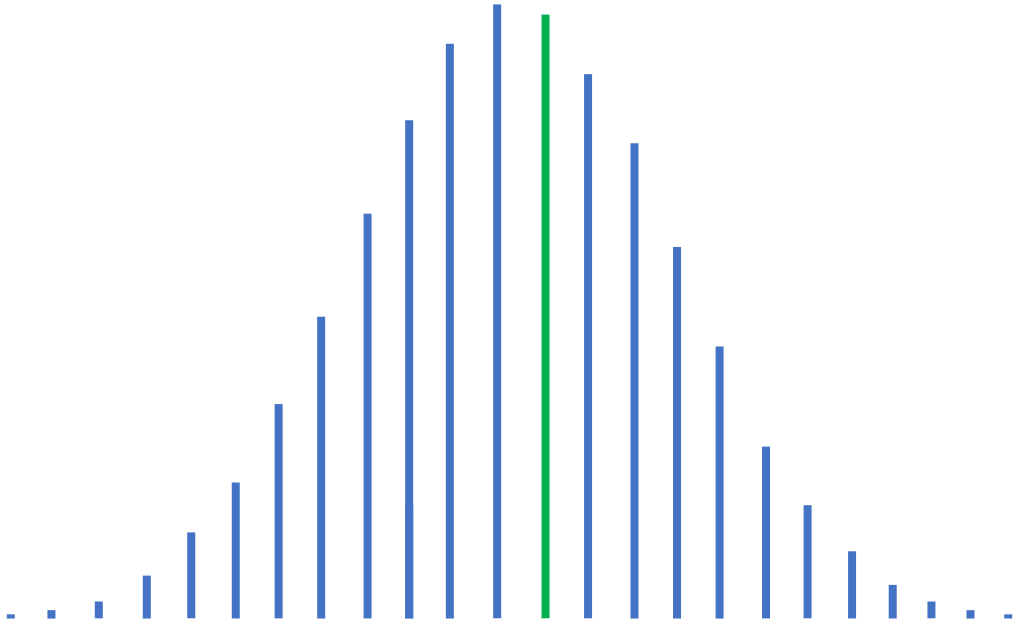
SampleOld

Threshold p

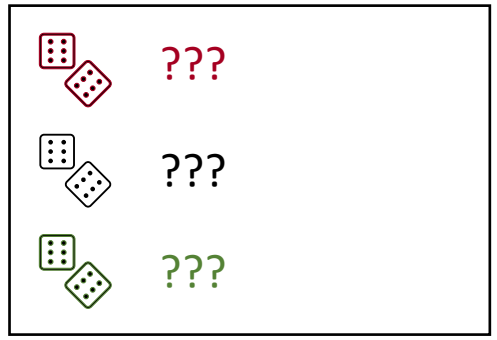
SampleUniform

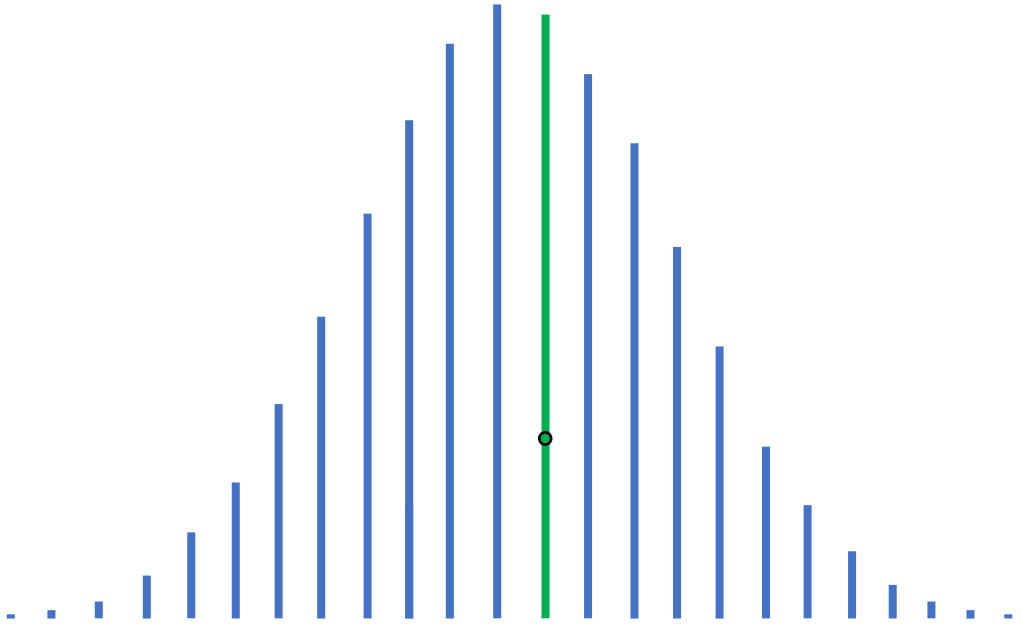
Large Range Explainer



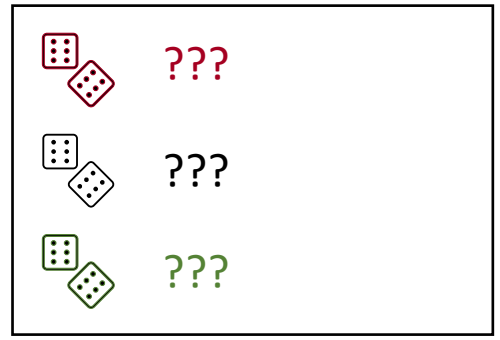


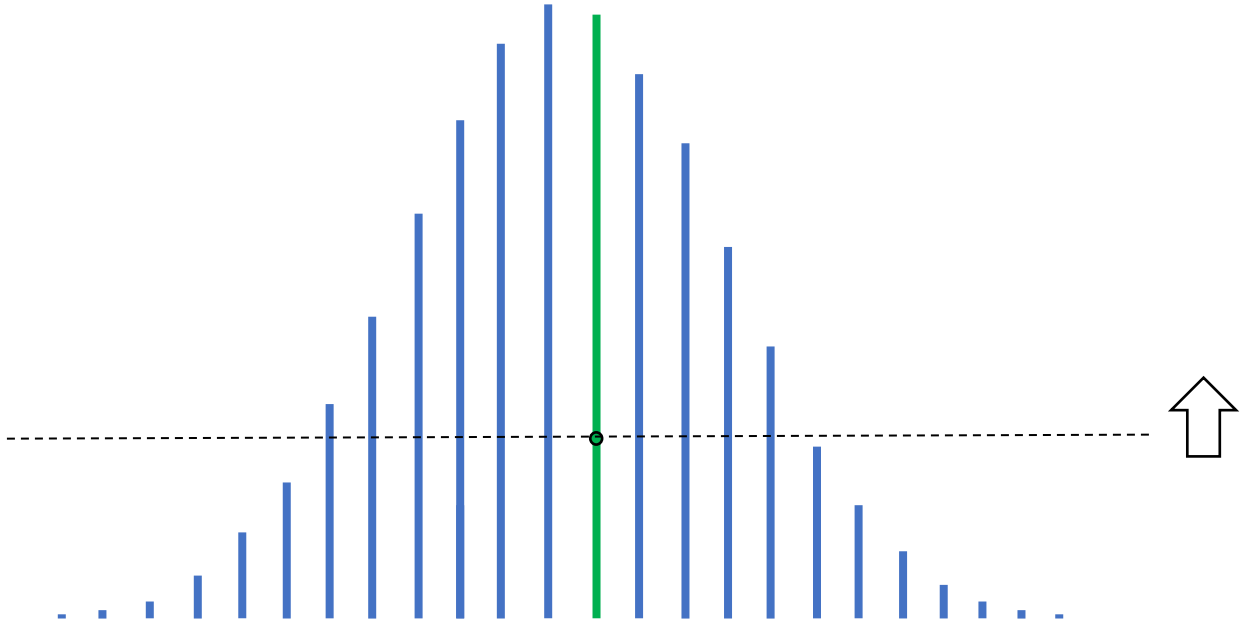
Large Range Explainer



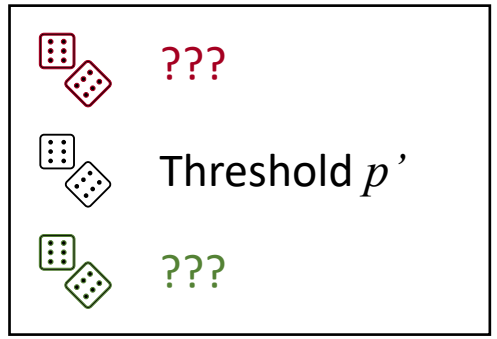


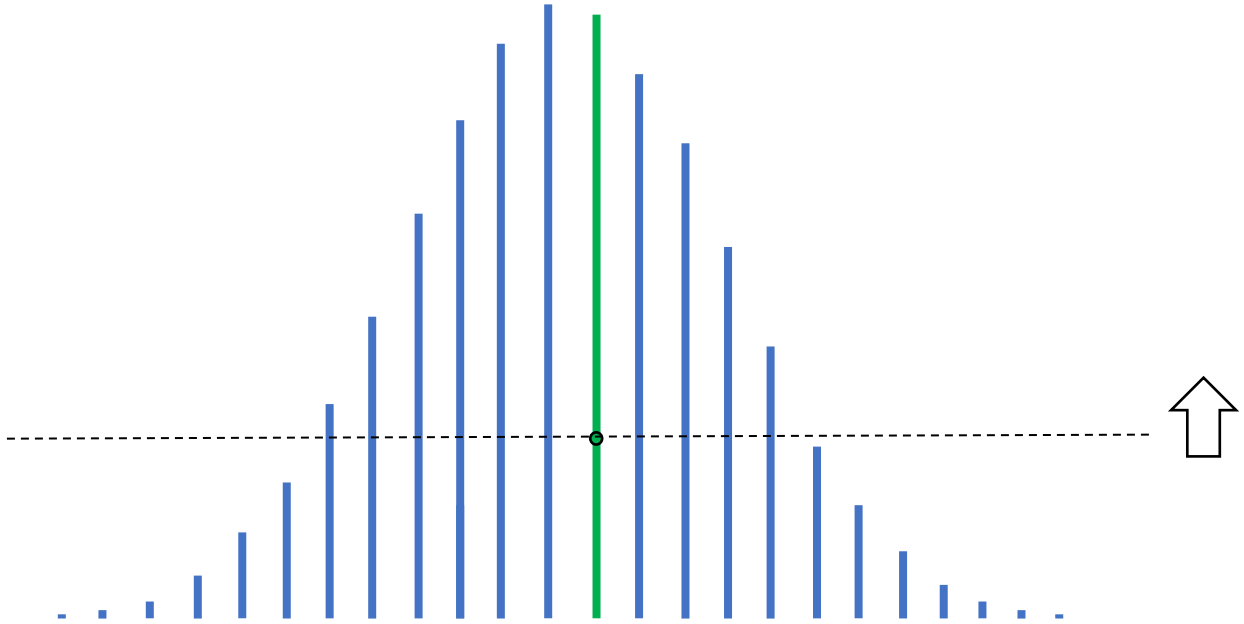
Large Range Explainer



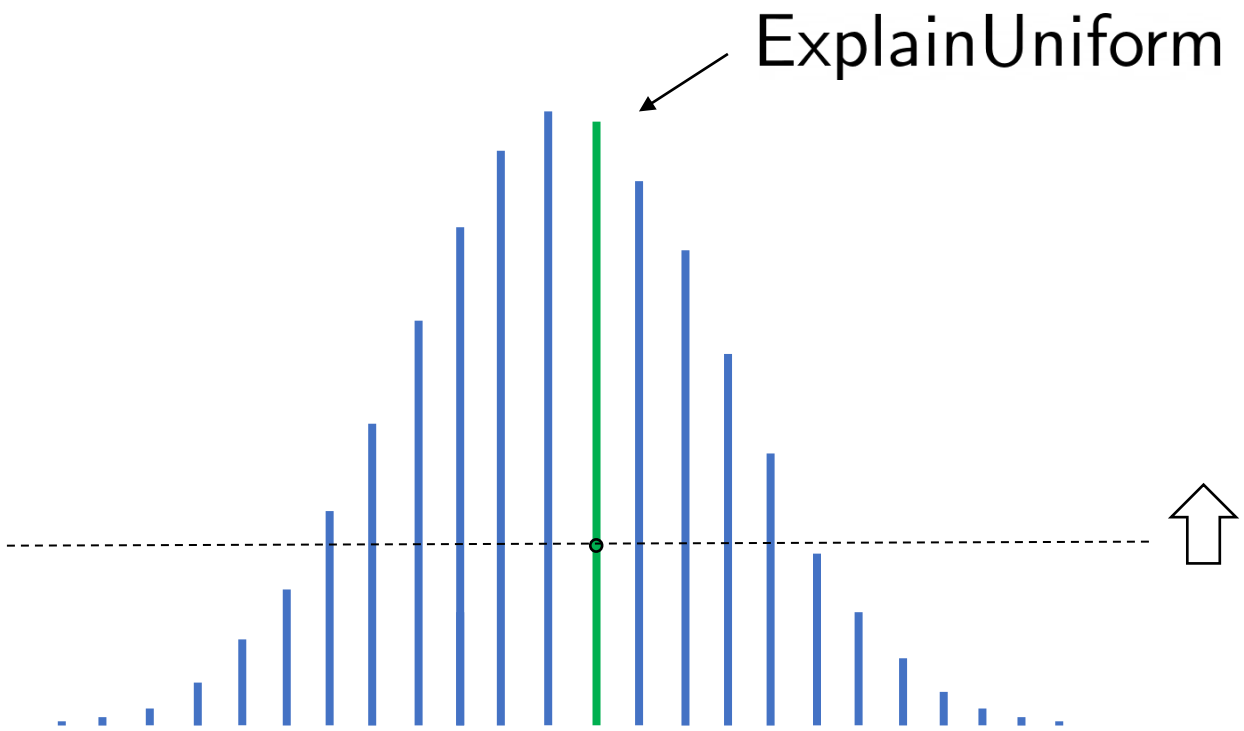


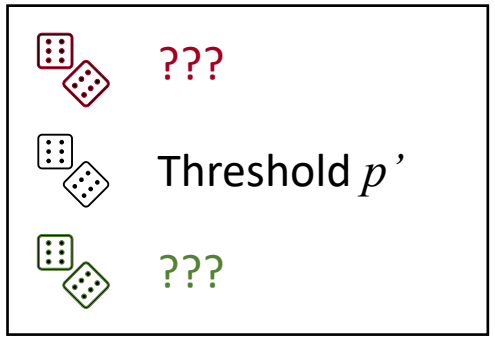
Large Range Explainer



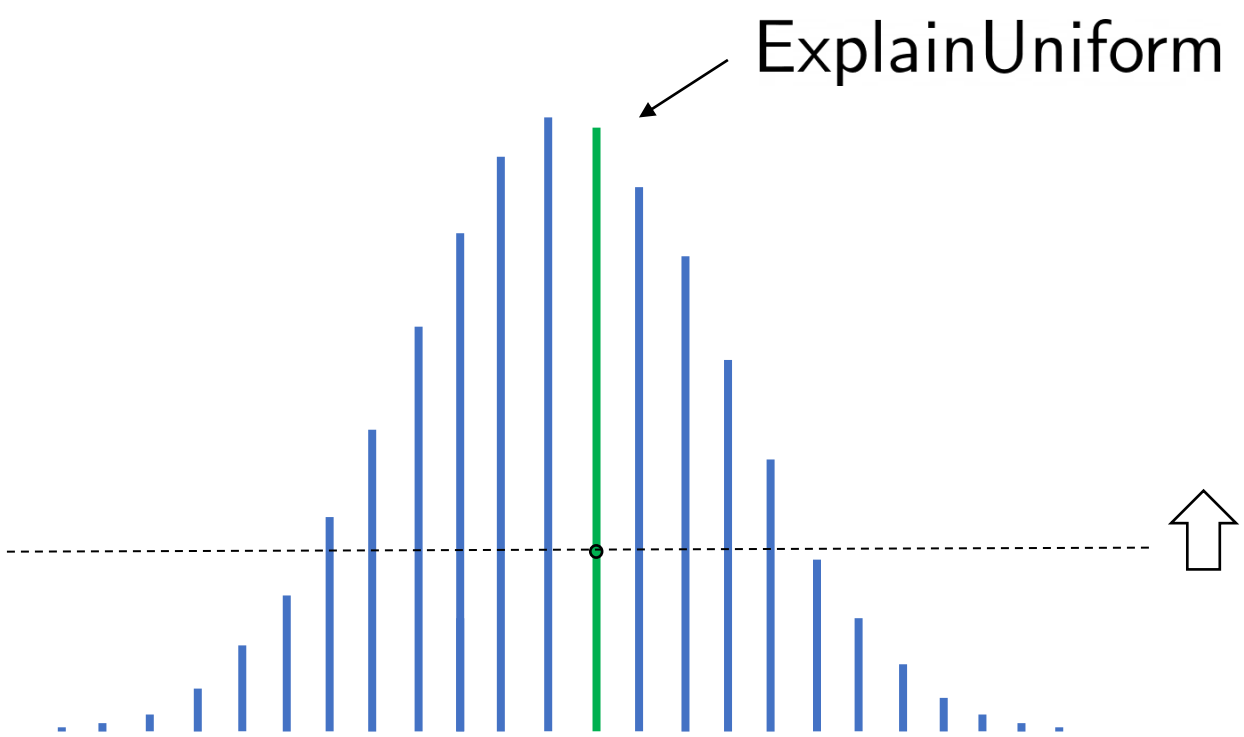


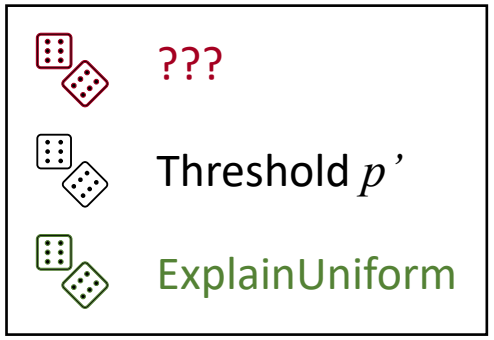
Large Range Explainer



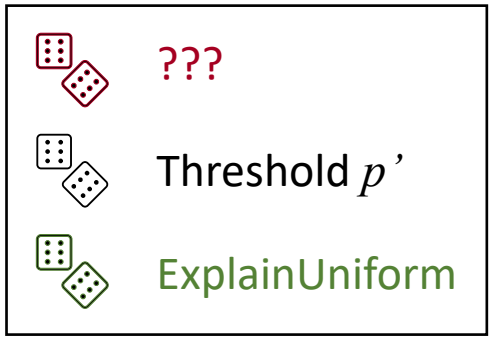


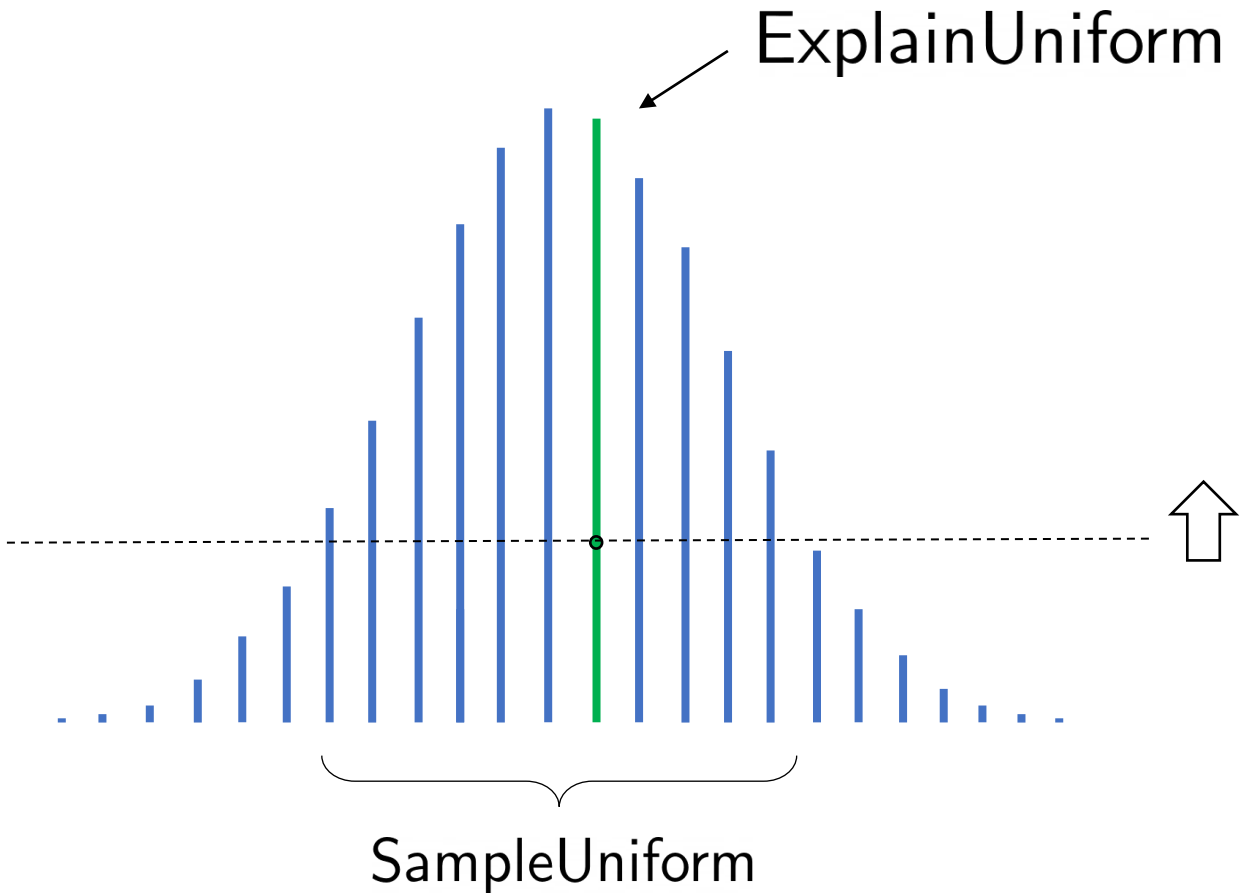
Large Range Explainer



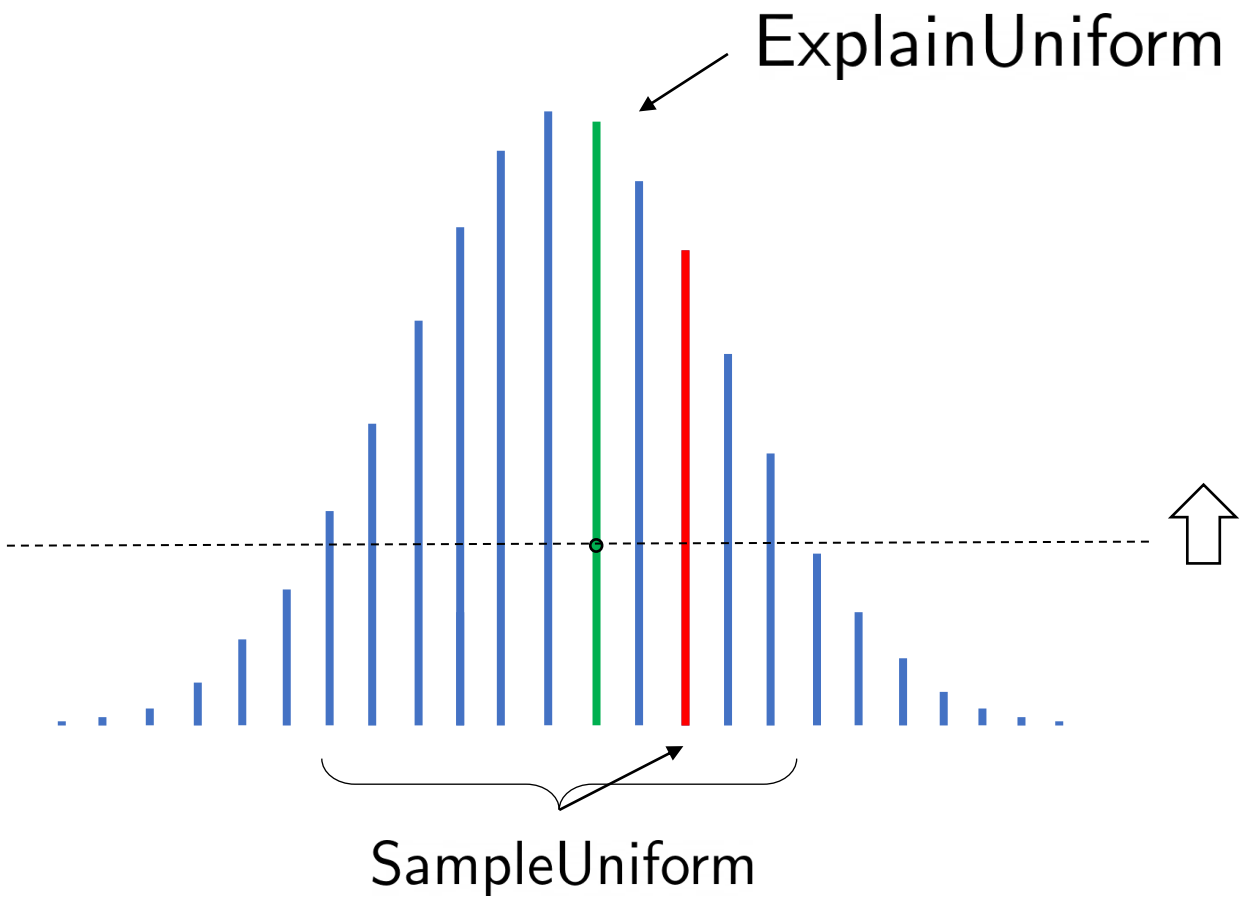


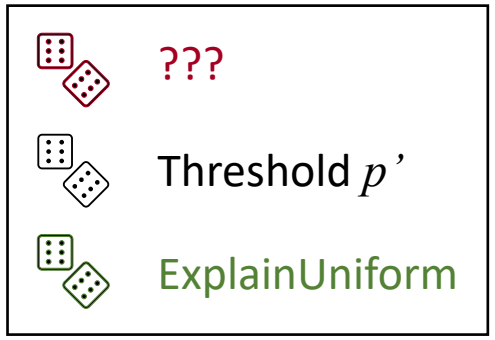
Large Range Explainer



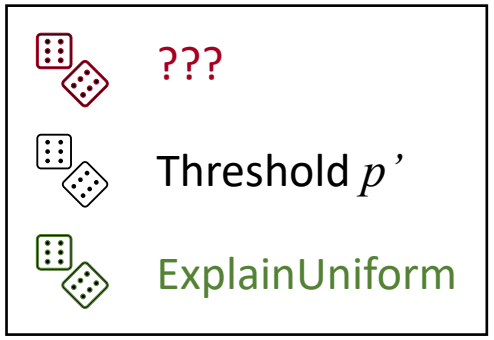


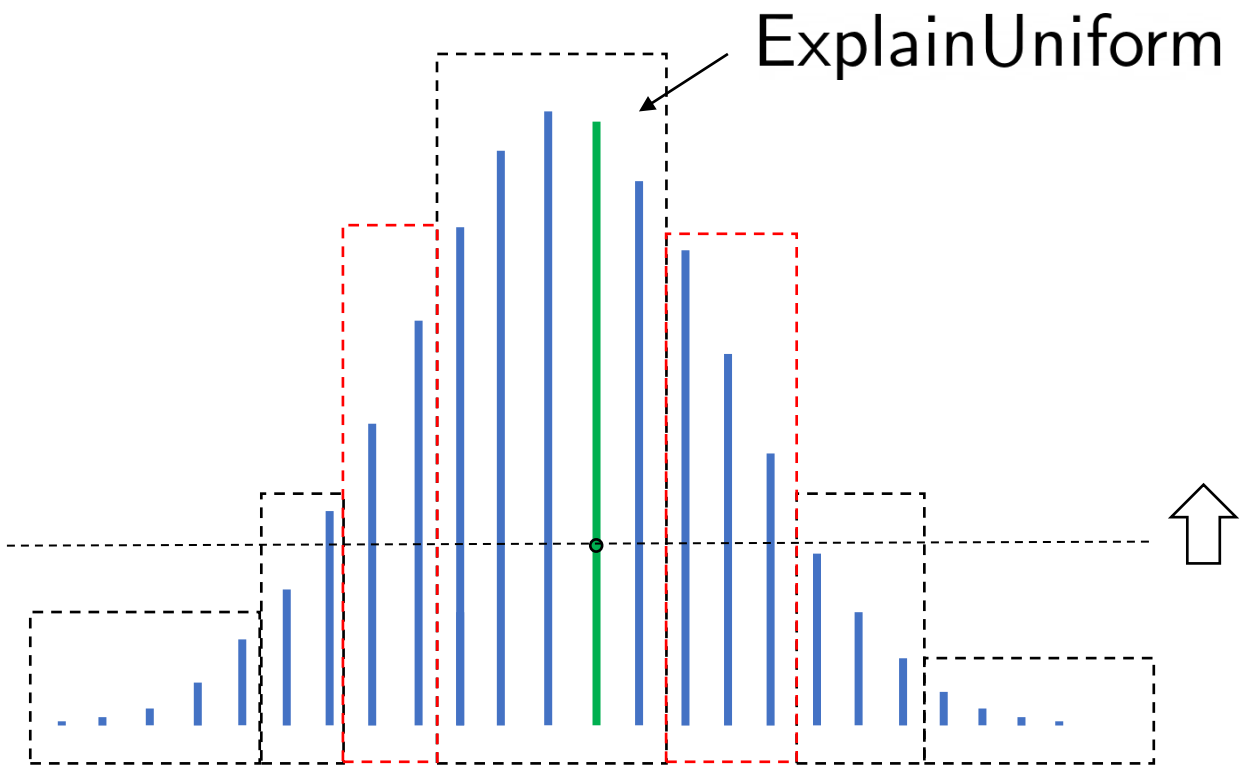
Large Range Explainer



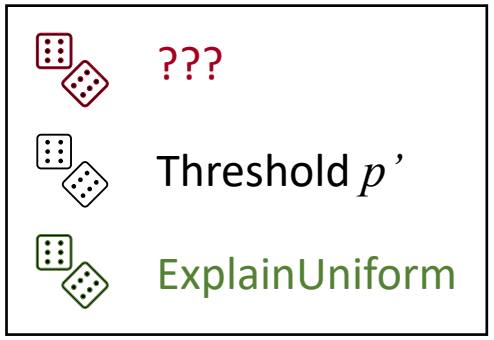


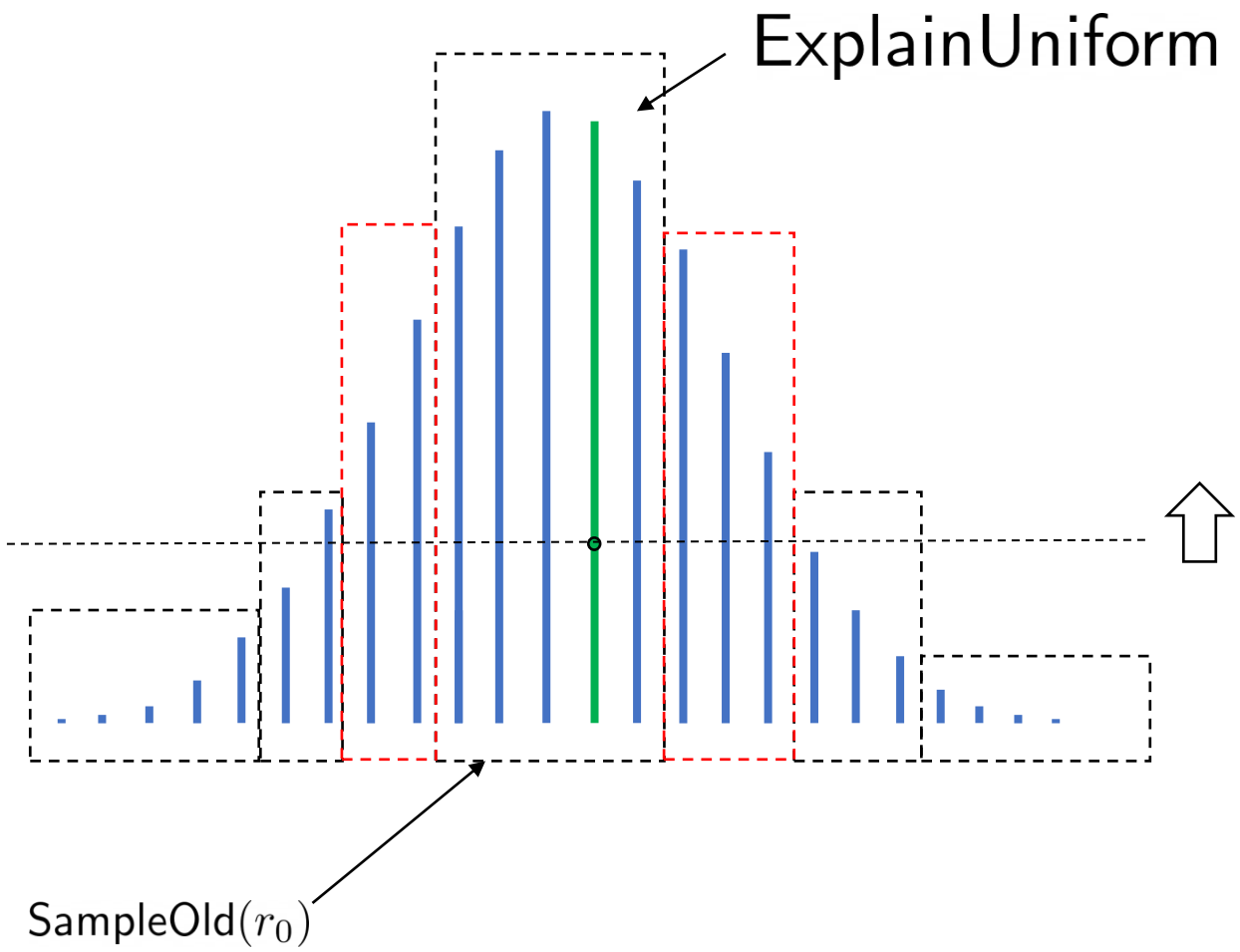
Large Range Explainer



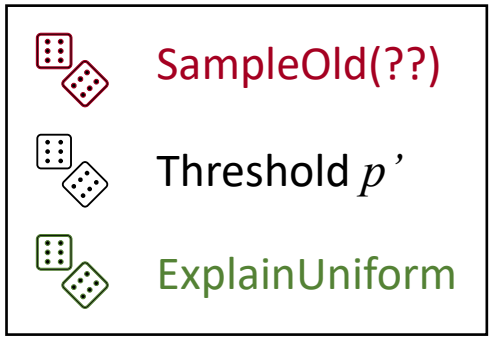


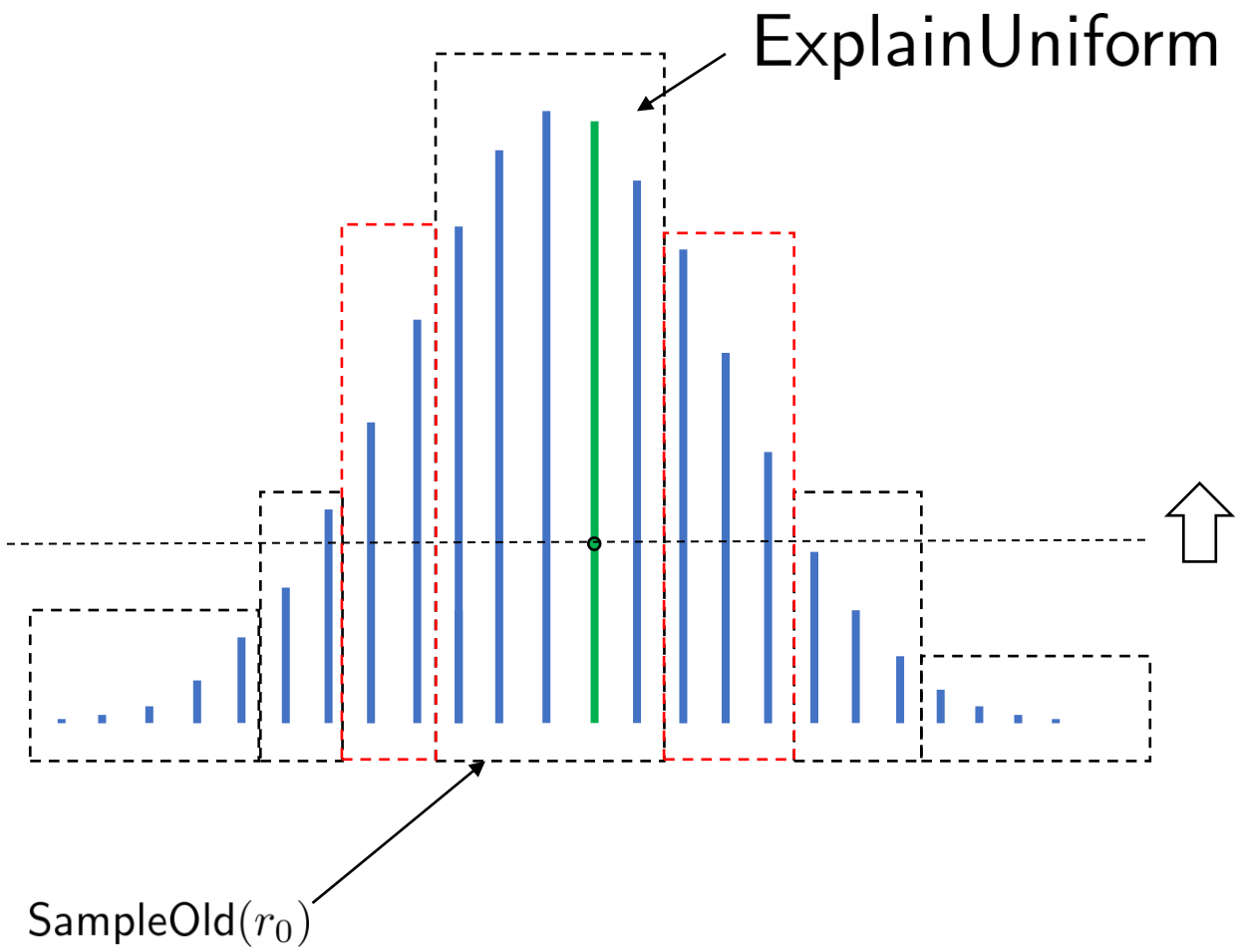
Large Range Explainer



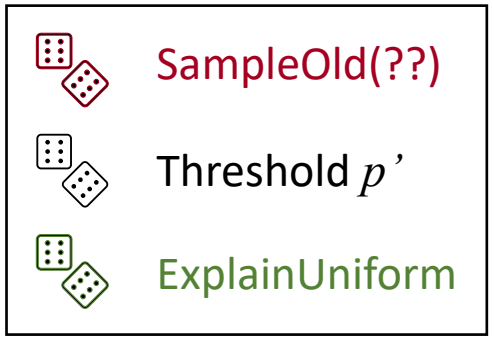


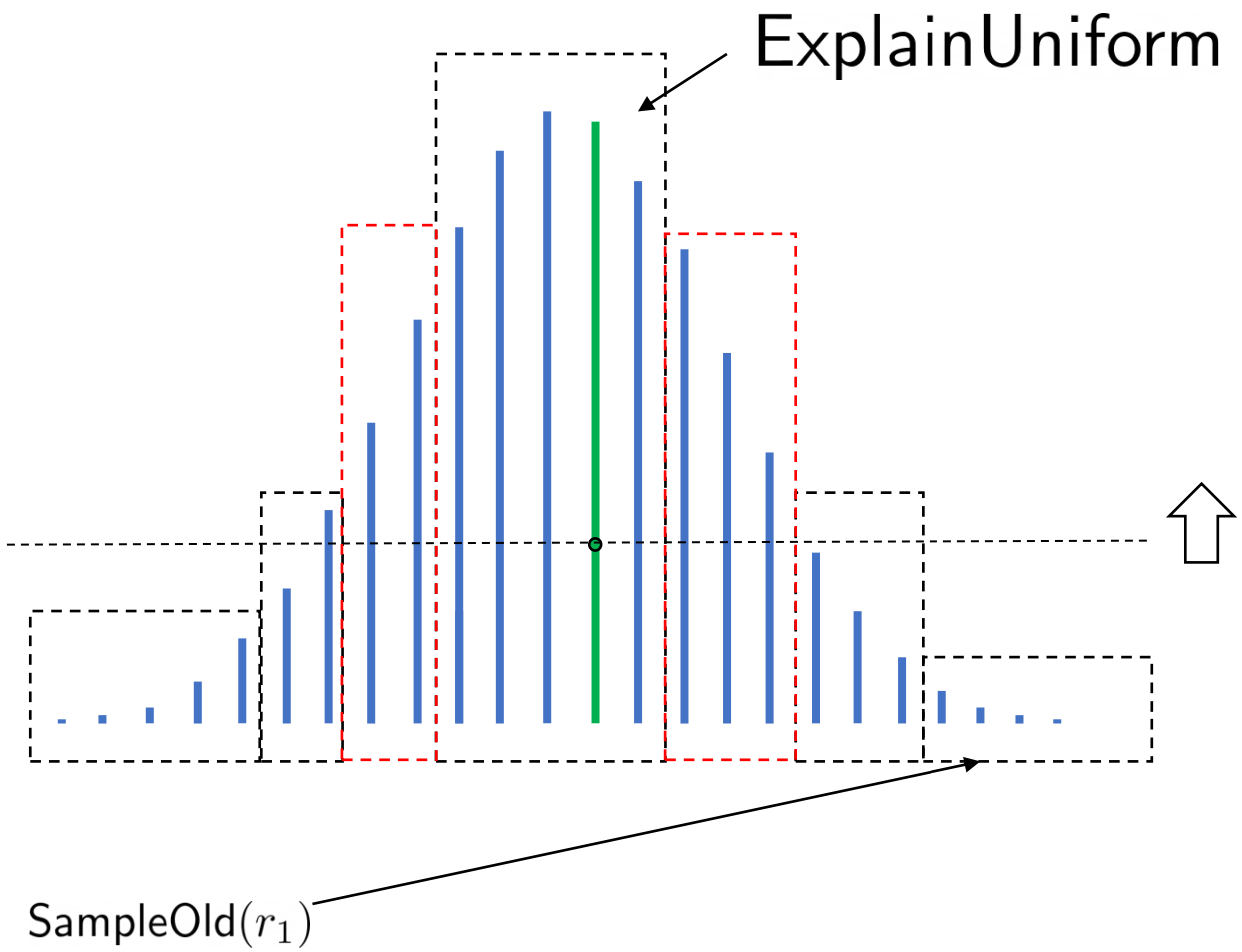
Large Range Explainer



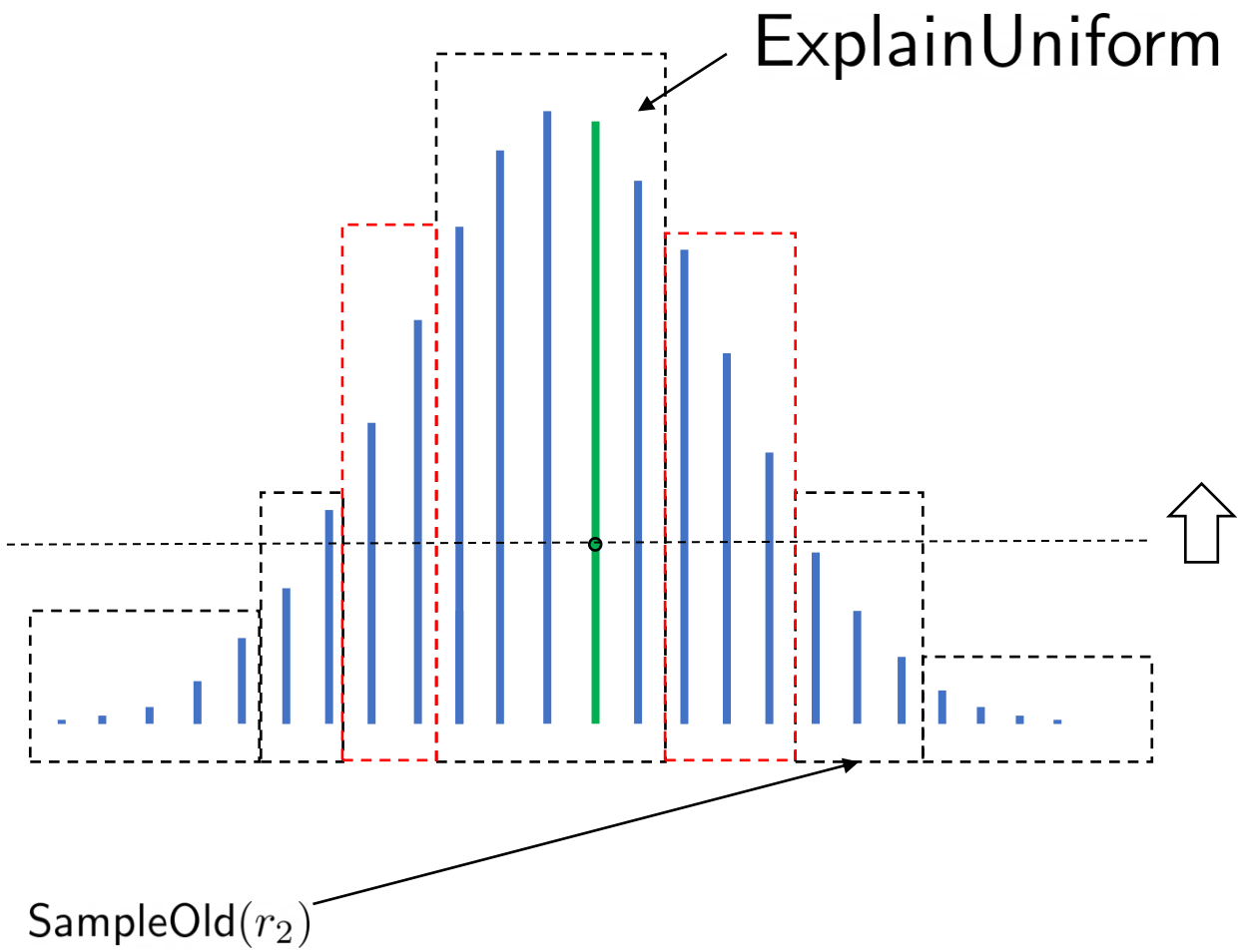


Large Range Explainer





Large Range Explainer

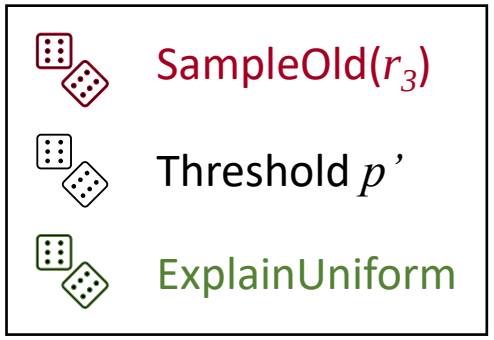


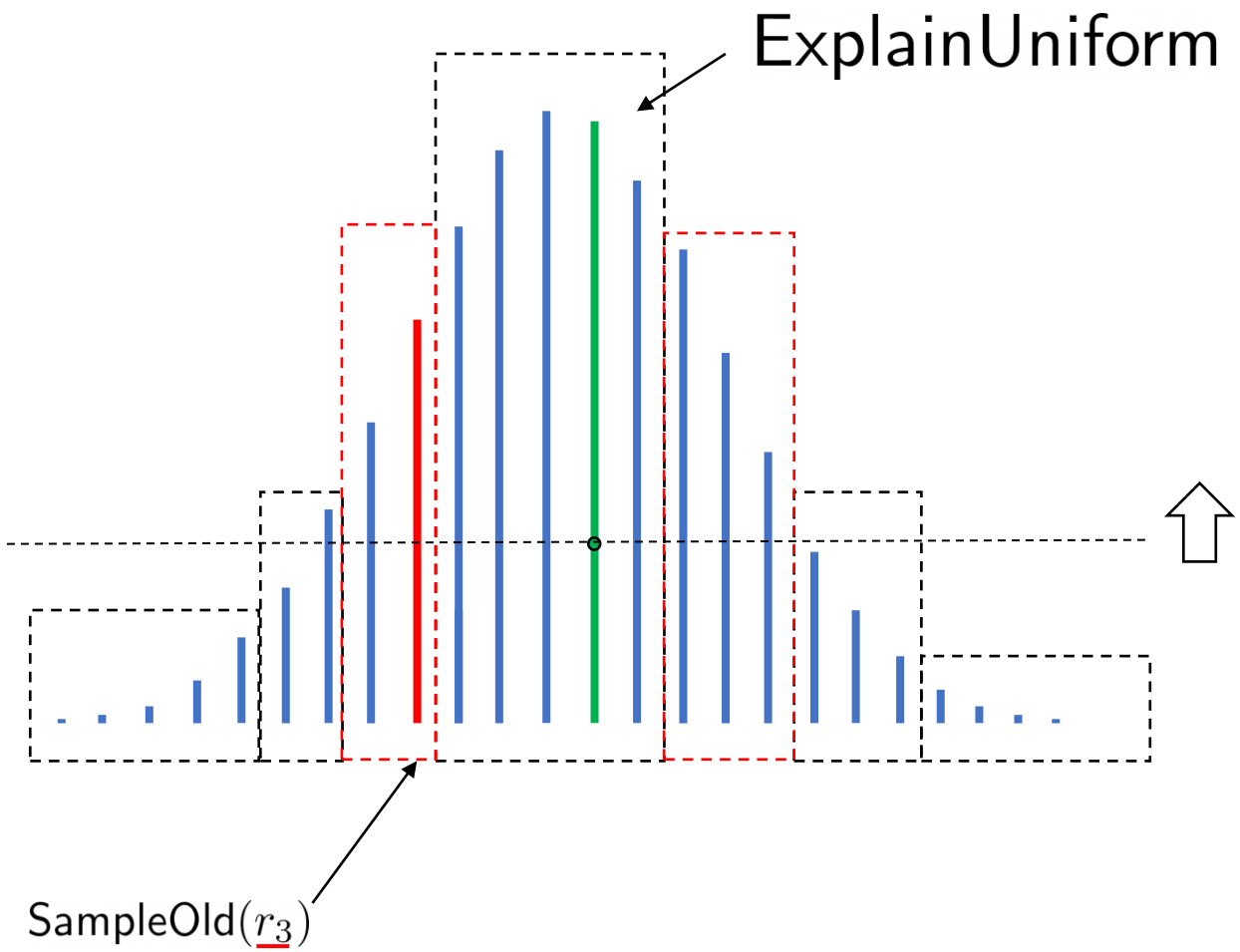
SampleOld(??)

Threshold p'

ExplainUniform

Large Range Explainer





Thank you!