# FESTA: Fast Encryption from Supersingular Torsion Attacks

Andrea Basso, Luciano Maino, Giacomo Pope

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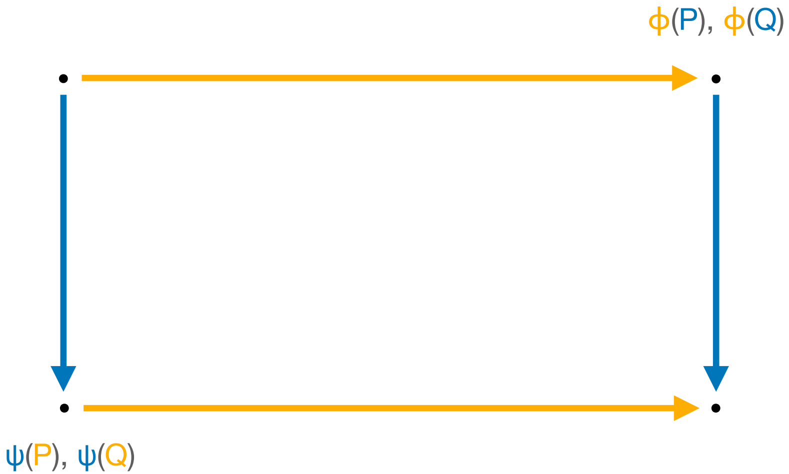
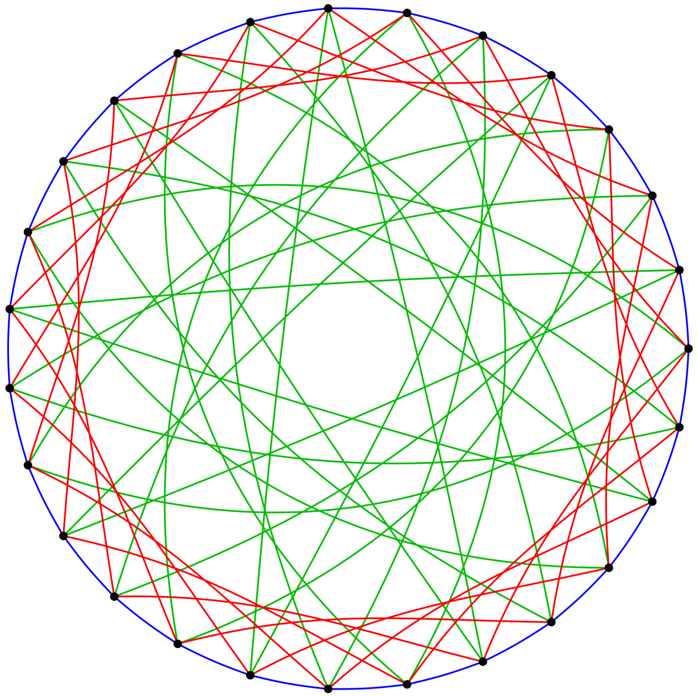
# Isogeny-based encryption



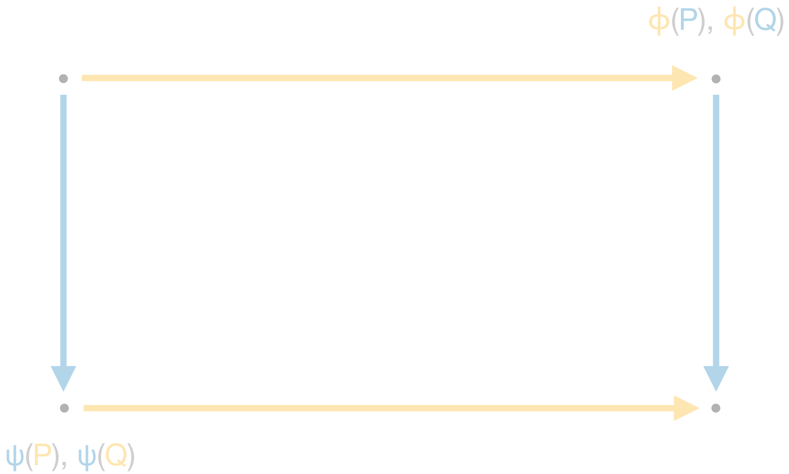

**SIDH**



**CSIDH**

The CSIDH image is based on Fig. 1 from *CSIDH: An Efficient Post-Quantum Commutative Group Action*

# Isogeny-based encryption

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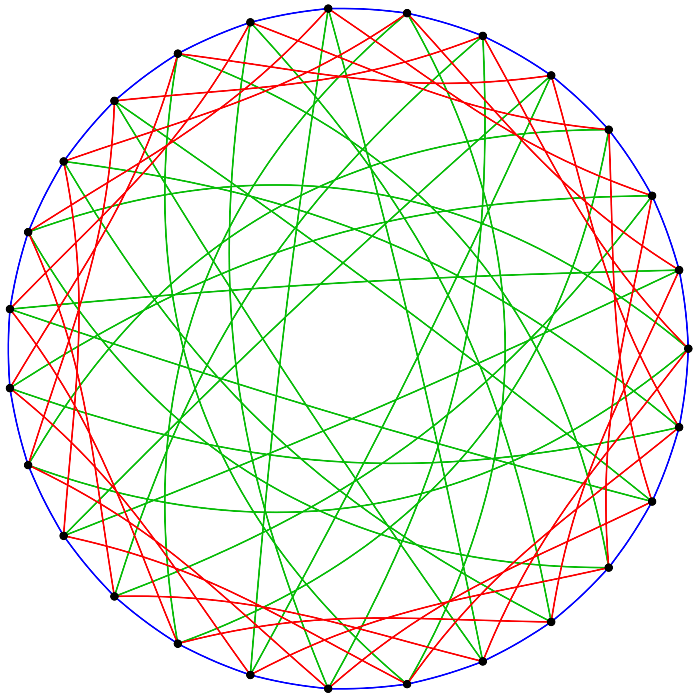

CSIDH

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# The attacks on SIDH

$E_0, E_1$
$\deg \phi$
$P, Q, \phi(P), \phi(Q)$

SIDH attacks

$\phi : E_0 \to E_1$

# A new assumption

## Scaling torsion points prevents attacks

$$\begin{matrix} P_0 \\ Q_0 \end{matrix} \;\bullet \xrightarrow{\quad \phi \quad} \bullet\; \begin{matrix} P_1 \\ Q_1 \end{matrix} = \begin{matrix} \phi(P_0) \\ \phi(Q_0) \end{matrix}$$

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$$\bullet \xrightarrow{\phi} \bullet$$

$$\begin{array}{l} P_0 \\ Q_0 \end{array} \qquad \begin{array}{l} P_1 \\ Q_1 \end{array} = \begin{array}{l} [\alpha]\ \phi(P_0) \\ [\beta]\ \phi(Q_0) \end{array}$$

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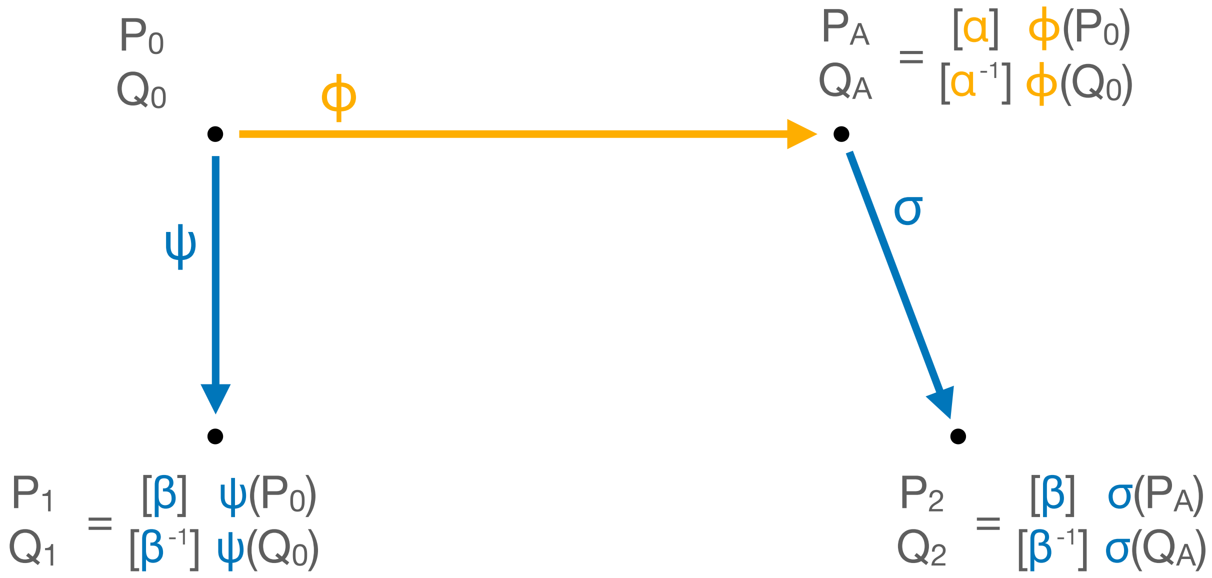


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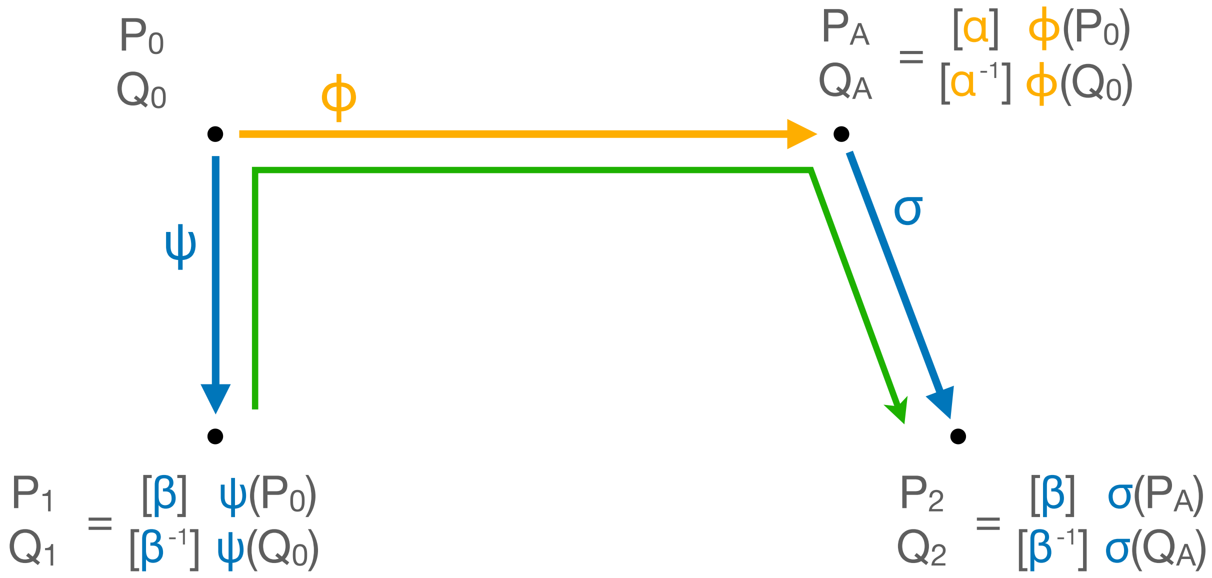


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$$\Rightarrow \begin{matrix} \hat{\psi}\phi\sigma(P_1) = [\deg\psi][\alpha^{-1}]P_2 \\ \hat{\psi}\phi\sigma(Q_1) = [\deg\psi][\alpha]Q_2 \end{matrix}$$

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$$f_{E_0,\, P_0,\, Q_0,\, E_A,\, P_A,\, Q_A}(\underbrace{\psi, \sigma, \beta}_{\text{partial-domain one-way}}) \;=\; E_1,\, P_1,\, Q_1,\, E_2,\, P_2,\, Q_2$$

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$$\Downarrow$$

**OAEP transform**

IND-CCA2 security in the QROM

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**OAEP transform**

IND-CCA2 security in the QROM

**Encrypt**

1. Sample random rnd
2. $\psi = (m \,||\, 0\ldots0) + H(\text{rnd})$
3. $\sigma,\ \beta = G(\psi) + \text{rnd}$
4. $\text{ct} = f(\psi,\sigma,\beta)$

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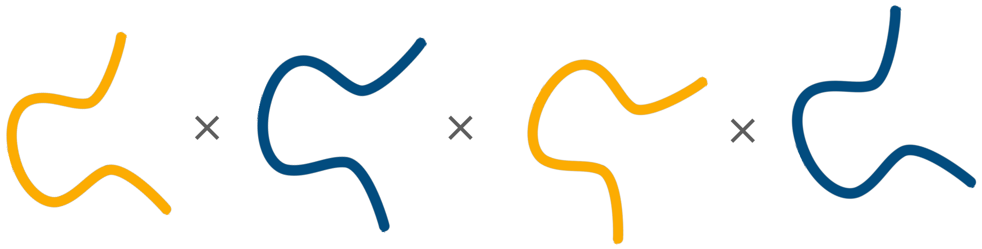
**Decrypt**

1. Compute $\psi, \sigma, \beta$
2. $\text{rnd} = G(\psi) - (\sigma, \beta)$
3. $(m \,||\, 0\ldots0) = \psi - H(\text{rnd})$

# There are attacks and attacks

Dimension two

Dimension four (and higher)

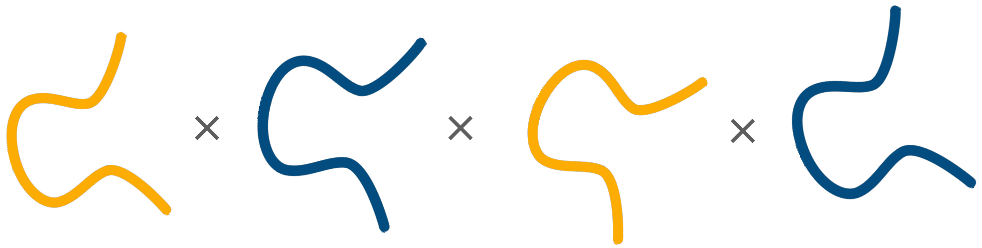


# There are attacks and attacks

**Dimension two**

- Fast and simple implementation
- Strict degree requirements

**Dimension four (and higher)**



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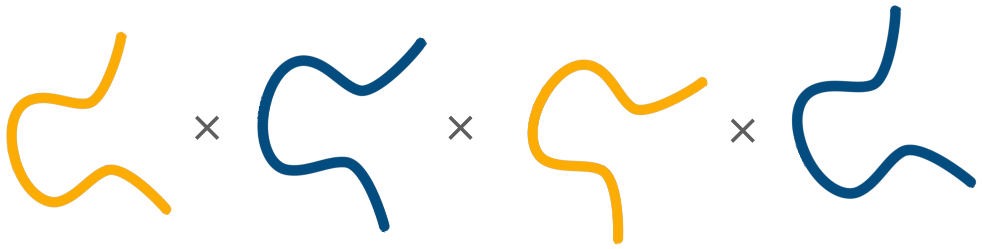

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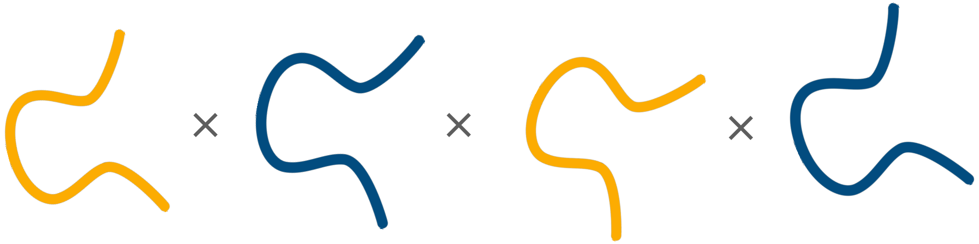

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- Small parameters ($p \approx 2^{400}$)
- Fast KeyGen and Encrypt, slow Decrypt

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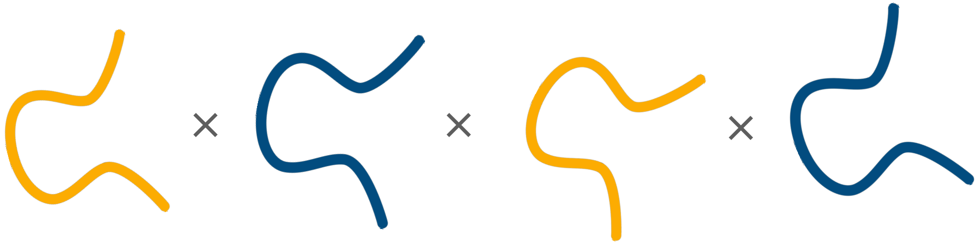

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# There are attacks and attacks

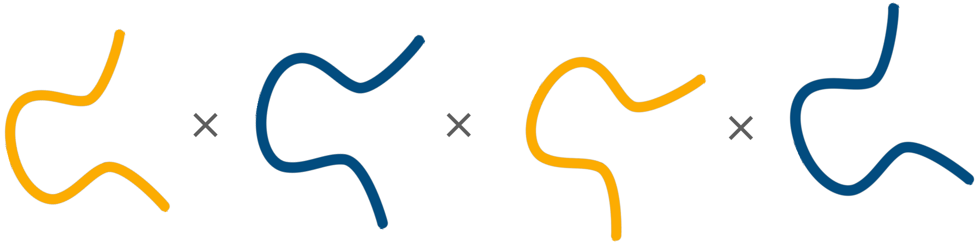
### Dimension two

- Fast and simple implementation
- Strict degree requirements

$\Downarrow$

- Larger parameters ($p \approx 2^{1300}$)
- Practical running times

### Dimension four (and higher)



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Running times of the attack dominated by the smoothness of the order of torsion points

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Attacks in dimension two require that

$$\deg(\psi\phi\sigma) + c = 2^b$$

for c smooth and computable

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$$\deg(\psi\phi_1) + \deg(\phi_2\sigma) = 2^b$$

with $\phi = \phi_1\phi_2$

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ord $P$ = ord $Q$ = $2^b$

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Ad-hoc approach based on Cornacchia's algorithm

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Ad-hoc approach based on Cornacchia's algorithm

$\Rightarrow$

$\deg(\psi)$, $\deg(\sigma)$ are $2^{12}$ smooth, $b = 632$,

$\deg(\phi)$ is $2^{16}$ smooth, $p \approx 2^{1292}$

# Results

```
andrea@MacBook-Pro FESTA-SageMath % sage example_festa.sage
================================================================================
                               Running FESTA_128
================================================================================

================================================================================
                           Keygen took: 4.467  seconds
================================================================================
--------------------------------------------------------------------------------
                        Compressed public key: 561 bytes
--------------------------------------------------------------------------------

================================================================================
                          Encrypt took: 3.057  seconds
================================================================================
--------------------------------------------------------------------------------
                       Compressed ciphertext: 1122 bytes
--------------------------------------------------------------------------------

================================================================================
                          Decrypt took: 10.102 seconds
================================================================================
```

# Conclusion

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New constructive framework based on the SIDH attacks

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2. New isogeny-based PKE scheme from more conservative assumptions

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**Paper**

https://eprint.iacr.org/2023/660.pdf

**Source Code**

https://github.com/FESTA-PKE/FESTA-SageMath