Doubly Efficient Interactive Proofs over Infinite and Non-Commutative Rings

Eduardo Soria-Vazquez

Crypto
Research Center

Doubly-Efficient Interactive proofs (a.k.a. “GKR-style”)

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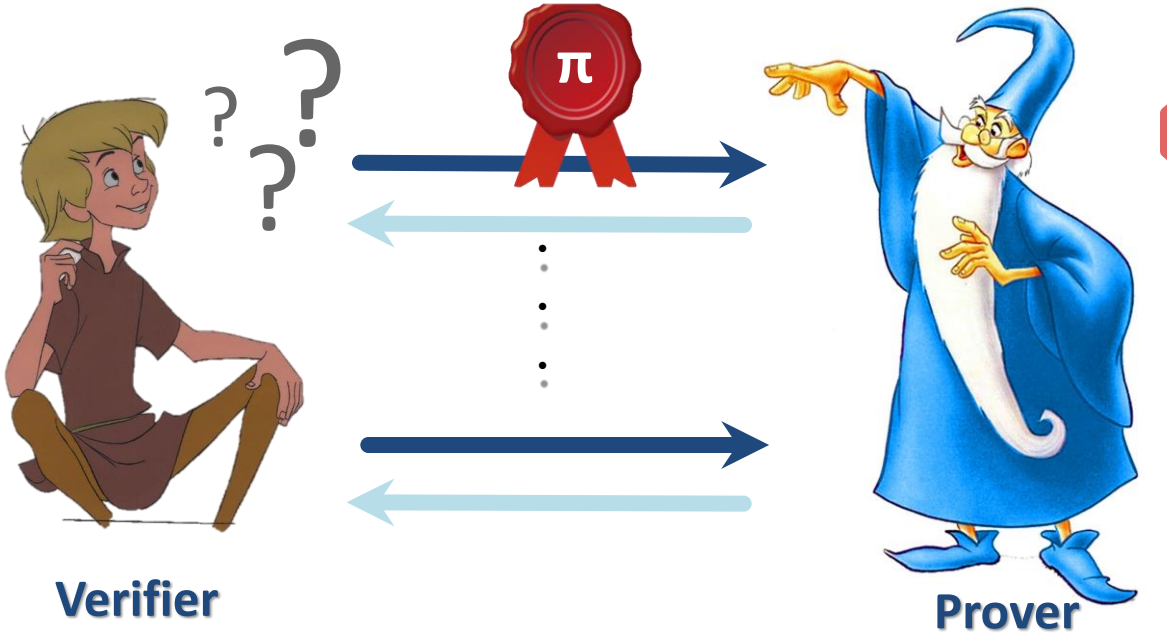

Soundness: False proofs should be rejected.

Correctness: Correct proofs should be accepted.

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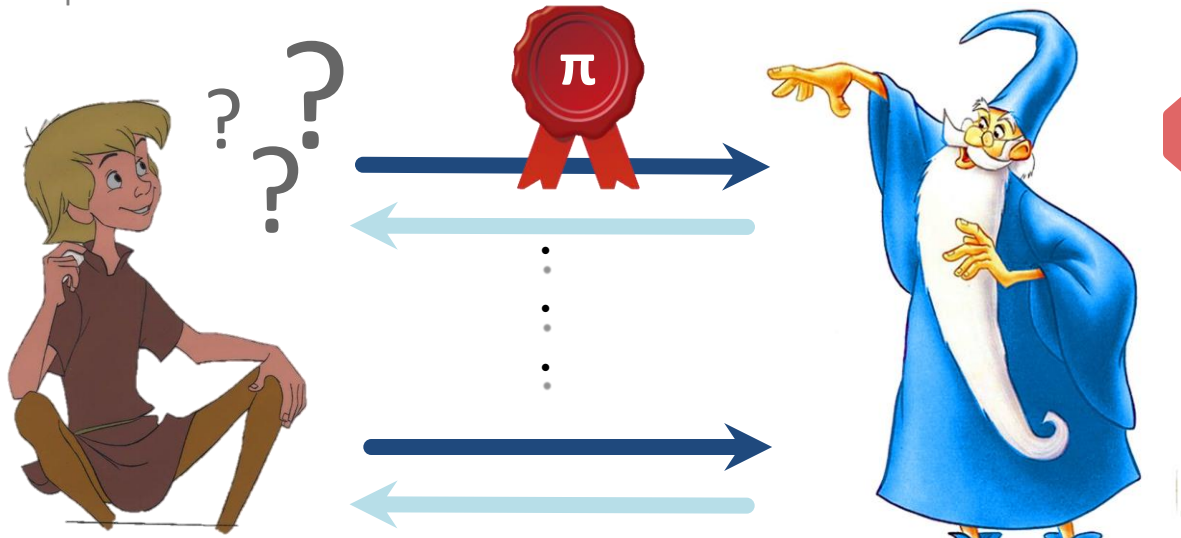
[XZZPS19] Xie, T., Zhang, J., Zhang, Y., Papamanthou, C., & Song, D. “Libra: Succinct zero-knowledge proofs with optimal prover computation.” CRYPTO’19.



Verifier
(quasi-linear in input size, log in circuit size)

Prover
(Linear time on circuit size)

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Correctness:
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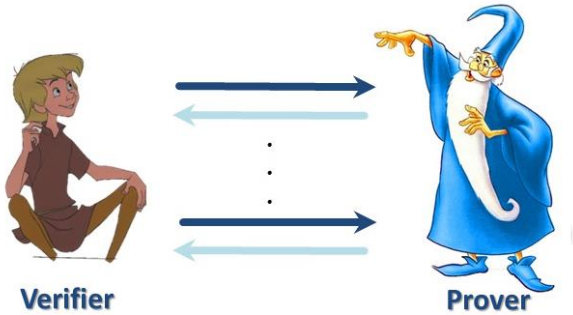
Verifiable computation

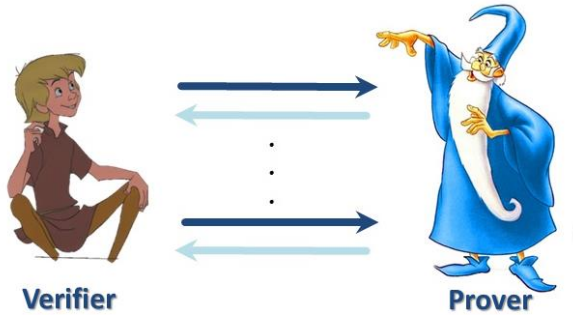


Input: Circuit C, circuit inputs x
Output: y, proof that $C(x) = y$

Verifiable computation (of complex programs, in practice)

```
main(){
 ...
}
```

C program



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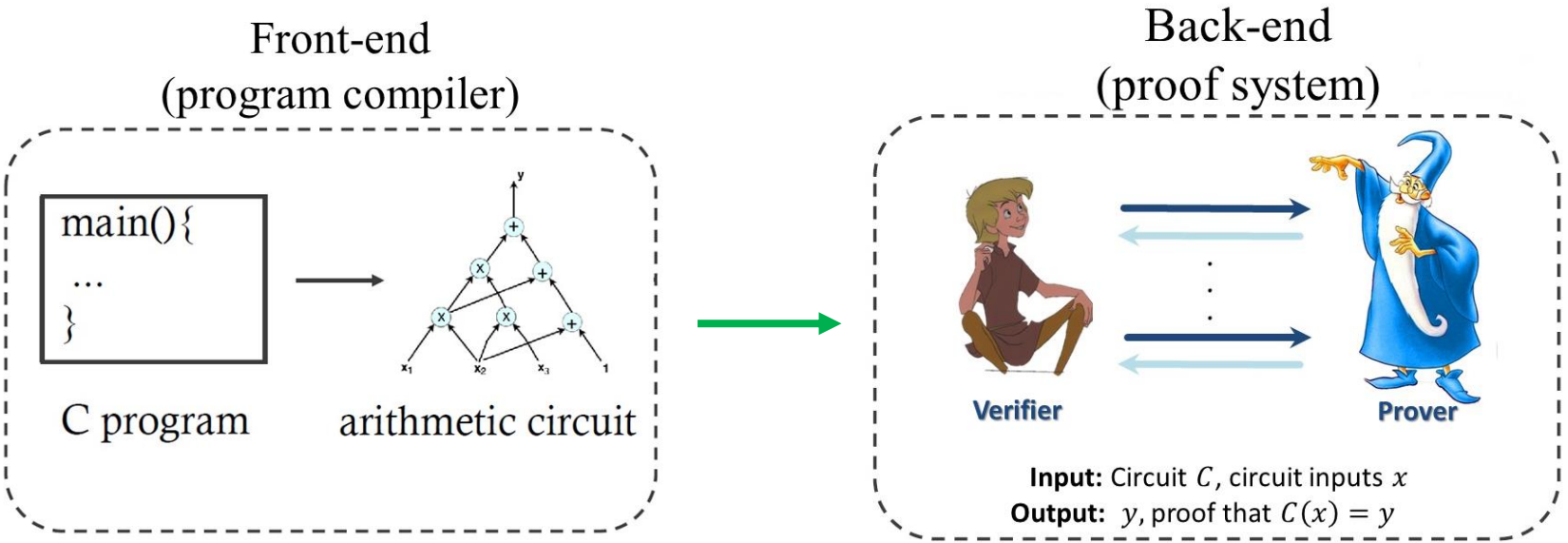

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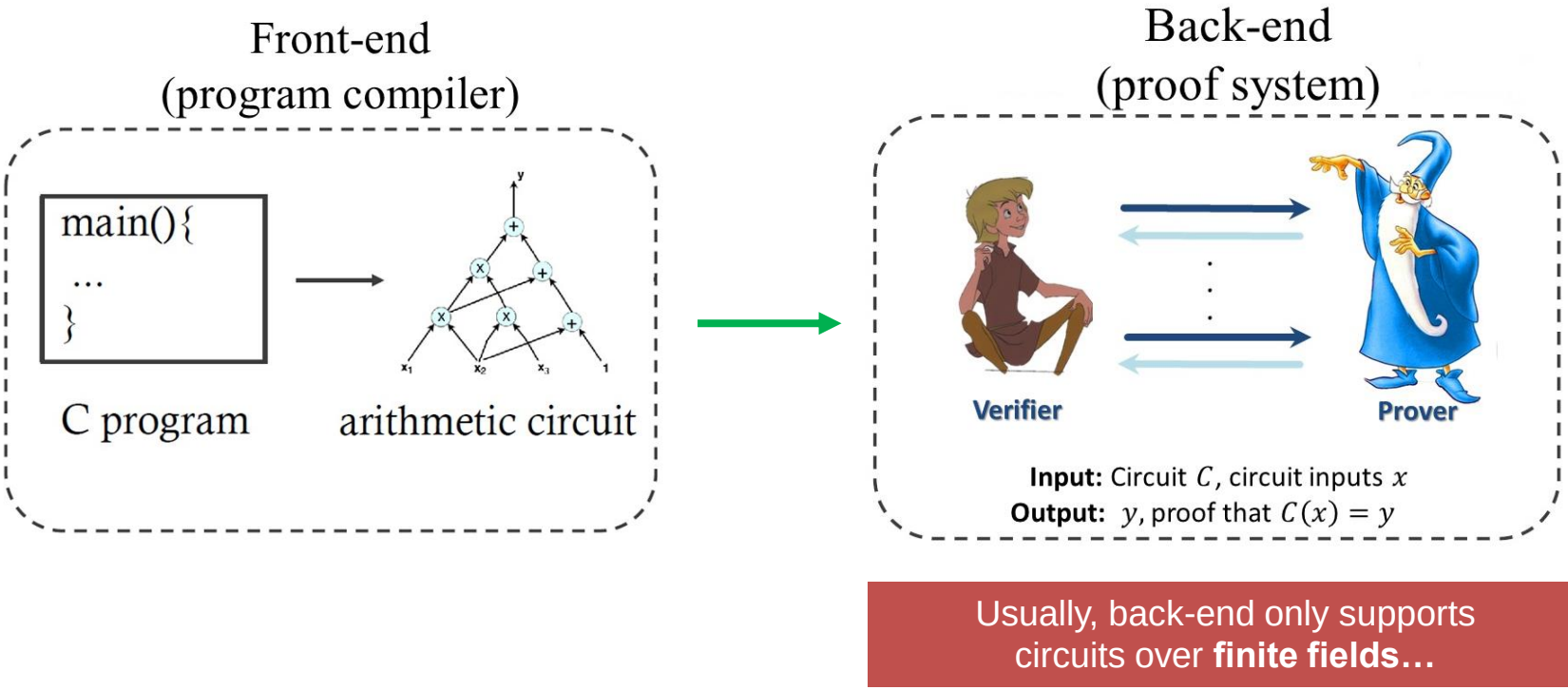

Usually, back-end only supports circuits over **finite fields…**

Verifiable computation (of complex programs, in practice)

Front-end
(program compiler)

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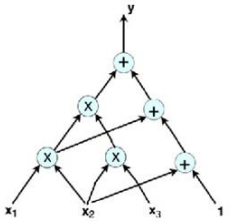
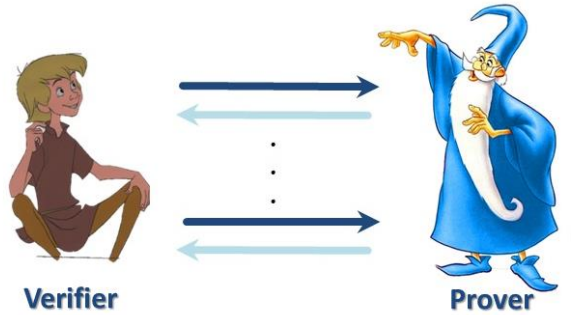


arithmetic circuit

Back-end
(proof system)



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… which becomes a limitation for the front-end.

Doubly Efficient Interactive Proofs: Beyond finite fields

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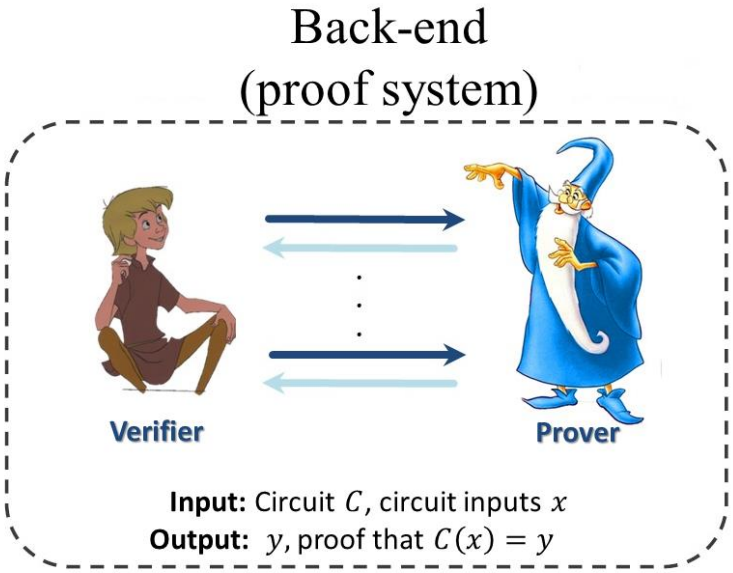

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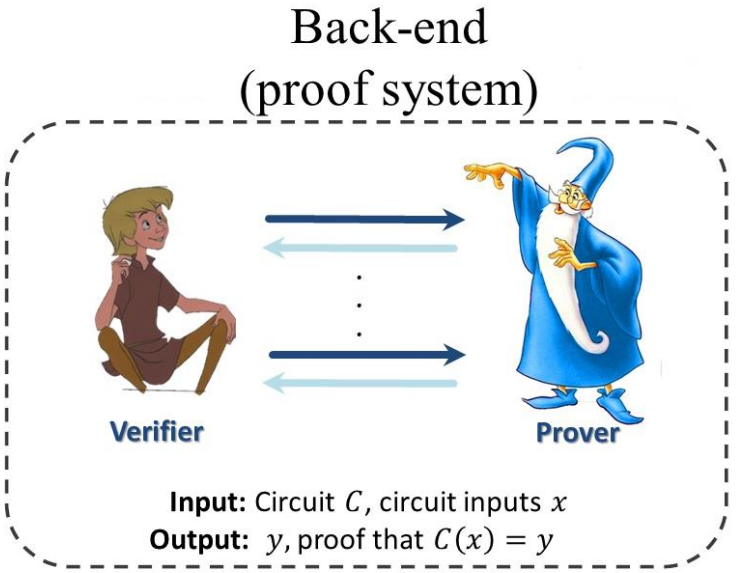

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This work: Extends to circuits over

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  Matrices, quaternions, Clifford algebras…
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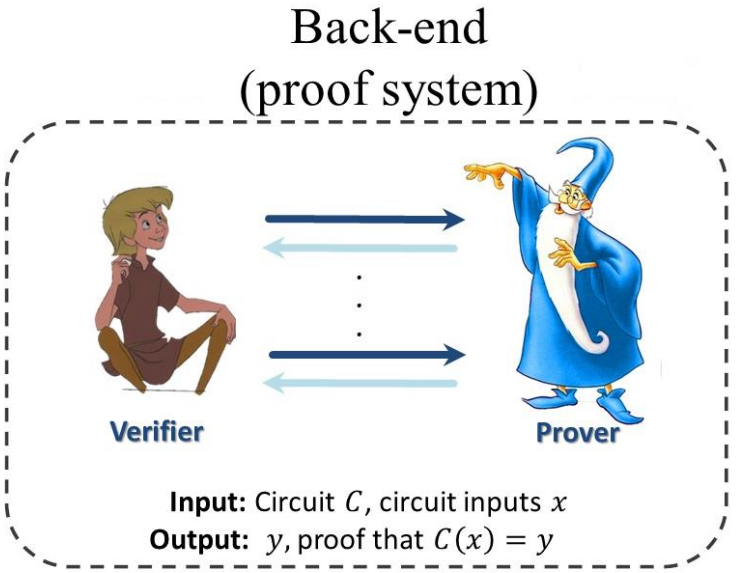
  Integers ($\mathbb{Z}$), Reals ($\mathbb{R}$), Polynomial Rings…



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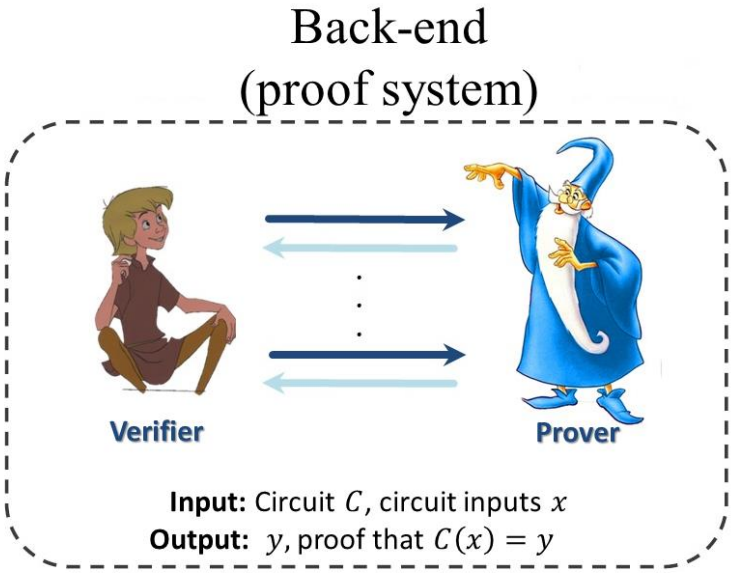

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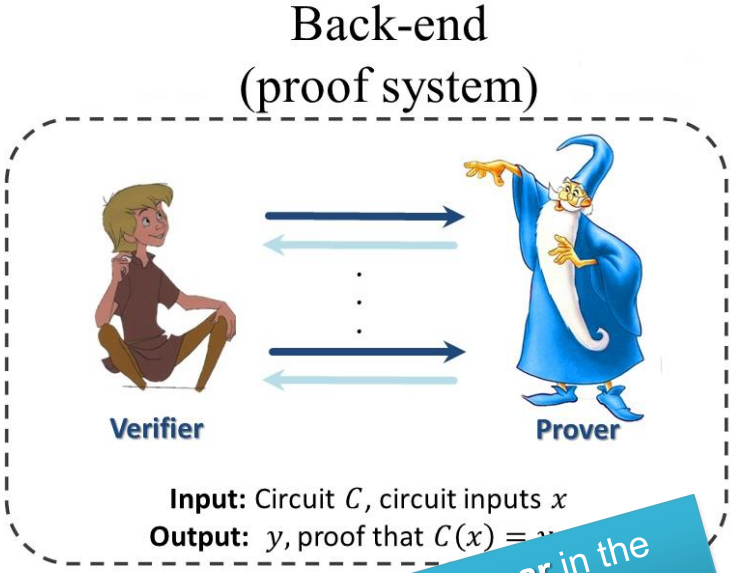

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↯

IEEE 754 floating point numbers.

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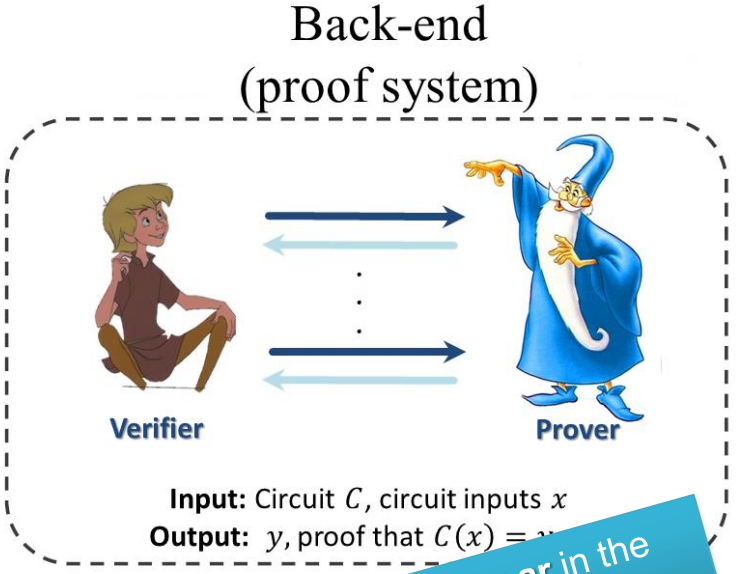

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Exact vs Approximate Computation

No prior back-end supported infinite rings (e.g. $\mathbb{R}$) $\Rightarrow$ Approximate arithmetic (e.g. floating point).

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ISSUES:

Approx. arith. does ***not*** constitute a ring:
Neither ***associative*** nor ***distributive***!
I.e. $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c; a \cdot (b + c) \neq ab + ac$

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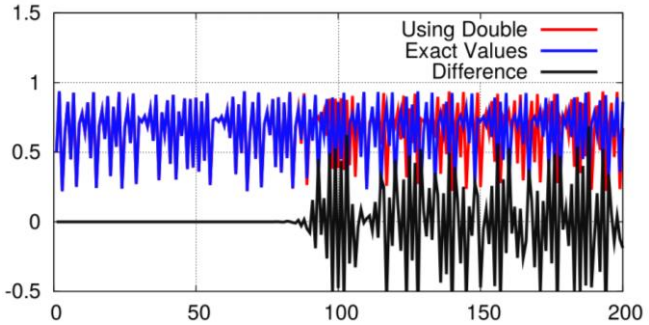

Logistic map: Double precision vs exact arithmetic.

$x_0 = 0.5;\ x_{n+1} = 3.75 \cdot x_n \cdot (1 - x_n).$

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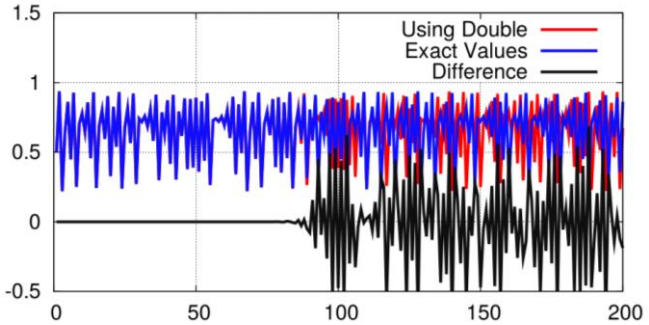

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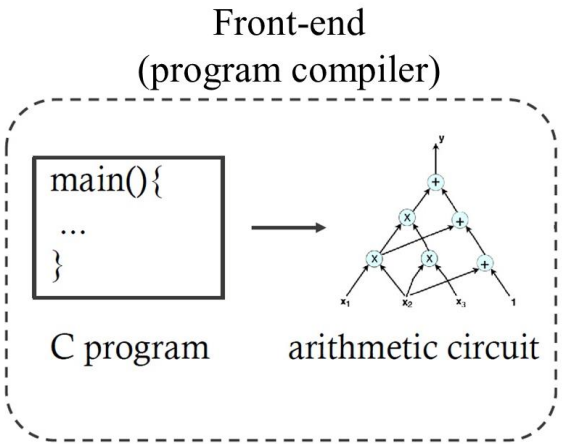
Malicious errors:

Circuit description biases the output.

How do we agree on the front-end?



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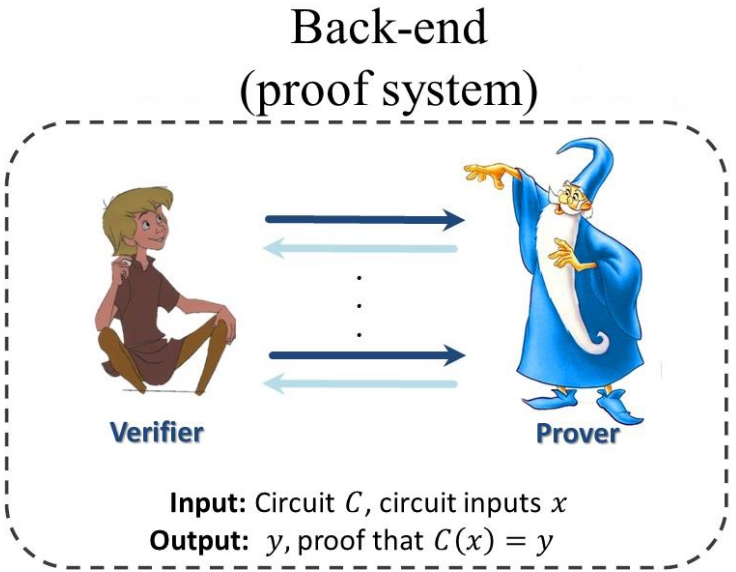

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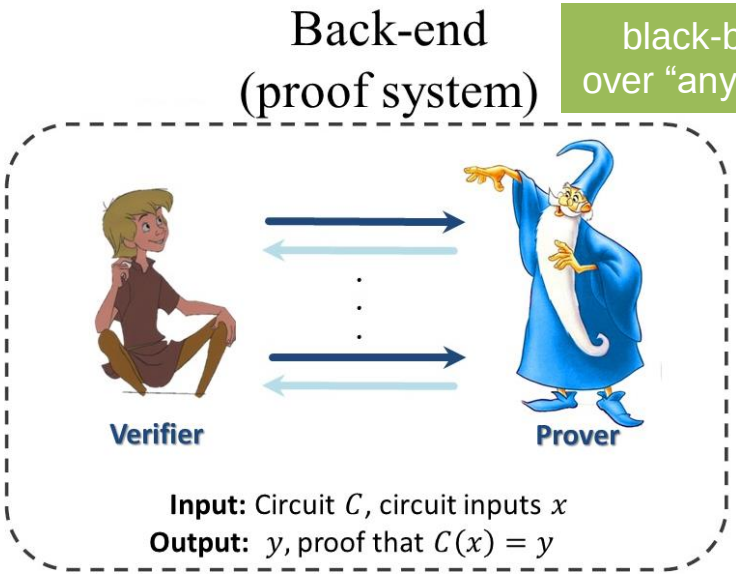

✓ **Theoretical** interest: Remove algebraic limitations, new rings with ***sublinear*** provers.

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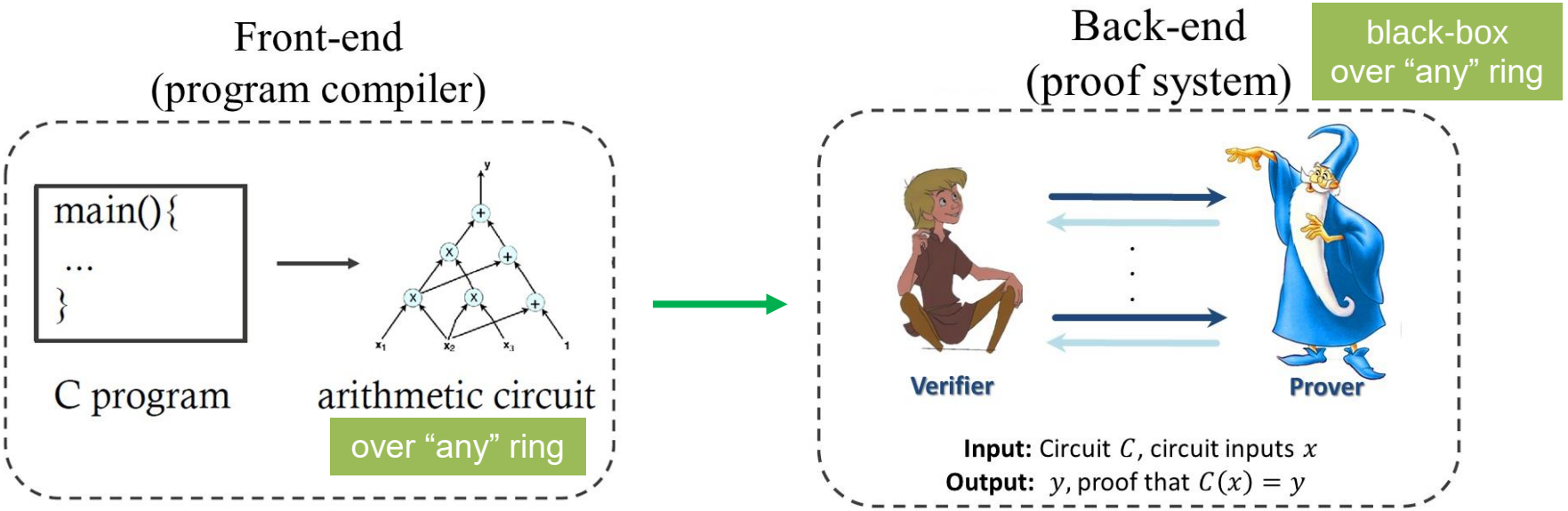

- ✓ **Theoretical** interest: Remove algebraic limitations, new rings with ***sublinear*** provers.
- ✓ ***Agile*** software*:* Simpler front-end, ring-specific libraries can be reused.

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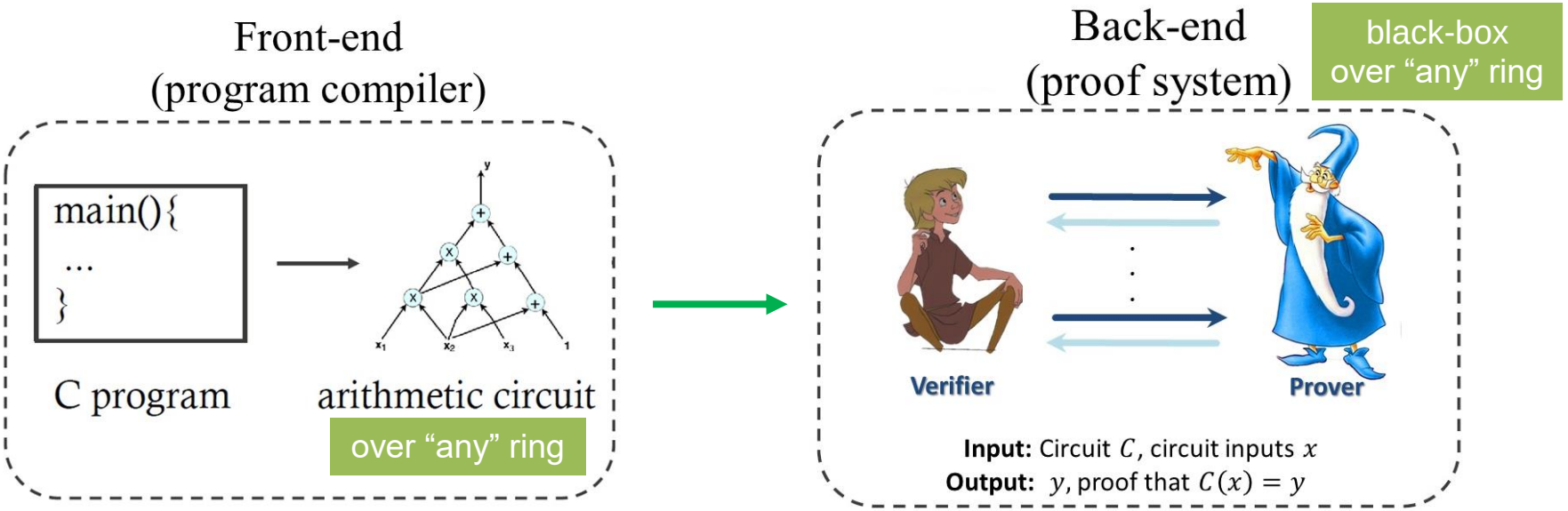

- ✓ **Theoretical** interest: Remove algebraic limitations, new rings with ***sublinear*** provers.
- ✓ ***Agile*** software*:* Simpler front-end, ring-specific libraries can be reused.
- ✓ Enable ***exact computation***: Impossible before, approx. arithmetic is an attack vector!

Technology Innovation Institute

Technical details sneak peek

“Any” ring R

black-box over “any” ring

R contains a big enough set $\mathbf{A} = \{a_1, \ldots a_n\}$ such that:
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Easily follow if $\mathbf{A} \subseteq Z(R)$
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What happens if we relax $\mathbf{A}$ to commutative set?
(i.e.$\forall a_i, a_j \in A, a_i \cdot a_j = a_j \cdot a_i$)

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For a commutative set $\mathbf{A}$, we define a new polynomial ring $\boldsymbol{R[X]}$

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- Actually, we can only bound roots in $\mathbf{A}$ for a ***subset*** of polys in our new $\boldsymbol{R[X]}$.
- Need to modify layer-consistency equation in GKR and sumcheck so as to always encounter polynomials in the right subset.

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For a commutative set $\mathbf{A}$, we define a new polynomial ring $\boldsymbol{R[X]}$

- **Core idea:** $\boldsymbol{X}$ commutes with whatever the set $\mathbf{A}$ does.
- Actually, we can only bound roots in $\mathbf{A}$ for a ***subset*** of polys in our new $\boldsymbol{R[X]}$.
- Need to modify layer-consistency equation in GKR and sumcheck so as to always encounter polynomials in the right subset.
- We provide a linear-time algorithm for the prover.

Euclidean division
for polynomials $\boldsymbol{f \in R[X]}$

Polynomial evaluation
$\boldsymbol{Ev_a: R[X] \to R}$ at any $\mathbf{a \in A}$
is a ring homomorphism

Contribution recap

Thm: Let **R** be a (possibly infinite, possibly non-commutative) ring containing a **commutative**, **regular difference set A**.
There exists a doubly-efficient IP for layered arithmetic circuits over **R**.

Contribution recap

Thm: Let **R** be a (possibly infinite, possibly non-commutative) ring containing a **commutative**, **regular difference set A**.
There exists a doubly-efficient IP for layered arithmetic circuits over **R**.

- ✓ **Theoretical** interest: Remove algebraic limitations, new rings with ***sublinear*** provers.
- ✓ ***Agile*** software*:* Simpler front-end, ring-specific libraries can be reused.
- ✓ Enable ***exact computation***: Impossible before, approx. arithmetic is an attack vector!

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Thanks for your attention!

Questions?

Full version: ia.cr/2022/587

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