Concrete Analysis of Quantum Lattice Enumeration

Shi Bai[1] **Maya-Iggy van Hoof**[2] Floyd B. Johnson[1]
Tanja Lange[3] **Tran Ngo**[1]

[1]Florida Atlantic University, United States.

[2]Ruhr University Bochum, Germany.

[3]Eindhoven University of Technology, the Netherlands.

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Motivation

- Lattices are a popular choice for post-quantum cryptography.
- 2 popular generic attacks:
  - Sieving (faster, but exponential space).
  - **Enumeration** (slower, but polynomial space).
- Most literature is about asymptotics.
- Our work: concrete quantum look.

History of Work

[Bel13]	Belovs's quantum walk
[AK17, Mon18]	Quantum tree backtracking algorithm
[ADPS16, ABB+ 17, PLP16]	Quantum Enumeration is mentioned
[ANS18]	Presenting quantum lattice enumeration and its variations are discussed at a high level without going into detail about quantum circuits and resource estimates

Our work

Proposed a concrete implementation of Montanaro's algorithm for lattice enumeration, and its circuit resource estimates.

Quantum Algorithms

- Paper: gate level.
- This talk: overview.
- Quantum computers use quantum bits, **qubits**.

- We interact with qubits using quantum **gates**.

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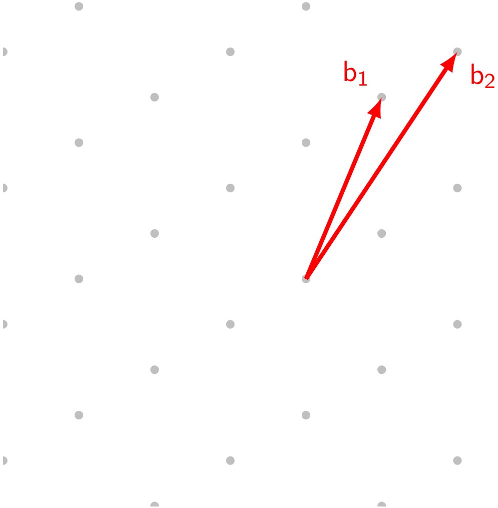
- Quantum efficiency: T-gates or T-depth.

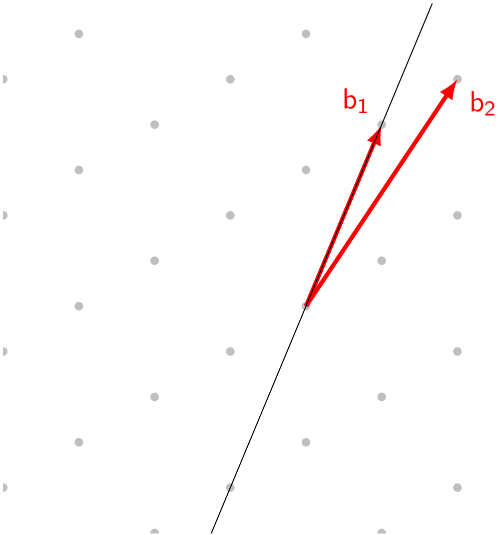
Enumeration



Visualization idea: Thijs Laarhoven.

Enumeration

- Pick one direction.



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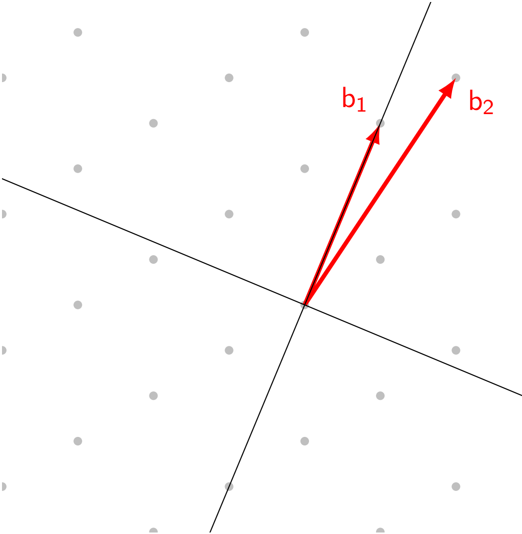
Enumeration

- Pick one direction.
- Consider directions orthogonal to it.



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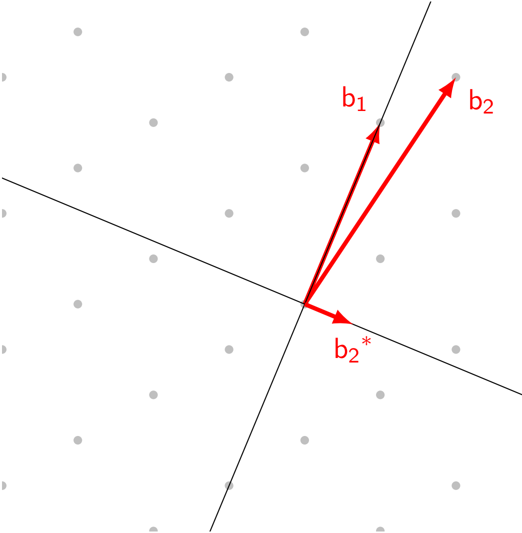
Enumeration

- Pick one direction.
- Consider directions orthogonal to it.
- Project other vector(s).



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Enumeration

- Pick one direction.
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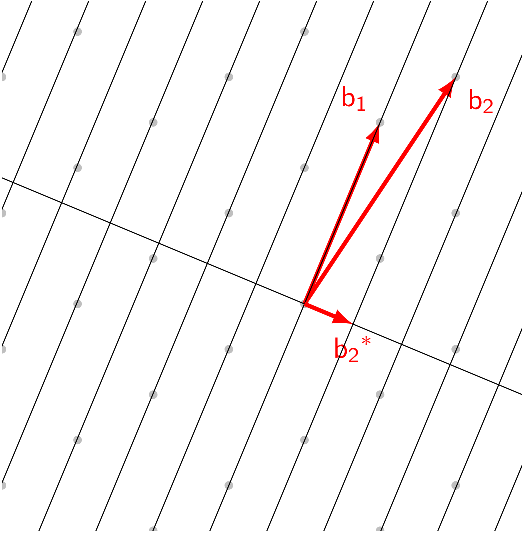
- Project other vector(s).
- Make a grid parallel to b_1 spaced by the length of ${b_2}^*$.



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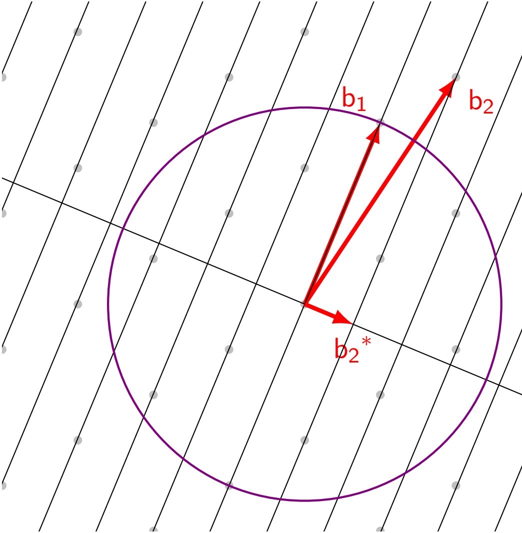
- Make a grid parallel to b_1 spaced by the length of $b_2{}^*$.
- Consider points within the sphere of radius $||b_1||$.



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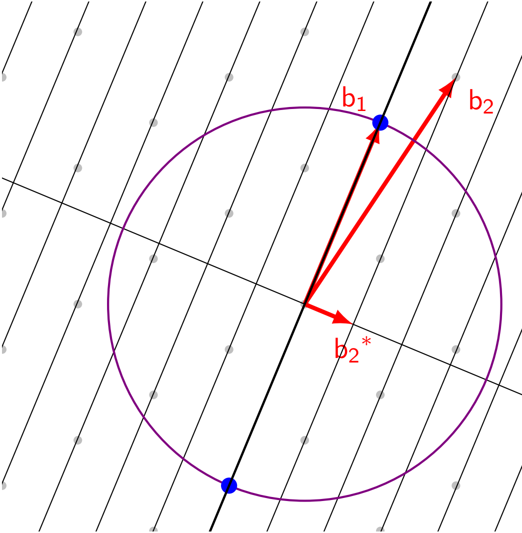
- Make a grid parallel to b_1 spaced by the length of $b_2{}^*$.
- Consider points within the sphere of radius $||b_1||$.
- For each multiple of $||b_2{}^*||$ find all lattice points on that line.



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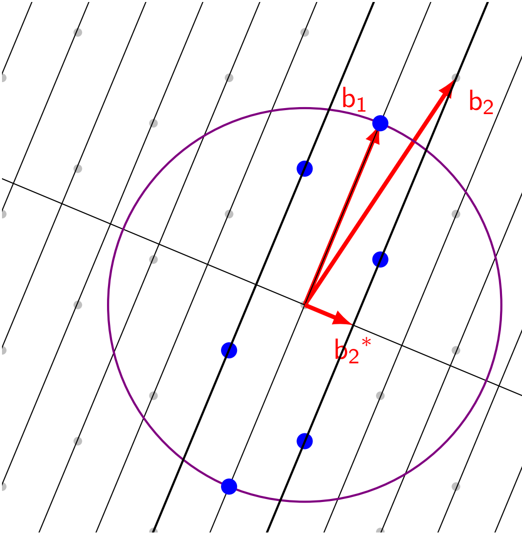

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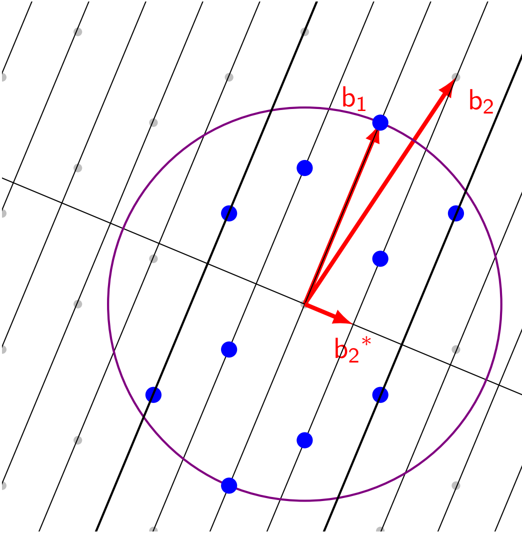

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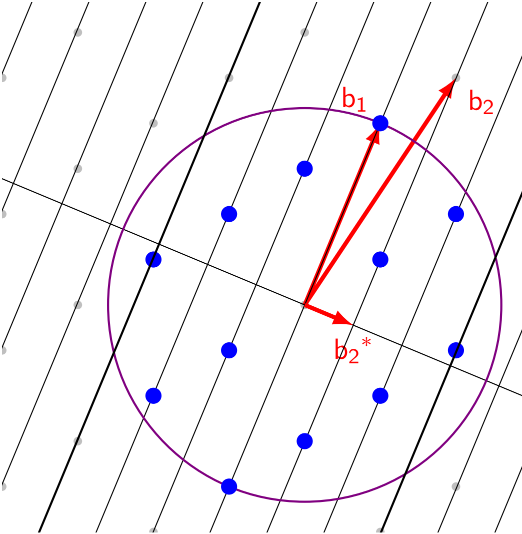
- Output the shortest vector in the sphere.



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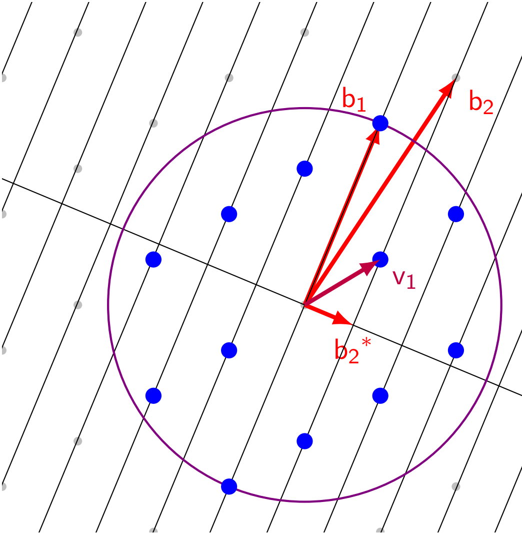

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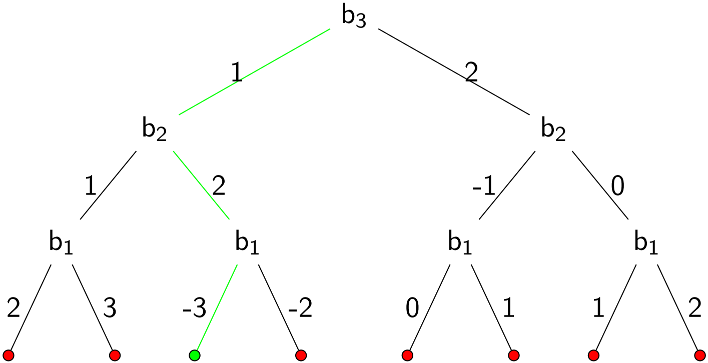
Naive tree search

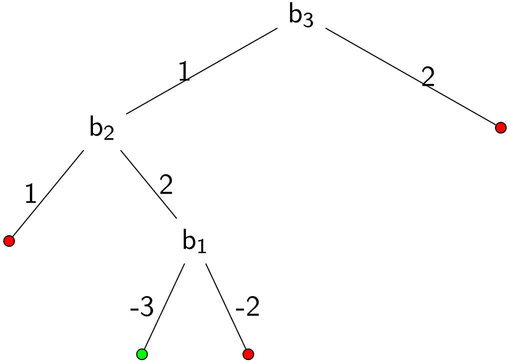
Our problem can be modelled as a tree.



Where the green node corresponds to the shortest vector $-3\mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3$.

Backtracking: idea

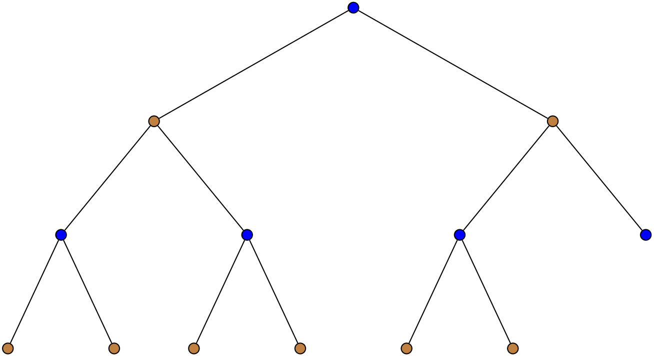
We need to find some way to reduce the number of nodes in the tree.



At each level call an oracle/predicate, this is called backtracking.

Quantum backtracking

Montanaro '18: split the tree into odd and even distance from the root.



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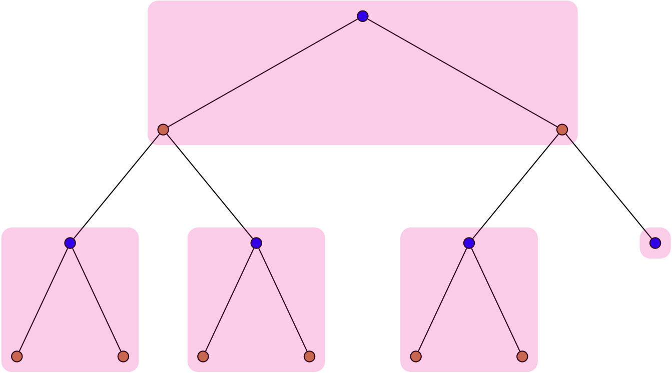
Each step consists of going over the even distance nodes and their children.



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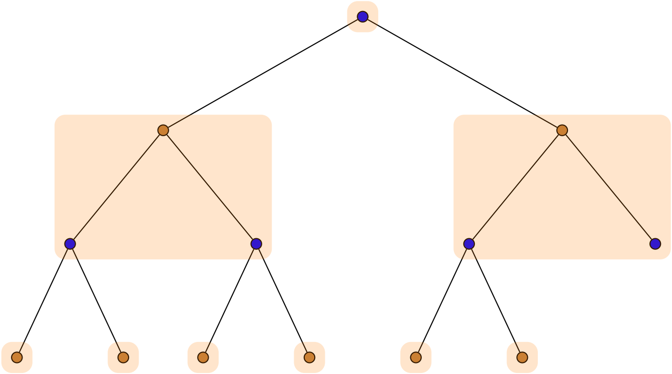

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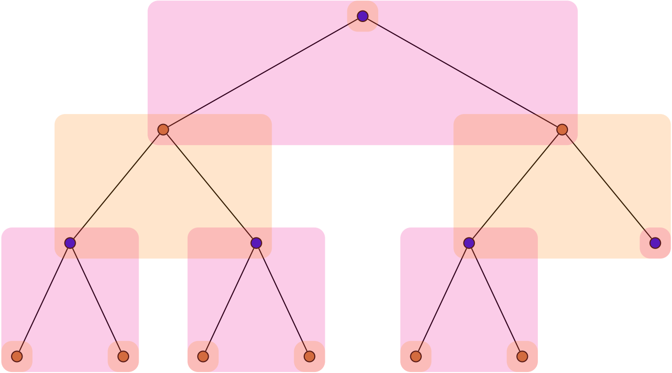
Part of our paper: circuit level implementation of this step, called $R_B R_A$.



Overall Circuit

Montanaro's algorithm uses the phase estimation of the operator $U = R_B R_A$ on the root of the substrees to detect a marked solution

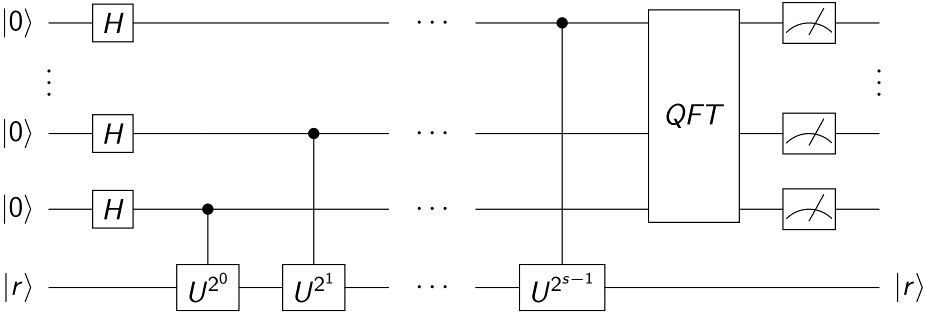


Figure: General circuit for phase estimation where $U := R_B R_A$.

Our circuit design and its resource estimation

Overall Picture

The circuit design contains two main components

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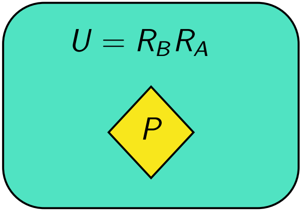

U: Quantum walk operator

P: Ensuring if the walk is correct

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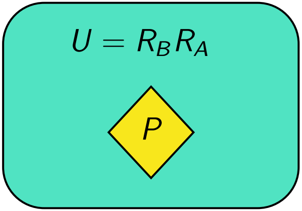

U: Quantum walk operator
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The predicate P will ensure the coefficients v_i satisfy

$$\sum_{j \geq n+1-l} \Big(\sum_{i \geq j} \mu_{i,j} \cdot v_i\Big)^2 \cdot \|\mathbf{b}_j^*\|^2 \leq R^2,$$

where $R, \mu_{ij}, \|\mathbf{b}_j^*\|$ are precomputed classically.

Predicate Design in a nutshell

$$\underbrace{\sum_{j \geq n+1-\ell}}_{(IV) Add_{FF}} \Big(\underbrace{\sum_{i \geq j}}_{(II) A_k} \underbrace{\mu_{i,j} \cdot v_i}_{(I)} \Big)^2 \underbrace{\cdot \|\mathbf{b}_j^*\|^2}_{(III) Mul_{FF}} \leq R^2,$$

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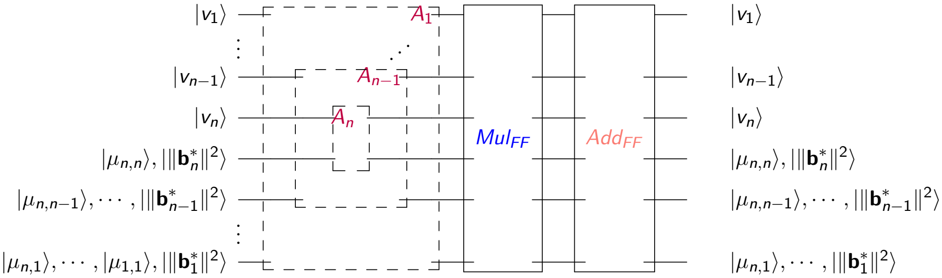

Figure: Components in the predicate circuit

Overall Circuit Design - Requirements

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n: $\dim(\mathcal{L})$ d: maximum degree of the tree B: bound of coefficients

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  - We know $\mathcal{T} \leq n^{n/(2e+o(n))}$
  - Better when the basis is preprocessed, e.g., in HKZ-reduced basis $d \approx n^{(\ln n)/4}$

Our results

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Overall Resource Estimates

— T-depth —

$$32\sqrt{\mathcal{T}n} \cdot \left[16np(\log B + 2\log n + p^{0.158}) + O(n\log B) + 8d^2\log(d\sqrt{\mathcal{T}n}) + 4d^2\log d + O(d^2)\right]$$

— T-size —

$$32\sqrt{\mathcal{T}n} \cdot \left[8(d+1)(14pn^2(B+1) + O(n^2B)) + 8d^2\log(d\sqrt{\mathcal{T}n}) + 16d^2\log d + O(d^2)\right]$$

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Practical Parameters

For cryptographic size n (e.g., $n \gtrapprox 400$), it's reasonable to model
$d \approx n, B \approx n^2, p \leq 3n$, $\log(\mathcal{T}) \approx cn \log n$

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T-depth: $(128cn^3 \log n + O(n^{2.158}))\sqrt{\mathcal{T}n}$

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For cryptographic size n (e.g., $n \gtrapprox 400$), it's reasonable to model $d \approx n, B \approx n^2, p \leq 3n$, $\log(\mathcal{T}) \approx cn \log n$

T-depth: $(128cn^3 \log n + O(n^{2.158}))\sqrt{\mathcal{T} n}$

T-size: $(10752n^6 + O(n^5))\sqrt{\mathcal{T} n}$

Heuristic 1: $d \approx n, B \approx n^2$

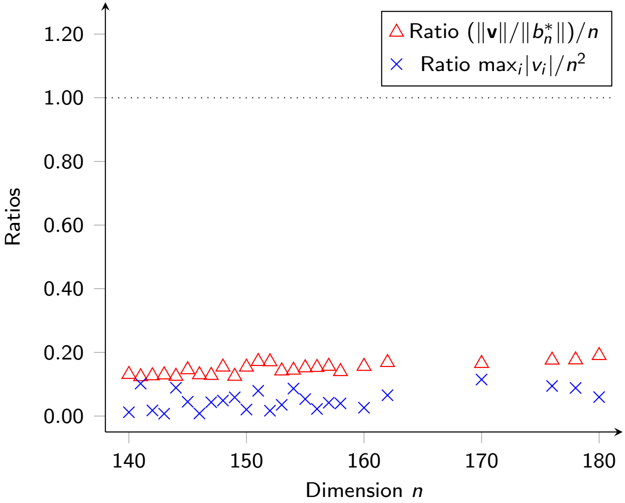

Figure: Bounds d and B, based on solved SVP Challenges.

Heuristic 2 - Bound $B \approx n^2$

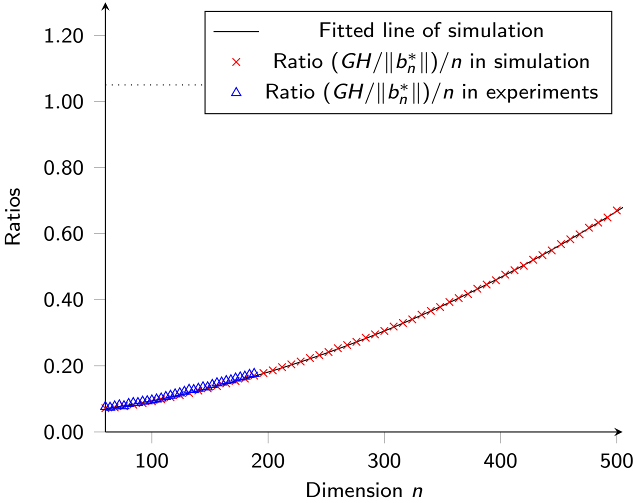

Figure: Bound B, based on BKZ experiments and simulations.

Heuristic 3 - $\mu_{i,j}$ have similar magnitude

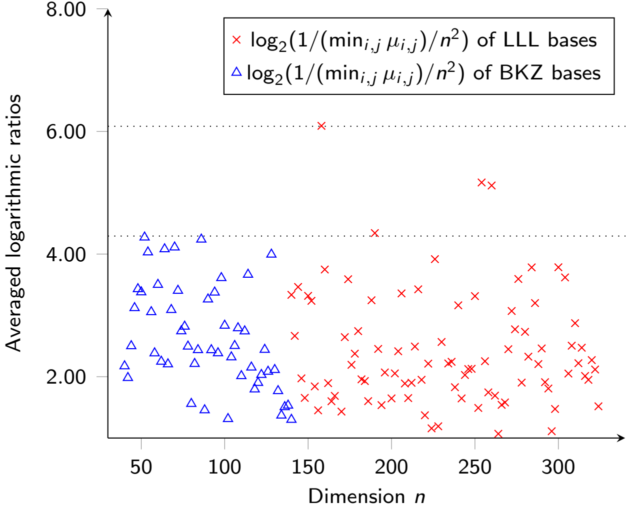

Figure: Experiments on the bound for $\mu_{i,j}$ in LLL/BKZ reduced basis.

Questions

- Can we make the phase estimation process parallel to reduce the depth?
- What are the resource estimates for extreme pruning?

The oracle receives questions now.
Do you have any questions?

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