Hidden Stabilizers, the Isogeny To Endomorphism Ring Problem and the Cryptanalysis of pSIDH

Asiacrypt 2023, Guangzhou

Mingjie Chen, Muhammad Imran, Gábor Ivanyos, Péter Kutas, Antonin Leroux, Christophe Petit

University of Birmingham

December 6, 2023

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Introducing the IsERP

Theoretical background

Roadmap

Reductions and the resolution of the IsERP

Hard problems in isogeny-based cryptography

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Problems A, B are equivalent.

Question: what if we consider a more general version of Problem B where we know (*the isogeny representation of*) an isogeny between E and E'?

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- Suborder representation — introduced in pSIDH key exchange [Leroux 2022], can be used to represent isogenies of **large prime degrees**.
- High dimension representation — introduced after the SIDH attacks [Robert 2022], can be used for **arbitrary degree** isogeny.

The IsERP

Problem (IsERP)

Let E_0, E be supersingular elliptic curves over $\mathbb{F}_{p^2}$ and $\varphi : E_0 \to E$ be an isogeny of degree N. Given the endomorphism ring $\mathrm{End}(E_0)$ and an isogeny representation of φ, compute $\mathrm{End}(E)$.

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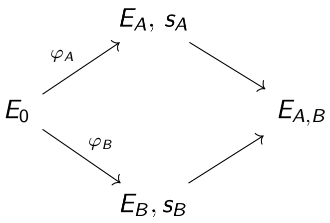

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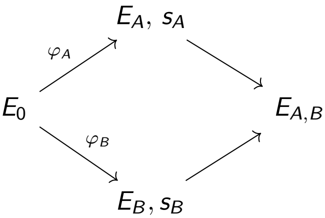

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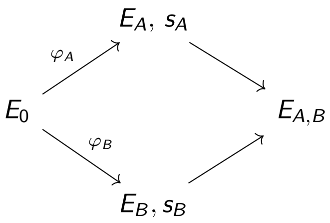

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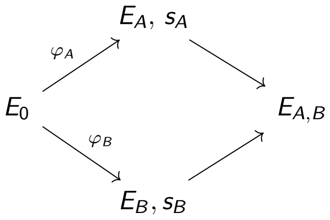

- E_0 is the public curve whose endomorphism ring is known
- $\varphi : E_0 \to E$ is the secret isogeny of large prime degree N that Alice (or Bob) computes
- s is the suborder representation of φ which is the embedding of $\mathbb{Z} + N\,\mathrm{End}(E_0)$ into $\mathrm{End}(E)$ induced by φ that allows Bob (or Alice) to compute $E_{A,B}$.

Deuring Correspondence

Supersingular elliptic curve **VS** Quaternion algebra

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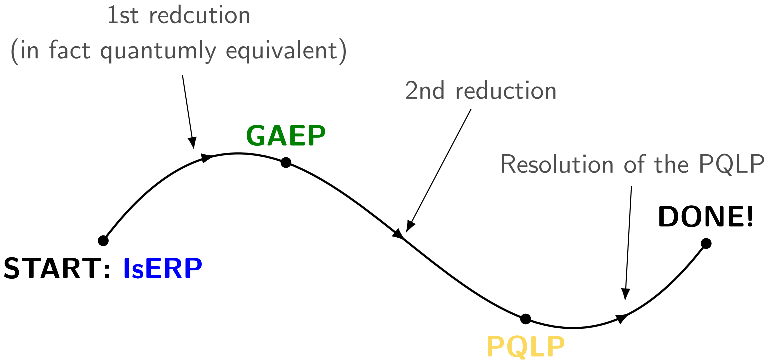
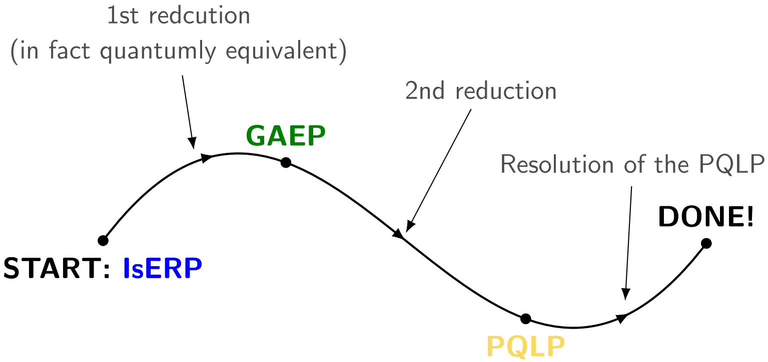
Supersingular elliptic curve **VS** Quaternion algebra

Supersingular j-invariants over $\mathbb{F}_{p^2}$ $j(E)$ (up to Galois conjugacy)	Maximal orders in $\mathcal{B}_{p,\infty}$ $\mathcal{O} \cong \mathrm{End}(E)$ (up to isomorphism)
(E_1, φ) with $\varphi : E_0 \to E_1$	I_φ integral left $\mathcal{O}_0$-ideal and right $\mathcal{O}_1$-ideal
$\theta \in \mathrm{End}(E_0)$	Principal ideal $\mathcal{O}_0\theta$
$\deg(\varphi)$	$n(I_\varphi)$

Roadmap


1st redcution
(in fact quantumly equivalent)
2nd reduction
Resolution of the PQLP
GAEP
DONE!
START: IsERP
PQLP

Roadmap



- Isogeny to Endomorphism Ring Problem (**IsERP**)

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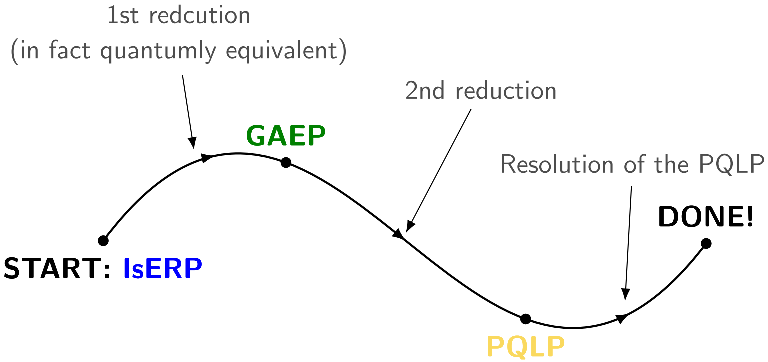

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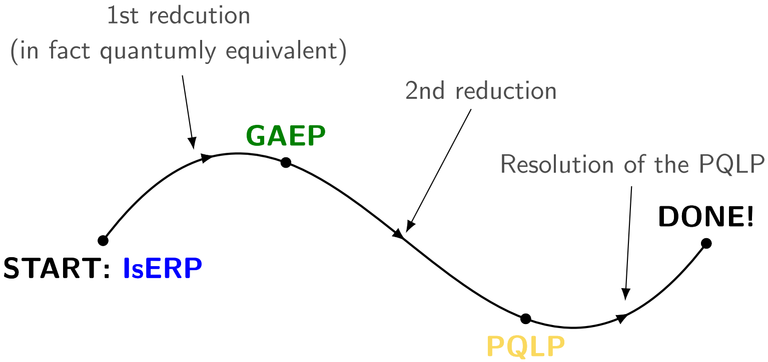

- Isogeny to Endomorphism Ring Problem (**IsERP**)
- Group Action Evaluation Problem (**GAEP**)
- Powersmooth Quaternion Lifting Problem (**PQLP**)

1st reduction

— the group action evaluation problem (GAEP)

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \curvearrowright \{\text{cyclic order } N \text{ subgroups of } \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}\}$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \star \langle (m, n) \rangle = \langle (am + bn, cm + dn) \rangle$$

$$\curvearrowright \{\text{cyclic subgroups of order } N \text{ of } E_0[N]\}$$

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Problem (GAEP)

Let E_0, E be supersingular elliptic curves over $\mathbb{F}_{p^2}$ and let $\varphi : E_0 \to E$ be an isogeny of degree N. Given $\mathrm{End}(E_0)$ and its corresponding quaternion order $\mathcal{O}_0$, a representation of φ and $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, find an isogeny representation of $g \star \varphi$.

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Given the embedding of $\mathbb{Z} + N\,\mathrm{End}(E_0)$ into $\mathrm{End}(E)$, compute the embedding of $\mathbb{Z} + N\,\mathrm{End}(E_0)$ into the endomorphism ring of the codomain curve of $g \star \varphi$.

1st reduction

— introducing the Stabilizer Subgroup

Consider the action

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Proposition

Let $\varphi : E_0 \to E$ be an isogeny of degree N. The stabilizer subgroup $Stab_\varphi$ is conjugate of the subgroup of upper triangular matrices (i.e., a Borel subgroup).

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— another look at the Stabilizer Subgroup

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Upon fixing a basis of $E_0[N]$, we have an isomorphism

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Note that here we are abusing notations by viewing $\theta \in \mathcal{O}_0$ under the isomorphism $\mathcal{O}_0 \cong \mathrm{End}(E_0)$.

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— reducing IsEPR to GAEP

Theorem

The **IsERP** *reduces to the* **GAEP** *in quantum polynomial time.*

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Knowing how to evaluate the action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ *on* $\varphi : E_0 \to E$ *of degree* N*, the goal is to compute the endomorphism ring of* E*.*

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III: *Compute the right order of* I_φ*.*

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Knowing $\mathrm{End}(E_0), \mathrm{End}(E), N, \varphi$ *and* $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, *we need to compute a representation of* $g \star \varphi$.

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- I: *Compute the ideal* I_φ *corresponding to* φ.
- II: *Find* $\theta \in \mathrm{End}(E)$ *such that* $g_\theta = g$.
- III: $I_{g\star\varphi} = \sigma(I_\phi \cap \mathcal{O}\sigma)\sigma^{-1} + N\mathcal{O}$ *(where we take* $\mathcal{O} \cong \mathrm{End}(E)$ *and* $\sigma \in \mathcal{O}$ *corresponds to* θ*).*

2nd reduction

— the powersmooth quaternion lifting problem (PQLP)

Problem (PQLP)

Let $\mathcal{O}$ be a maximal order in $\mathcal{B}_{p,\infty}$. Given an integer N and an element $\sigma_0 \in \mathcal{O}$ such that $(n(\sigma_0), N) = 1$, find $\sigma = \lambda\sigma_0 \bmod N\mathcal{O}$ of powersmooth norm with some λ coprime to N.

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Fix a curve E_0 and let $\mathcal{O}_0 \cong \mathrm{End}(E_0)$:

$$\sigma \in (\mathcal{O}_0/N\mathcal{O}_0)^\times \longleftrightarrow \theta \in (\mathrm{End}(E_0)/N\,\mathrm{End}(E_0))^\times$$

$$(\mathcal{O}_0/N\mathcal{O}_0)^\times \longleftrightarrow g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \longleftrightarrow (\mathrm{End}(E_0)/N\,\mathrm{End}(E_0))^\times$$

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Note that $\ker(g \star \varphi) = \theta(\ker \varphi)$.

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Consider the commutative diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{\theta} & E_0 \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle [\theta]^*\varphi} \\ E & \xrightarrow{[\varphi]^*\theta} & E' \end{array}$$

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Theorem

The **GAEP** *reduces to the* **PQLP** *in classical polynomial time.*

Resolution of the PQLP

Problem (PQLP)

Let $\mathcal{O}$ be a maximal order in $\mathcal{B}_{p,\infty}$. Given an integer N and an element $\sigma_0 \in \mathcal{O}$ such that $(n(\sigma_0), N) = 1$, find $\sigma = \lambda\sigma_0 \bmod N\mathcal{O}$ of powersmooth norm with some λ coprime to N.

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    - Find $\alpha_1, \alpha_2, \alpha_3 \in Rj$ such that $\sigma_0 = \alpha_1\gamma\alpha_2\gamma\alpha_3 \bmod N\mathcal{O}$.

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Thank you!