Unified View for Notions of Bit Security

ASIACRYPT@Guangzhou, China
December, 2023

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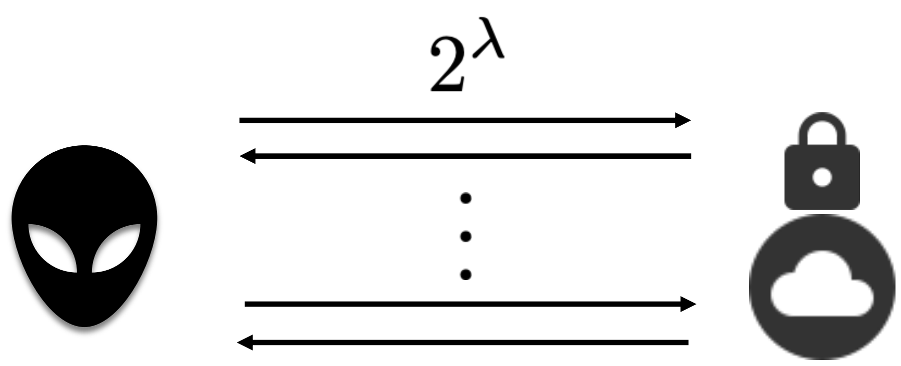
What is bit security?

We shall quantify how much security a certain system provide…

Roughly, a system is λ bit secure if 2^λ operations are needed to break the system.



Bit security of one-way function

a representative of search primitive

Given one-way function (permutation)

$$f : \{0,1\}^n \to \{0,1\}^n$$

and an attack with cost T such that

$$\Pr\big(A(f(x)) = x\big) = \varepsilon_A$$

how much bit security is guaranteed?

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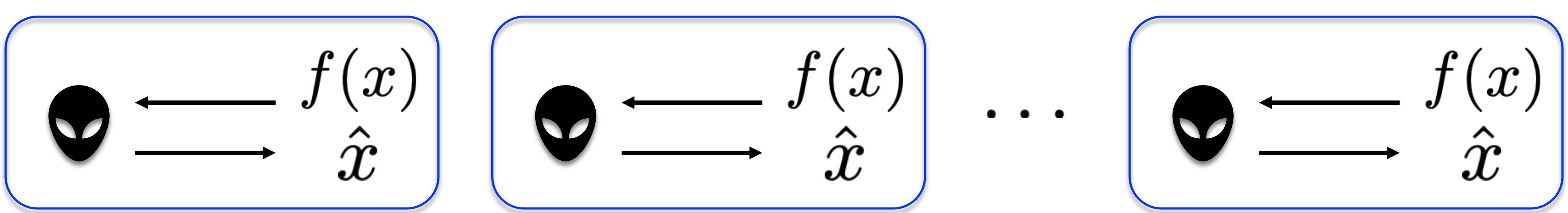
how much bit security is guaranteed?

The success probability can be amplified to $\simeq N\varepsilon_A$



N times

Total cost is $\mathcal{O}(N \cdot T_A) = \mathcal{O}\left(\frac{T_A}{\varepsilon_A}\right) \implies \mathrm{BS} = \min_A \left\{ \log_2 \left(\frac{T_A}{\varepsilon_A} \right) \right\}$

Prior work and outline of our results

How should we define bit security of decision primitives/assumptions.

(PRG, encryption, DDH)

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Result 1: a slight modification of WY21 is equivalent to MW18.

Result 2: Goldreich-Levin reduction is tight in WY21.

Result 3: via canonical games, it suffices to consider decision game in WY21

Result 4: application to distribution replacement theorem

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Motivating question 1

Attack with success probability 40 %

Attack with success probability 50 %

Prediction Games

Game	1	2	3	4	5	6	7	8	9	10
Prediction	1	0	0	0	1	0	0	0	1	0
Outcome	0	0	1	0	1	1	0	1	0	1

Game	1	2	3	4	5	6	7	8	9	10
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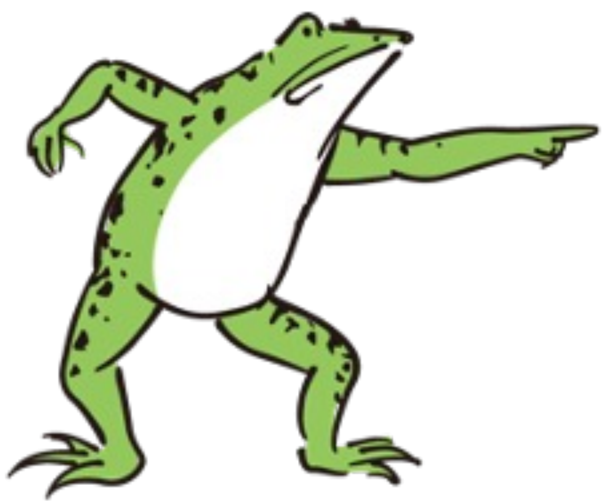
Motivating question 2

Attack with success probability 60 %



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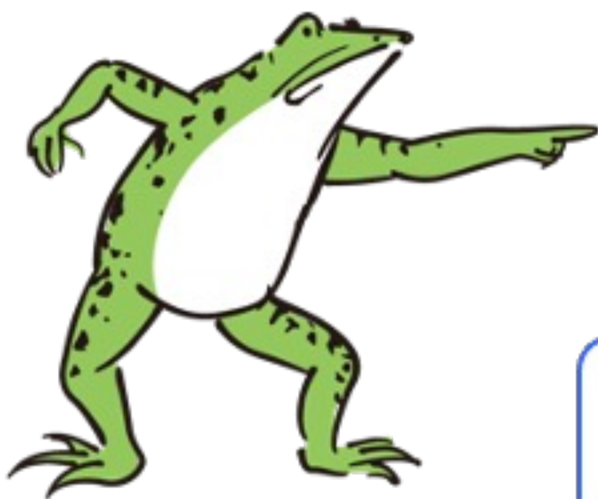

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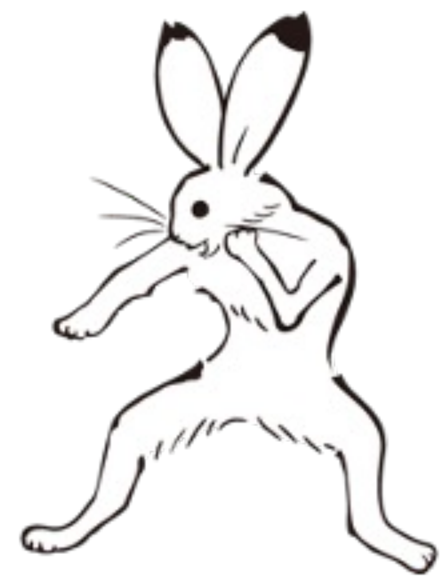

Arranged based on the outcomes

100%

20%

Game	1 2 4 7 9	3 5 6 8 10
Prediction	0 0 0 0 0	0 1 0 0 0
Outcome	0 0 0 0 0	1 1 1 1 1

Biased adversary

60%

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Game	1 2 4 7 9	3 5 6 8 10
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Unbiased adversary

Two kinds of adversaries of PRG

Consider a construction of PRG using one-way permutation.

Given one-way permutation

$$f : \{0,1\}^n \to \{0,1\}^n$$

and its hard-core predicate

$$h : \{0,1\}^n \to \{0,1\}$$

Seed: $x \in_R \{0,1\}^n$ Output: $G(x) = (f(x), h(x))$

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Indistinguishability game:

PRG: $u = 0$ $(y, z) = (f(x), h(x))$

TRG: $u = 1$ $(y, z) = (f(x), \sigma)$ $\sigma \in_R \{0,1\}$

Two kinds of adversaries of PRG

There are a few possible attacks:

1) Linear test attack:

For a fixed vector $v \in \{0,1\}^{n+1}$, output $\hat{u} = 0$ if $\langle v, (y, z) \rangle = 0$

$$A_0 = \left(1/2 + \varepsilon_1, 1/2 - \varepsilon_1\right) \qquad A_1 = \left(1/2, 1/2\right)$$

There exists v such that $\varepsilon_1 \geq 2^{-n/2}$ [Alon-Goldreich-Hastad-Peralta 92].

2) Inversion attack:

Invert $f(x)$, and output $\hat{u} = 0$ if it succeed and $h(x) = z$.

If the success probability of inversion is $2\varepsilon_2$,

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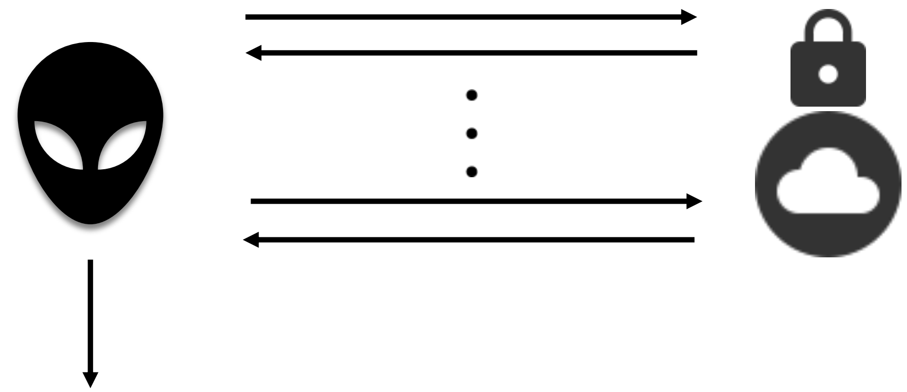
The standard advantage cannot capture the difference of biased and unbiased adversaries.

Bit security framework of Micciancio-Walter



$$y \in \{0,1\} \cup \{\perp\}$$

Bit security is defined as $$\min_A \left\{ \log \frac{T_A}{\mathrm{adv}_A^{\mathrm{MW}}} \right\}$$

mutual information

$$\mathrm{adv}_A^{\mathrm{MW}} := \frac{I(U \wedge Y)}{H(U)} = 1 - \frac{H(U|Y)}{H(U)}$$

Shannon entropy

$U \in \{0,1\}$ is a random secret of game

Y is the adversary's output

Bit security framework of Micciancio-Walter

For decision game,

conditional square advantage

$$\mathrm{adv}_A^{\mathrm{MW}} \simeq \alpha_A \cdot (2\beta_A - 1)^2 =: \mathrm{adv}_A^{CS}$$

where

$$\alpha_A := \Pr\left(A \text{ outputs } Y \neq \perp\right) \quad \beta_A := \Pr\left(Y = U | A \text{ outputs } Y \neq \perp\right)$$

1) Linear test attack:

$$\alpha_A = 1, \quad \beta_A = \varepsilon_1^2 \Longrightarrow \mathrm{adv}_A^{\mathrm{CS}} = \varepsilon_1^2$$

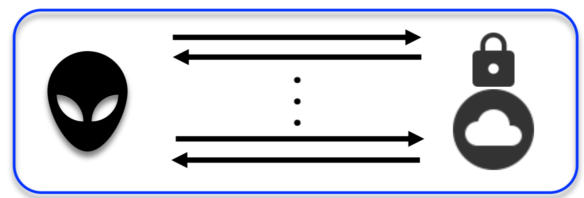
2) Inversion attack:

$$\alpha_A = 2\varepsilon_2, \quad \beta_A = 1/4 \Longrightarrow \mathrm{adv}_A^{\mathrm{CS}} = \varepsilon_2/2$$

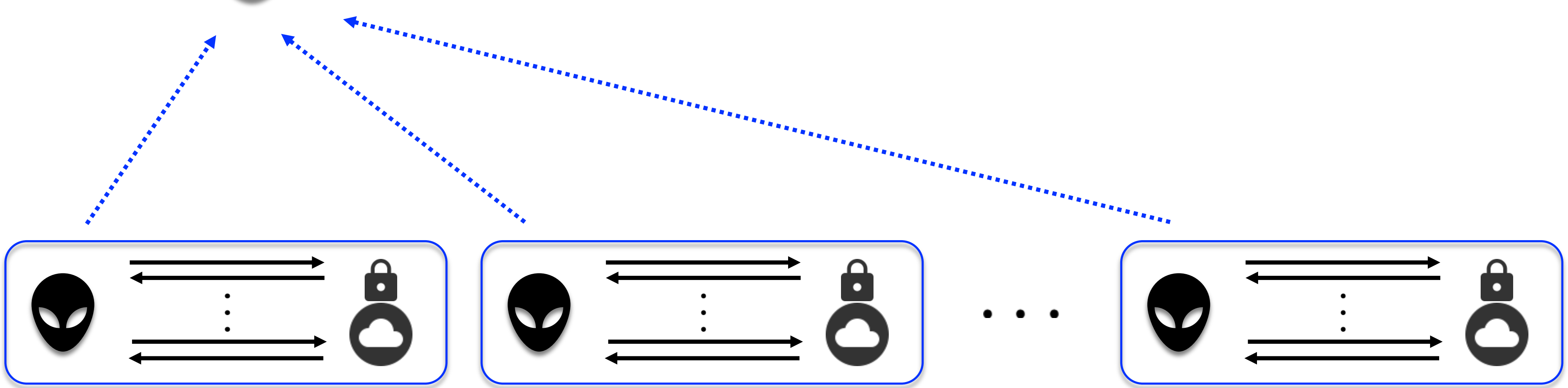
Bit security framework of WY21

Consider two types of adversaries: inner  and outer 

A B

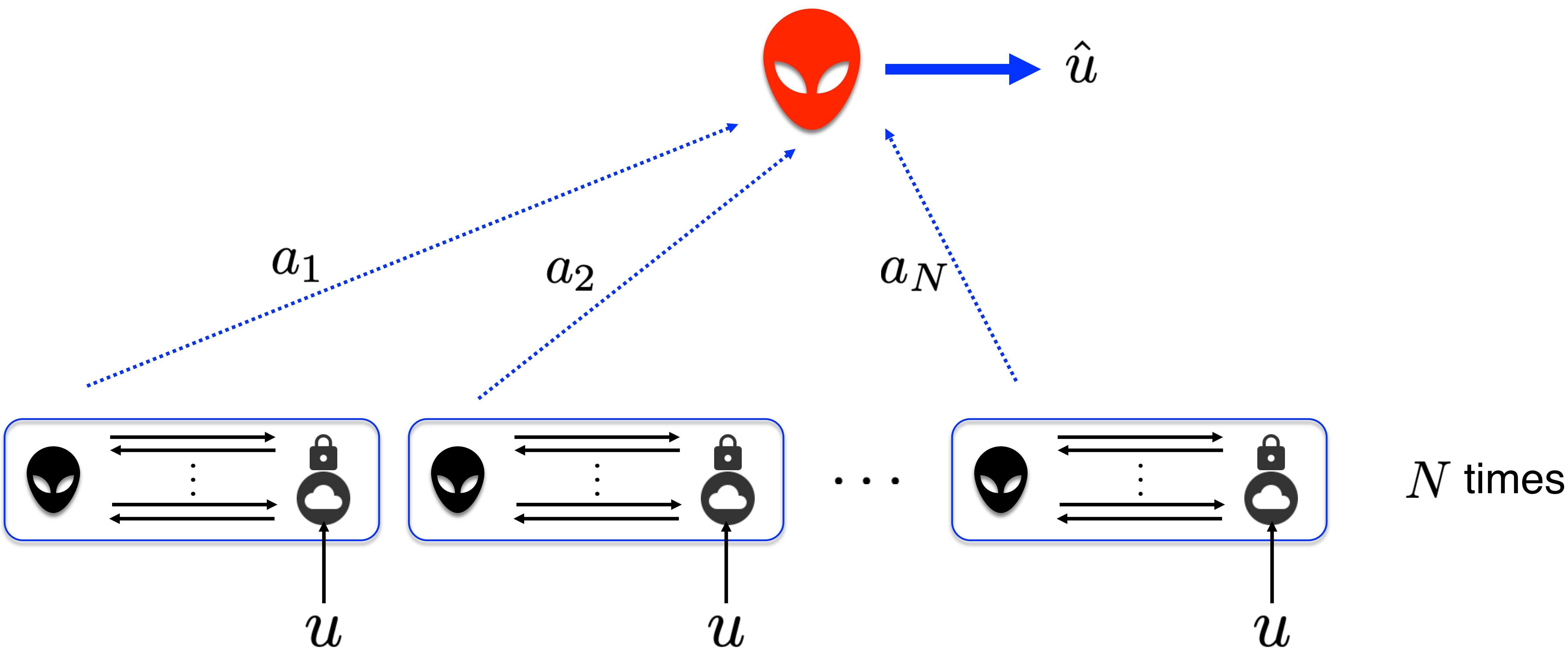
Inner  plays a usual game 

outer  invokes inner adversaries to amplify the winning probability



N times

Winning condition of outer adversary (decision)



Each inner 👽 plays an independent game with consistent u .

If $\hat{u} = u$, outer 👽 wins the game.

Bit security definition of WY21

Bit security is defined as

$$\mathrm{BS}_G^{\mu} := \min_{A,B} \left\{ \log\left(N \cdot T_A\right) : \Pr\left(B \text{ wins }\right) \geq 1-\mu \right\}$$

inner

outer

eg) $\mu = 0.01$

Characterization of Bit security of WY21

Theorem [WY21]

Bit security can be characterized as

$$\mathrm{BS}_G^{\mu} := \min_A \left\{ \log \left(\frac{T_A}{\mathrm{adv}_A} \right) \right\} + \mathcal{O}(1)$$

where $\mathrm{adv}_A = \mathrm{adv}_A^{\mathrm{Renyi}} := D_{1/2}(A_0 \| A_1)$

A_u : probability distribution of output a by A when secret is u

$$D_{1/2}(A_0 \| A_1) = -2 \ln \sum_a \sqrt{A_0(a) A_1(a)}$$ Rényi divergence of order 1/2

WY21 did not consider $\perp$, but the result in unchanged even if we consider $\perp$.

- Upper bound is derived using the likelihood ratio test for Bayesian hypothesis testing.
- Lower bound is derived using an inequality between Rényi divergence and TV distance.

(Fuchs-van de Graaf inequality)

Open problems in WY21

WY bit security behaves mostly the same as MW bit security.

Particularly,

1) Linear test attack: $\mathrm{adv}_A^{\mathrm{Renyi}} = \Theta(\varepsilon_1^2)$

2) Inversion attack: $\mathrm{adv}_A^{\mathrm{Renyi}} = \Theta(\varepsilon_2)$

However,

-tightness of Goldreich-Levin reduction

-connection between WY bit security and MW bit security

remained unsolved.

Relation between MW and WY (1)

Theorem 1

For any adversary A,

$$\mathrm{Adv}_A^{\mathrm{CS}} \leq 4\mathrm{Adv}_A^{\mathrm{Renyi}}$$

Implication of Theorem 1:

If a decision game G is λ bit secure in the sense of WY,

then, up to a constant bit, G is λ bit secure in the sense of MW.

The proof is via bounding the CS advantage by the Hellinger distance, and

then using a connection between Rényi divergence and Hellinger distance.

Relation between MW and WY (2)

Theorem 2

For an adversary A such that $\mathrm{Adv}_A^{\mathrm{Renyi}} \leq 1$, there exists an adversary $\tilde{A}$ such that

$$\mathrm{Adv}_A^{\mathrm{Renyi}} \leq 12\ \mathrm{Adv}_{\tilde{A}}^{\mathrm{CS}}$$

and $\tilde{A}$ has the same cost as A.

Implication of Theorem 2:

The assumption $\mathrm{Adv}_A^{\mathrm{Renyi}} \leq 1$ is not restrictive:

Even if $\mathrm{Adv}_A^{\mathrm{Renyi}} > 1$, we can consider $\theta A + (1-\theta) A_{\text{triv}}$

and apply Theorem 2 to A_θ .

If a decision game G is λ bit secure in the sense of MW,

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Proof outline of Theorem 2

Lemma (relabeling [MW18])

For an adversary A and $z \in \{0, 1, \perp\}$, let $\tilde{A}^z$ be an adversary such that

run A and obtain output a

output 0 if $a = z$ and $A_0(z) \geq A_1(z)$

output 1 if $a = z$ and $A_0(z) < A_1(z)$

only one of these occurs

otherwise ($a \neq z$), output $\perp$

Then,

$$\mathrm{Adv}^{\mathrm{CS}}_{\tilde{A}^z} = \frac{1}{2}\frac{(A_0(z) - A_1(z))^2}{A_0(z) + A_1(z)}$$

Non-verifiable primitive

Computational Diffie-Hellman (CDH):

$$(g^x, g^y) \longrightarrow \boxed{A'} \longrightarrow \text{estimate } g^{xy}$$

non-verifiable search game

$\varepsilon_{A'}^{\mathrm{cdh}}$: success probability

Decision Diffie-Hellman (DDH):

$u = 0 \quad (g^x, g^y, g^{xy})$

or

$u = 1 \quad (g^x, g^y, g^z)$

$$\longrightarrow \boxed{A'} \longrightarrow \text{estimate } u$$

Attack DDH using CDH oracle: probability of $\perp$ is 0

$$A_0 = (\varepsilon_{A'}^{\mathrm{cdh}}, 1 - \varepsilon_{A'}^{\mathrm{cdh}}, 0) \qquad A_1 = (\varepsilon_{A'}^{\mathrm{cdh}}/p, 1 - \varepsilon_{A'}^{\mathrm{cdh}}/p, 0)$$

$$\mathrm{adv}_A^{\mathrm{CS}} = (\mathrm{adv}_A^{\mathrm{TV}})^2 = (1 - 1/p)^2 (\varepsilon_{A'}^{\mathrm{cdh}})^2 \quad \mathrm{adv}_A^{\mathrm{Renyi}} = \Omega(\varepsilon_{A'}^{\mathrm{cdh}})$$

The CS advantage is much smaller than the Renyi advantage

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$$\longrightarrow \boxed{A'} \longrightarrow \text{estimate } u$$

Apply Lemma (relabeling) with $z=0$

$$A_0 = (\varepsilon^{\mathrm{cdh}}_{A'}, 1-\varepsilon^{\mathrm{cdh}}_{A'}, 0) \qquad A_1 = (\varepsilon^{\mathrm{cdh}}_{A'}/p, 1-\varepsilon^{\mathrm{cdh}}_{A'}/p, 0)$$

$$\tilde{A}^z_0 = (\varepsilon^{\mathrm{cdh}}_{A'}, 0, 1-\varepsilon^{\mathrm{cdh}}_{A'}) \qquad \tilde{A}^z_1 = (\varepsilon^{\mathrm{cdh}}_{A'}/p, 0, 1-\varepsilon^{\mathrm{cdh}}_{A'}/p)$$

$$\mathrm{adv}^{\mathrm{CS}}_{\tilde{A}^z} = \Omega(\varepsilon^{\mathrm{cdh}}_{A'})$$

Summary of advantages for various attacks

Attacks	$\mathsf{Adv}^{\mathrm{TV}}$	$\mathsf{Adv}^{\mathrm{CS}}$	$\mathsf{Adv}^{\mathrm{Renyi}}$
Balanced attack without $\perp$ $A_0 = (1/2+\delta, 1/2-\delta)$ $A_1 = (1/2, 1/2)$ e.g.) Linear test attack for PRG	δ	δ^2	$\Theta(\delta^2)$
Unbalanced attack with $\perp$ $A_0 = (\delta, 0, 1-\delta)$ $A_1 = (\delta/2, \delta/2, 1-\delta)$ e.g.) Inversion attack for PRG	$\delta/2$	$\delta/2$	$\Theta(\delta)$
Unbalanced attack without $\perp$ $A_0 = (\delta, 1-\delta)$ $A_1 = (\delta/p, 1-\delta/p)$ e.g.) CDH oracle attack for DDH	$(1-1/p)\delta$	$(1-1/p)^2\delta^2$	$\Theta(\delta)$
Balanced 0/1-unbalanced $\perp$ attack $A_0 = (1/2-\delta/2, 1/2-\delta/2, \delta)$ $A_1 = (1/2-\delta/4, 1/2-\delta/4, \delta/2)$ e.g.) Inversion attack using $\perp$	$\delta/2$	0	$\Theta(\delta)$

The CS advantage is sensitive to “labeling”, but after relabeling,

the CS advantage and the Rényi advantage lead to essentially the same bit security.