Revisiting Higher-Order Differential-Linear Attacks from an Algebraic Perspective

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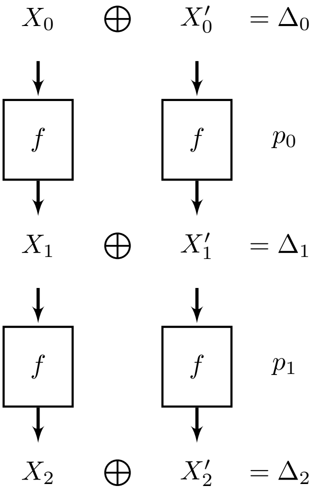
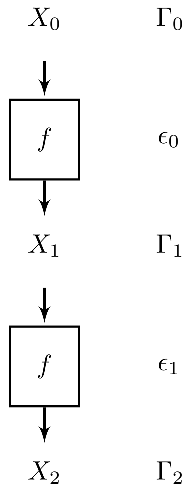
Contents

Differential Cryptanalysis



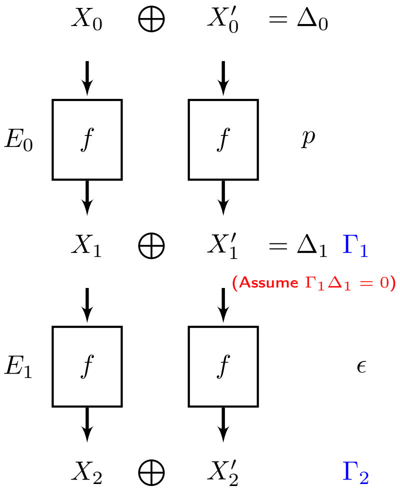
- Proposed by [BS,Crypto'91]
- Probability: $\Delta_0 \to \Delta_2$
- Traditionally studied using statistical method
  - Probability: $\Delta_0 \to \Delta_1$ with p_0
  - Probability: $\Delta_1 \to \Delta_2$ with p_1
  - Probability: $p = p_0 p_1$

Linear Cryptanalysis



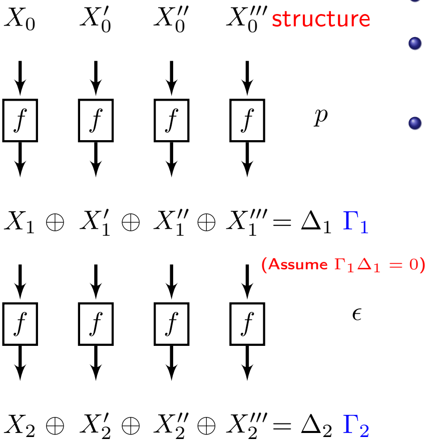
- Proposed by [Mat,Eurocypt'93]
- Correlation: $\Gamma_0 \rightarrow \Gamma_1$
- Traditionally studied using statistical method
  - Correlation: $\Gamma_0 \rightarrow \Gamma_1$ with ϵ_0
  - Correlation: $\Gamma_1 \rightarrow \Gamma_2$ with ϵ_1
  - Correlation: $\epsilon = \epsilon_0 \epsilon_1$

Differential-Linear Cryptanalysis



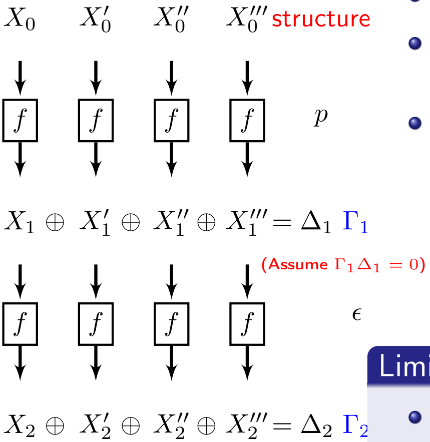
- Proposed by [LH,Crypto'94]
- Cor.: $\Gamma_2(X_2 \oplus X_2')$ w/ $X_0 \oplus X_0' = \Delta_0$
- Traditionally studied using statistical 2-phase method
  - $E = E_1 \circ E_0$
  - Probability: $\Delta_0 \to \Delta_1$ with p
  - Correlation: $\Gamma_1 \to \Gamma_2$ with ϵ
  - DL correlation: $p\epsilon^2$

Higher-Order Differential-Linear Cryptanalysis



- Proposed by [BDK,FSE'05]
- Cor.: $\Gamma_2 \left(\bigoplus X_2\right)$ with $X_0, X_0', \ldots$ being a HD structure
- Traditionally studied using statistical 2-phase method
  - $E = E_1 \circ E_0$
  - Probability of HD of E_0 is p
  - Correlation: $\Gamma_1 \to \Gamma_2$ with ϵ
  - Correlation of HDL: $p\epsilon^{2^d}$

Higher-Order Differential-Linear Cryptanalysis



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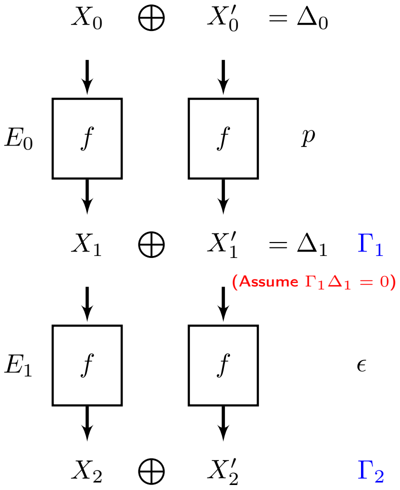
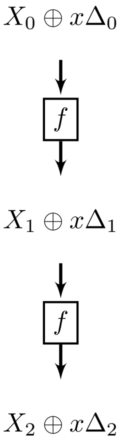
Limitations

- No method for a probabilistic HD
- If $\epsilon < 1$, HDL correlation goes to zero



- Proposed by [LLL,Crypto'21]
- An algebraic perspective
  - DL cor. $=$ cor. of $\Gamma_2(X_2 \oplus X_2')$

Algebraic Transitional Form



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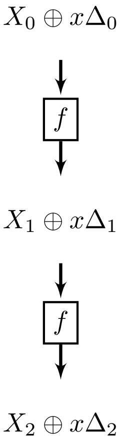

- Proposed by [LLL,Crypto'21]
- An algebraic perspective
  - DL cor. $=$ cor. of $\Gamma_2(X_2 \oplus X_2')$
- The form of output difference can be derived from a recursive method
  - X_1, Δ_1 are functions of X_0
  - $\Gamma_2(X_2 \oplus X_2')$ is a function of X_1, Δ_1

Contents

Contributions

- HATF: to generalize the ATF to the higher-order case
  - HATF can predict the probabilistic bias of a HDL approximation
  - New distinguishers/key-recovery attacks on Ascon and Xoodyak
- DSF: to linearize Ascon permutation
  - Improved zero-sum distinguishers for Ascon permutations

Contents

HD of a Boolean function [Lai, 1994]

$f : \mathbb{F}_2^n \to \mathbb{F}_2$ and an ℓ^{th}-order input difference $\mathbf{\Delta} = (\Delta_0, \ldots, \Delta_{\ell-1})$ for a certain input $X \in \mathbb{F}_2^n$. The ℓ^{th} derivative of f is calculated as

$$\mathcal{D}_{\mathbf{\Delta}} f(X) = \bigoplus_{a \in X \oplus \mathsf{span}(\mathbf{\Delta})} f(a)$$

HDL Cryptanalysis from an Algebraic Perspective

HD of a Boolean function [Lai, 1994]

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Prop. (Algebraic Perspective on HD/HDL)

Let

$$\mathcal{M} : \mathbb{F}_2^\ell \to X \oplus \mathsf{span}(\boldsymbol{\Delta})$$
$$(x_0, x_1, \ldots, x_{\ell-1}) \mapsto X \oplus x_0\Delta_0 \oplus \cdots \oplus x_{\ell-1}\Delta_{\ell-1} \triangleq X \oplus \boldsymbol{x}\boldsymbol{\Delta}$$

We have

$$\mathcal{D}_{\boldsymbol{\Delta}} f(X) = \bigoplus_{\boldsymbol{x} \in \mathbb{F}_2^\ell} f(X \oplus \boldsymbol{x}\boldsymbol{\Delta}) = D_{\boldsymbol{x}} f(X \oplus \boldsymbol{x}\boldsymbol{\Delta})$$

HDL Cryptanalysis from an Algebraic Perspective

HD of a Boolean function [Lai, 1994]

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Expression of HD: Coe $(f(X \oplus \boldsymbol{x}\boldsymbol{\Delta}), \boldsymbol{x})$

Higher-Order Algebraic Transitional Form

Iterative Cipher

Ciphers are iterative composed of simple round functions

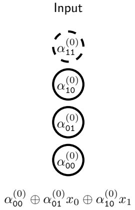
$$E = E_{R-1} \circ E_{R-2} \circ \cdots E_1 \circ E_0, \quad E_r : \mathbb{F}_2^n \to \mathbb{F}_2^n$$

We can construct the expression of HD in an iterative method

Construction of Higher-Order Algebraic Transitional Form

Write $X \oplus \boldsymbol{x}\boldsymbol{\Delta}$ as $\bigoplus_{u \in \mathbb{F}_2^\ell} \alpha_u \boldsymbol{x}^u$:

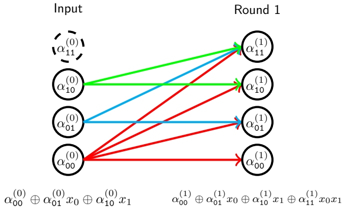
$$\alpha_u = \begin{cases} X, & u = 0 \\ \Delta_i, & u = e_i \\ \mathbf{0}, & \text{otherwise} \end{cases}$$



Construction of Higher-Order Algebraic Transitional Form

Apply E_r to $\bigoplus_{u \in \mathbb{F}_2^\ell} \alpha_u^{(r)} \boldsymbol{x}^u$

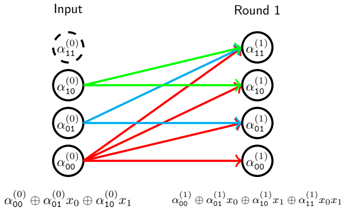
$$\bigoplus_{u \in \mathbb{F}_2^\ell} \alpha_u^{(r+1)} \boldsymbol{x}^u = E_r \left(\bigoplus_{u \in \mathbb{F}_2^\ell} \alpha_u^{(r)} \boldsymbol{x}^u \right)$$



Construction of Higher-Order Algebraic Transitional Form

$\alpha_u^{(r+1)}$ is a function of $\alpha_u^{(r)}$

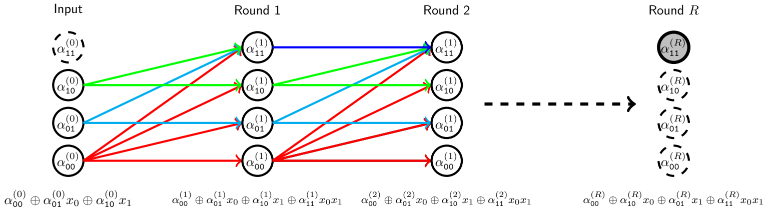
$$\alpha_u^{(r+1)} = \mathsf{Coe}\left(E_r\left(\bigoplus_{u\in\mathbb{F}_2^\ell}\alpha_u^{(r)}\boldsymbol{x}^u\right), \boldsymbol{x}^u\right)$$



Construction of Higher-Order Algebraic Transitional Form

Connecting all round functions, we obtain HATF of E,

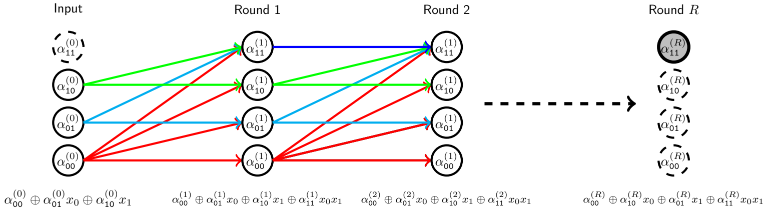
$$\mathcal{E} = \mathcal{E}_{R-1} \circ \mathcal{E}_{R-2} \circ \cdots \circ \mathcal{E}_0, \quad \mathcal{E}_r : (\mathbb{F}_2^n)^{2^\ell} \to (\mathbb{F}_2^n)^{2^\ell}$$



Construction of Higher-Order Algebraic Transitional Form

Time complexity of constructing the HATF:

- Dominated by the calculations of ANFs round by round
- Most time-consuming step is to calculate the d-degree monomials for $\bigoplus_{u\in\mathbb{F}_2^\ell} \alpha_u^{(r)} \boldsymbol{x}^u$
- $2^{d\ell}$ multiplications/additions
- Final time complexity: $\mathcal{O}(2^{d\ell})$ (detailed analysis can be found in the paper)



Computing the Bias of HDL

$\alpha_{\mathbf{1}}^{(R)}$ is a composite form:

$$\left(\alpha_u^{(0)}, u \in \mathbb{F}_2^n\right) \xrightarrow{\mathcal{E}_0} \cdots \xrightarrow{\mathcal{E}_{R-2}} \left(\alpha_u^{(R-1)}, u \in \mathbb{F}_2^n\right) \xrightarrow{\mathcal{E}_{R-1}} \alpha_{\mathbf{1}}^{(R)}$$

Lemma (LLL, Crypto'21)

Assume the bias of $x_0, x_1, \ldots, x_{n-1}$ are $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$, respectively.

$$\text{Bias}(f) = \sum_{\substack{x_0, x_1, \ldots, x_{n-1} \\ s.t. f(x_0, \ldots, x_{n-1})=0}} \prod_{i=0}^{n-1} \left(\frac{1}{2} + (-1)^{x_i} \varepsilon_i\right) - \frac{1}{2}$$

- Time complexity is exponential in the number of variables in the ANF
- The number of variables is at most $d \times 2^\ell$
- Final time complexity: $\mathcal{O}(2^{\ell + d \times 2^\ell})$ (detailed analysis can be found in the paper)

Reduce the Complexity for Primitives with Quadratic Round Functions

- Primitives with quadratic round functions are more and more popular
- Higher-order differential related attacks are one of the main threats

Quadratic Boolean function can be transformed into a disjoint form [JA, 1977]

$$f = x_0x_1 + x_2x_3 \quad (\checkmark)$$
$$f = x_0x_1 + x_0x_2 \quad (\times)$$

Reduce the Complexity for Primitives with Quadratic Round Functions

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Quadratic Boolean function can be transformed into a disjoint form [JA, 1977]

$$f = x_0x_1 + x_0x_2(\times) \rightarrow f = x_0(x_1 + x_2) \xrightarrow{\text{Sub}} f = t_0t_1(\checkmark)$$

Reduce the Complexity for Primitives with Quadratic Round Functions

A quicker method

- Apply a linear substitution to all the variables to make f be disjoint

$$f = g \circ M(x_0, x_1, \ldots, x_{n-1})$$

- Compute the correlation of new variables by Piling-up lemma

$$y = x_0 \oplus x_1 \oplus x_2 \oplus \cdots$$

- Compute the correlation of each individual part

$$g = x_0 x_1 + x_0 + x_1 + 1$$

- Compute the correlation of f

$$f = g_0 \oplus g_1 \oplus g_2 \cdots$$

Reduce the Complexity for Primitives with Quadratic Round Functions

A quicker method

- The variable substitution is the most time-consuming: $\mathcal{O}(n^{3.8})$
  (n is the number of variables)
- The number of variables in an ANF is $2 \times 2^{\ell}$
- Final time complexity: $\mathcal{O}(2^{3.8\ell})$

Assumption Made for the Method

Assumption

- The construction of HATF does not require assumptions
- The calculation of bias of variables requires the variables to be independent

Trouble and Solution

- If a variable is linear, then it is more risky not to be independent

$$\alpha_{\boldsymbol{u}}^{(r+1)}[i] = \alpha_{\boldsymbol{u}}^{(r)}[i_0] \oplus \alpha_{\boldsymbol{u}}^{(r)}[i_1] \oplus \cdots$$

Assumption Made for the Method

Assumption

- The construction of HATF does not require assumptions
- The calculation of bias of variables requires the variables to be independent

Trouble and Solution

- If a variable is linear, then it is more risky not to be independent

Not introduce new variables

Assumption Made for the Method

Assumption

- The construction of HATF does not require assumptions
- The calculation of bias of variables requires the variables to be independent

Trouble and Solution

- Different bits of $\alpha_{\boldsymbol{u}}^{(r)}$ can be highly related

$$\alpha_{\boldsymbol{u}}^{(r)}[i] = \alpha_{\boldsymbol{u}}^{(r)}[j] \text{ or } \alpha_{\boldsymbol{u}}^{(r)}[i] = \alpha_{\boldsymbol{u}}^{(r)}[j] + 1$$

Assumption Made for the Method

Assumption

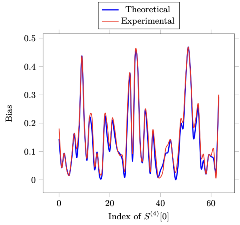
- The construction of HATF does not require assumptions
- The calculation of bias of variables requires the variables to be independent

Trouble and Solution

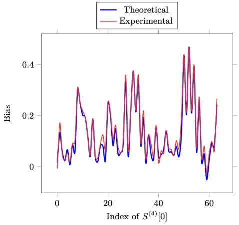
- Different bits of $\alpha_{\boldsymbol{u}}^{(r)}$ can be highly related

$$\alpha_{\boldsymbol{u}}^{(r)}[i] \text{ can be represented by } \alpha_{\boldsymbol{u}}^{(r)}[j]$$

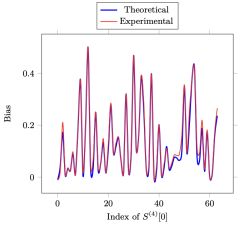
Some curves for 2nd order HDL of 4-round Ascon initialization



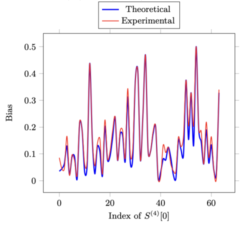
(a) $\Delta(0,1)$



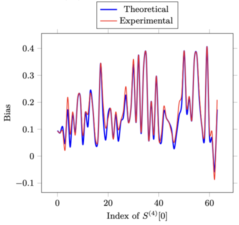
(b) $\Delta(0,2)$



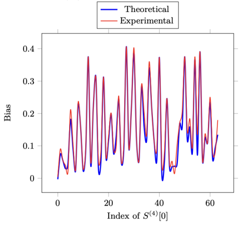
(c) $\Delta(0,3)$



(a) $\Delta(0,4)$



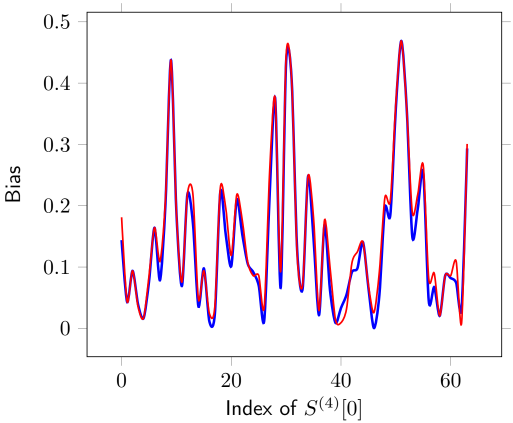
(b) $\Delta(0,5)$



(c) $\Delta(0,6)$

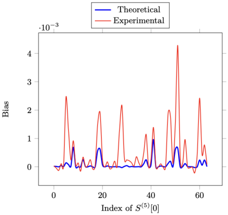
Precision of HATF

Curve of one 2^{th}-order HDL for 4-round Ascon initialization

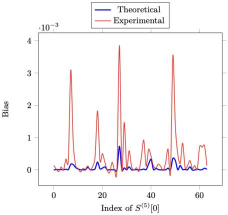


Precision of HATF

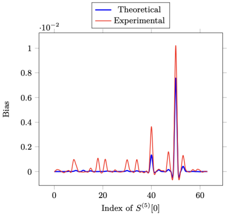
Some curves for 2nd order HDL of 5-round Ascon initialization:



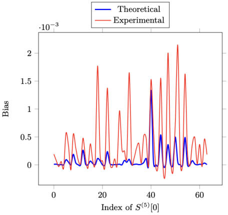
(a) $\Delta(0,1)$



(b) $\Delta(0,2)$



(c) $\Delta(0,3)$



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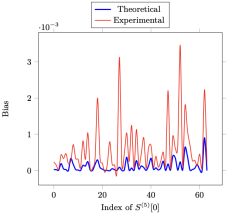
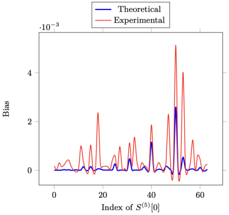


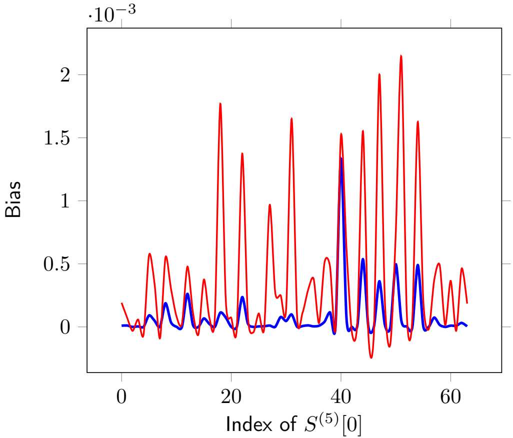
(b) $\Delta(0,5)$



(c) $\Delta(0,6)$

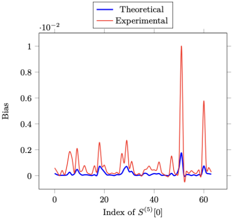
Precision of HATF

Curve of one 2^{nd}-order HDL for 5-round Ascon initialization

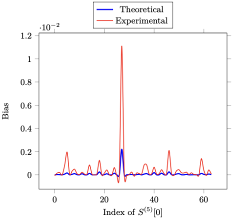


Precision of HATF

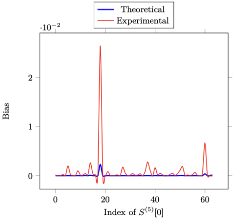
Some curves for 3rd to 8th order HDL of 5-round Ascon initialization:



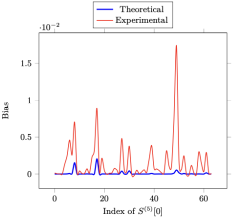
(a) $\Delta(0, 24, 33)$



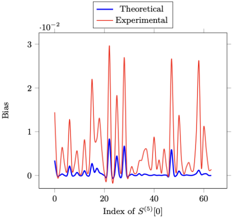
(b) $\Delta(0, 9, 15, 41)$



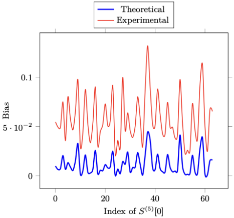
(c) $\Delta(0, 9, 24, 51, 55)$



(a) $\Delta(1, 12, 18, 22, 21, 52)$



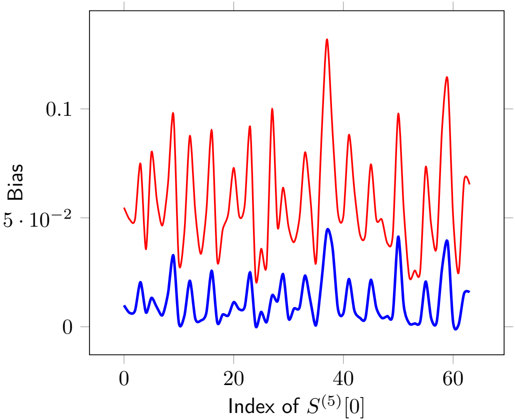
(b) $\Delta(10, 13, 21, 31, 49, 55, 61)$



(c) $\Delta(0, 8, 9, 13, 14, 26, 43, 60)$

Precision of HATF

Curve of one 8^{th}-order HDL for 5-round Ascon initialization



Precision of HATF

Discussion on Precision

- HATF CANNOT provide any upper/lower bound for HDL biases
- Quite precise to predict biased bits
- When the reported bias is high, the real bias is also high
- We have not observed any counterexamples during our experiments

Results

Results for Ascon initialization

Primitive	Round	Order	Bias Expr.	Bias Theory	Method	Reference
Ascon Init.	4	1^{st}	2^{-2}	2^{-20}	Classical	[DEMS, CT-RSA'15]
				2^{-5}	DLCT	[BDKW, Eurocrypt'19]
				$2^{-2.365}$	ATF	[LLL, Crypto'21]
				$\mathbf{2^{-2.09}}$	HATF	Here
		2^{nd}	2^{-1}	$\mathbf{2^{-1}}$	HATF	Here
	5	1^{st}	2^{-9}	–	Experimental	[DEMS, CT-RSA'15]
				$\mathbf{2^{-10}}$	HATF	Here
		2^{nd}	$2^{-6.60}$	$\mathbf{2^{-7.05}}$	HATF	Here
		8^{th}	$2^{-3.35}$	$\mathbf{2^{-4.73}}$	HATF	Here
	6	3^{rd}	$2^{-22}\dagger$	$\mathbf{2^{-25.97}}\dagger$	HATF	Here

† This bias holds when 24 conditions are satisfied

Results

Primitive	Round	Order	Bias		Method	Reference
			Expr.	Theory		
Xoodyak Init.	4	1^{st}	$2^{-9.7}$	–	Experimental	[DW, SAC'22]
				$\mathbf{2^{-9.67}}$	HATF	Here
			$-2^{-5.36}$	–	Experimental	[DW, SAC'22]
				$\mathbf{-2^{-6.0}}$	HATF	Here
		2^{nd}	$2^{-5.72}$	$\mathbf{2^{-5.72}}$	HATF	Here
		4^{th}	2^{-1}	$\mathbf{2^{-1}}$	HATF	Here
	5	2^{nd}	–	$\mathbf{2^{-45}}$	HATF	Here
Xoodoo	4	-	2^{-1}	2^{-1}	Rot. DL	[LSL, Eurocrypt'21]
		4^{th}	2^{-1}	$\mathbf{2^{-1}}$	HATF	Here
	5	3^{rd}	$2^{-8.79}$	$\mathbf{2^{-8.96}}$	HATF	Here

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Differential Supporting Function

We know:

$$\mathcal{D}_{\mathbf{\Delta}} f(X) = \bigoplus_{\boldsymbol{x} \in \mathbb{F}_2^{\ell}} f(X \oplus \boldsymbol{x}\mathbf{\Delta}) = D_{\boldsymbol{x}} f(X \oplus \boldsymbol{x}\mathbf{\Delta})$$

Differential Supporting Function

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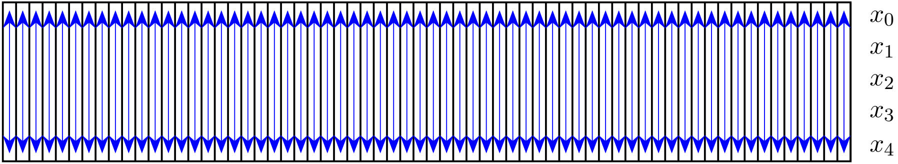
$$\mathcal{D}_{\boldsymbol{\Delta}} f(X) = \bigoplus_{\boldsymbol{x} \in \mathbb{F}_2^{\ell}} f(X \oplus \boldsymbol{x}\boldsymbol{\Delta}) = D_{\boldsymbol{x}} f(X \oplus \boldsymbol{x}\boldsymbol{\Delta})$$

- X and $\boldsymbol{\Delta}$ are parameters
- With X and $\boldsymbol{\Delta}$ being properly chosen, $D_{\boldsymbol{x}} f(X \oplus \boldsymbol{x}\boldsymbol{\Delta})$ can be made simpler

DSF on Ascon Permutation



- Intuition: Let all Sboxes have the same $\bar{X} + x\bar{\Delta}$
- $32 \times 31 = 992$ choices
- Evaluate the algebraic degree of r-round Ascon with $X = \bar{X}^{64}$, $\mathbf{\Delta} = \bar{\Delta}^{64}$

DSF on Ascon Permutation

$$(\bar{X}, \bar{\Delta}) \in \left\{ \begin{array}{l} (\texttt{0x6}, \texttt{0x13}), (\texttt{0xa}, \texttt{0x13}), (\texttt{0xc}, \texttt{0x17}), (\texttt{0xf}, \texttt{0x18}), \\ (\texttt{0x15}, \texttt{0x13}), (\texttt{0x17}, \texttt{0x18}), (\texttt{0x19}, \texttt{0x13}), (\texttt{0x1b}, \texttt{0x17}) \end{array} \right\}$$

Round r	**Upper bounds on the algebraic degree**				
	$S^{(r)}[0]$	$S^{(r)}[1]$	$S^{(r)}[2]$	$S^{(r)}[3]$	$S^{(r)}[4]$
4	3	3	2	2	3
5	6	5	5	6	6
6	11	11	12	12	11
7	23	24	23	23	22
8	47	47	45	46	47

Improved Zero-Sum Results for Ascon Permutation

New zero-sum distinguishers on Ascon permutation:

Type	Rnd	Data(log)	Time (log)	Method	Reference
From Start	8	130	130	Integral	[Todo, Eurocrypt'15]
		48	**48**	HD	Here
Best	11	315	315	Integral	[Todo, Eurocrypt'15]
Inside-outside	12	130	130	Zero-Sum	[Todo, Eurocrypt'15]
		55	**55**	Zero-Sum	Here

Discussion on the new zero-sum distinguishers

- The inputs (outputs) are fixed, so they are different from/weaker than the previous zero-sum distinguishers (derived from division property)
- More information is captured

Contents

Conclusion

- A generalization of the algebraic perspective on DL to HDL cases
- The first theoretical method for a probabilistic HDL distinguisher: HATF
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- A systematic method for linearization and finding zero-sum distinguishers for Ascon: DSF

Conclusion

- A generalization of the algebraic perspective on DL to HDL cases
- The first theoretical method for a probabilistic HDL distinguisher: HATF
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Thank You!