Secure Multiparty Computation from Threshold Encryption based on Class Groups

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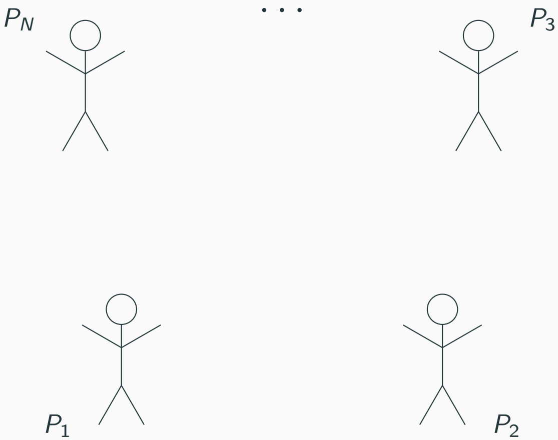
August 23, 2023 – Crypto'23

Aarhus University

Introduction and Preliminaries

Secure Multiparty Computation



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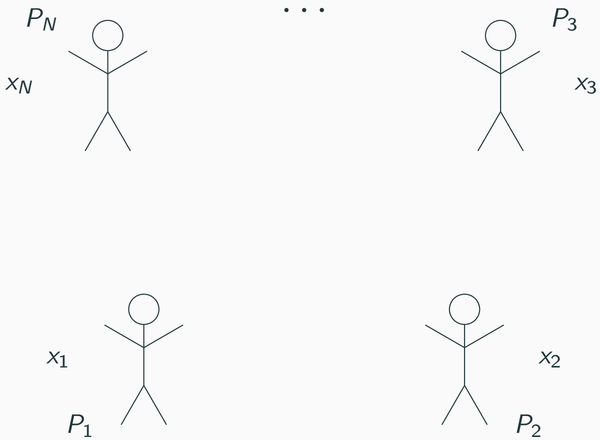

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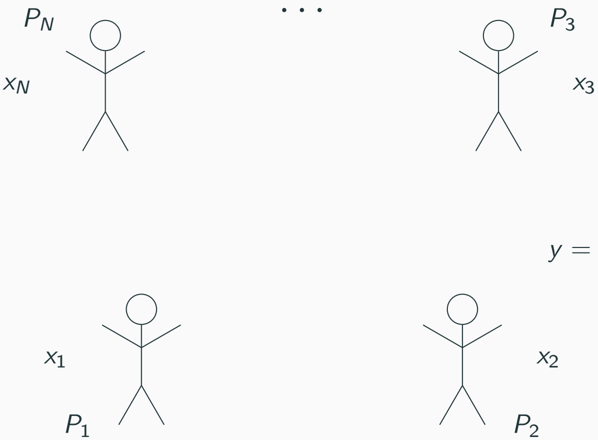

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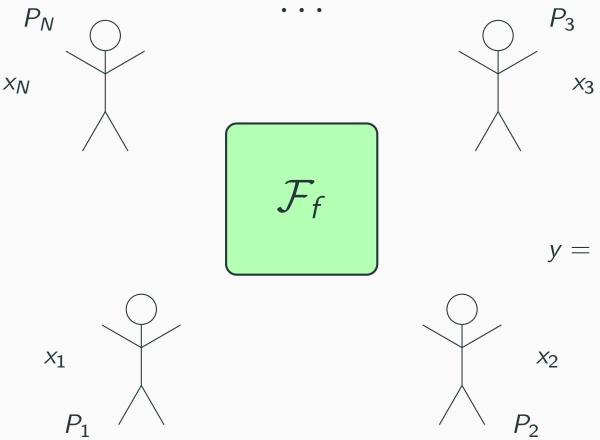

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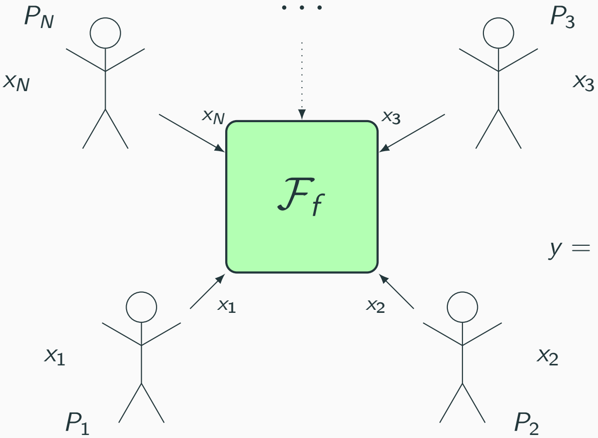

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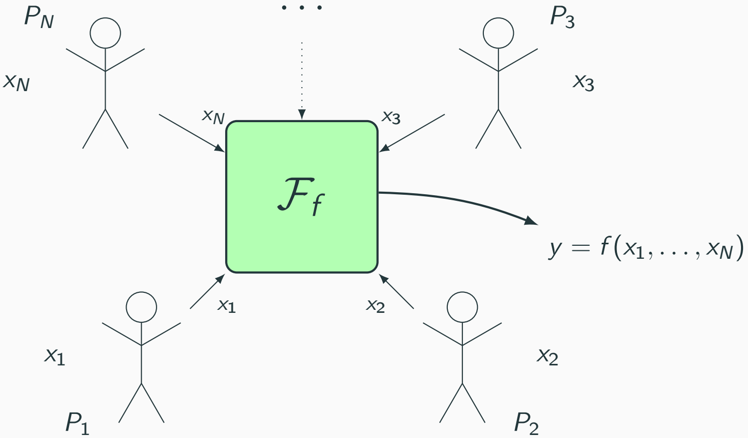

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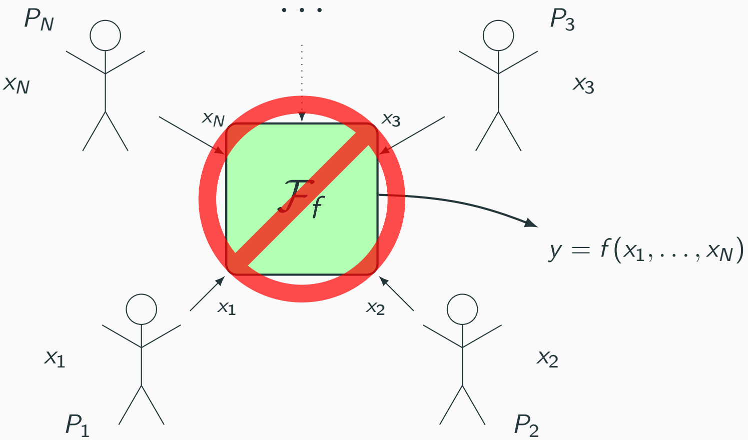

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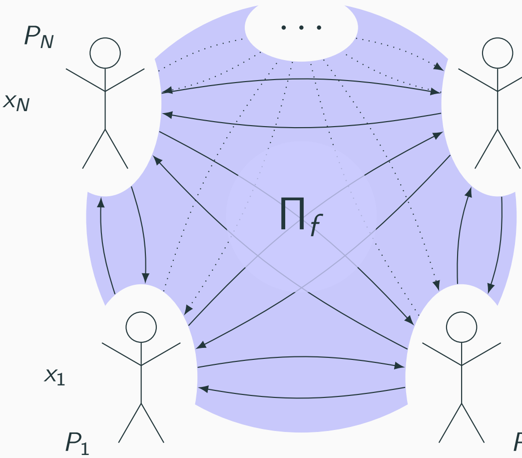

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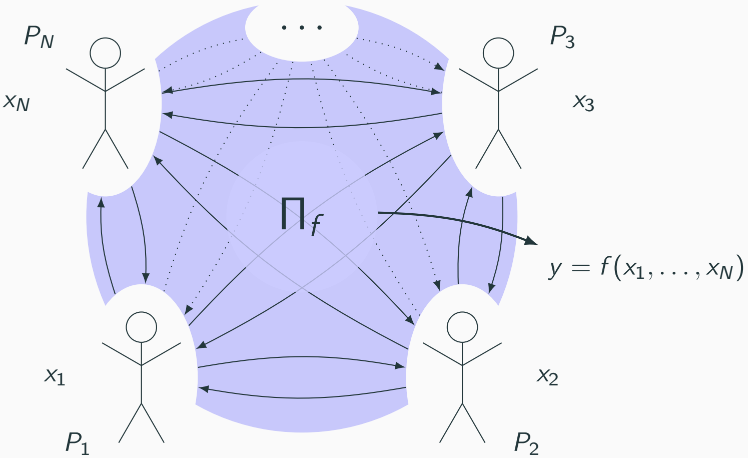
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x_N
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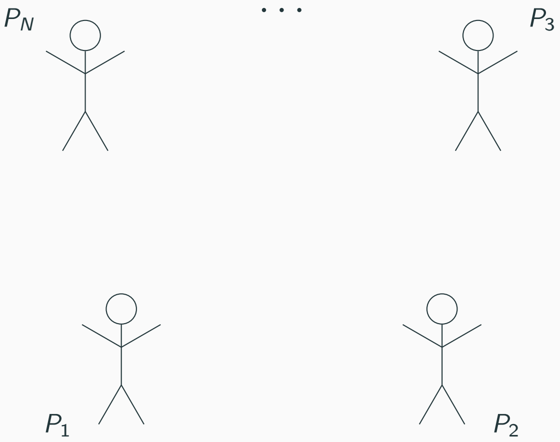

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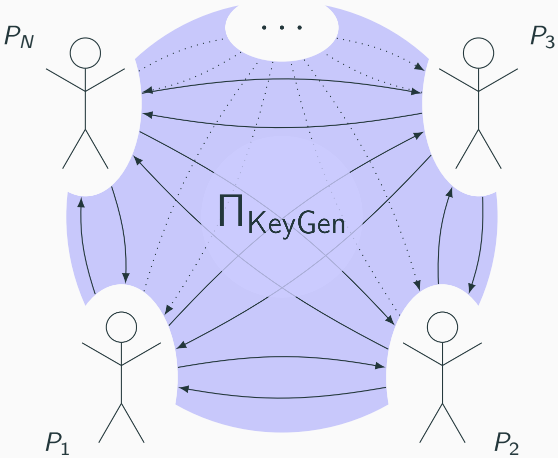

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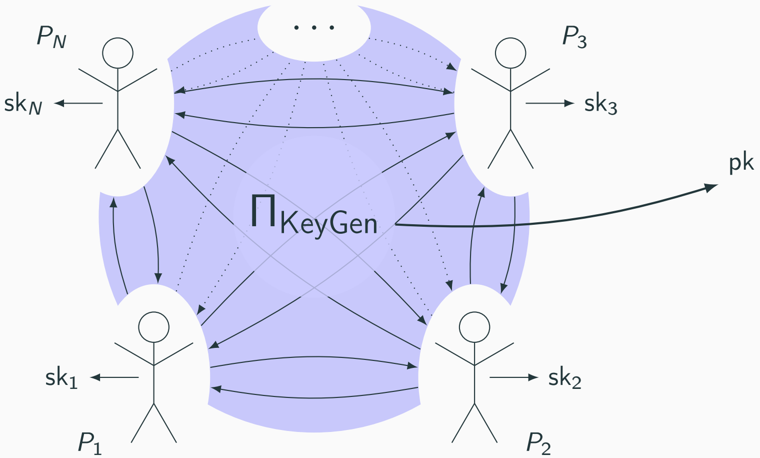

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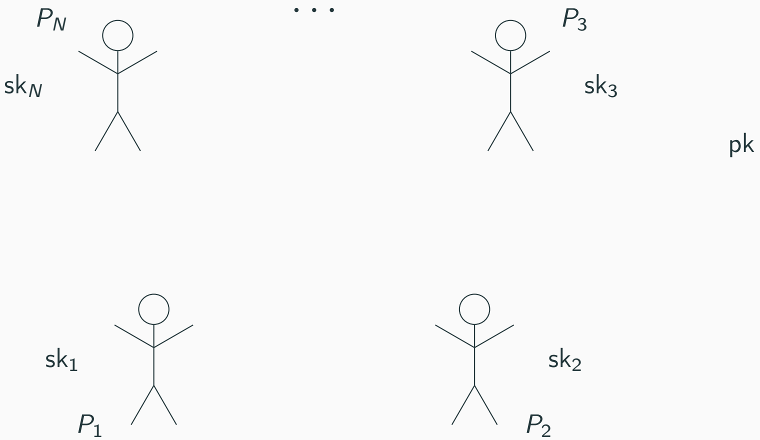

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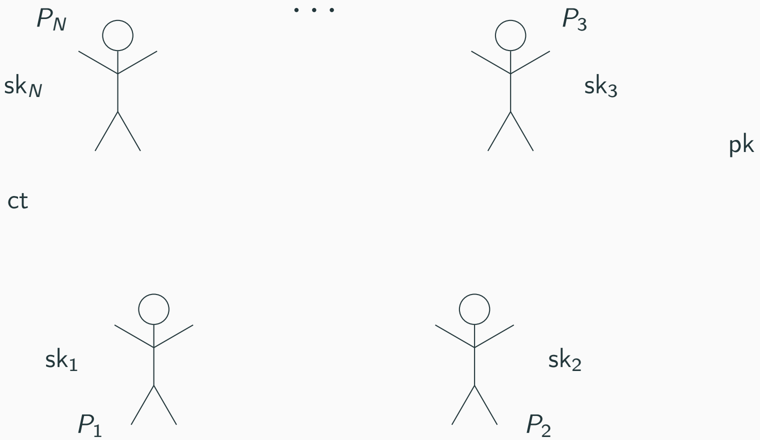

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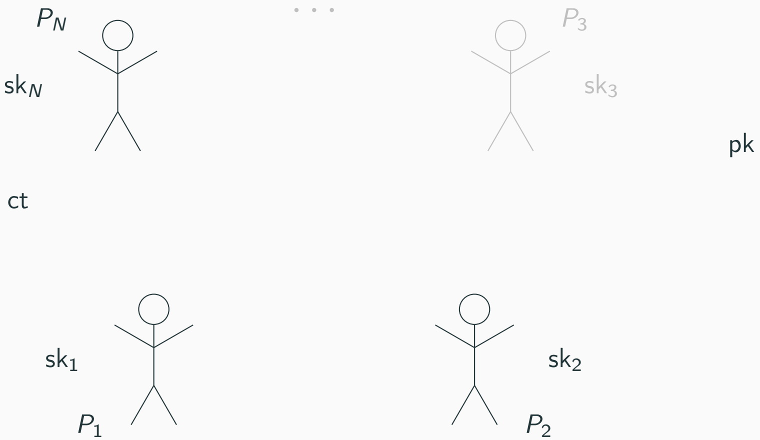

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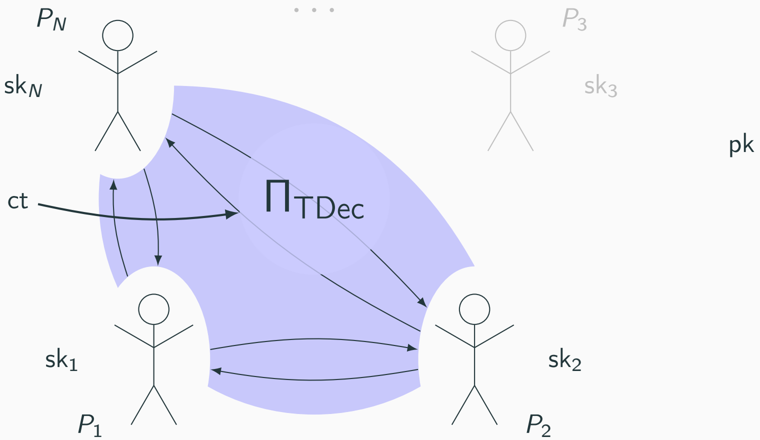

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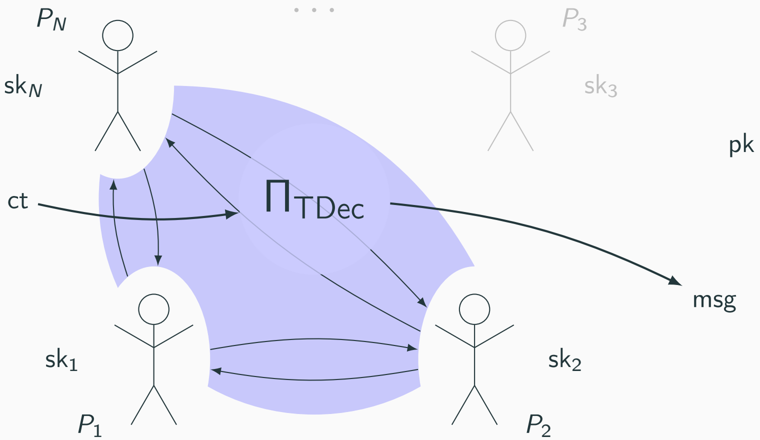

Class Groups

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- Hardness assumptions
  - *ORD*: hard to find the order of any $h \in G \setminus F$
  - *HSM*: hard to distinguish random elements of G and G^q
- Advantages
  - can choose q freely as large prime
  - transparent setup
  - faster and smaller than Paillier ($\rightsquigarrow$ BICYCL by Bouvier et al. [BCIL22])

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- IND-CPA secure by the *HSM* assumption
- Analogue of Camenisch-Shoup encryption for the CL framework

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Ingredients

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- Multiplication $\mathsf{ct}_z \leftarrow \mathsf{ct}_x \cdot \mathsf{ct}_y$:
  1. jointly sample mask $\mathsf{ct}_d, [\![d]\!]$ such that $d \in_r \mathbb{F}_q$
  2. create additive sharing $[\![x]\!] \leftarrow [\![d]\!] - \mathsf{TDec}(\mathsf{ct}_x + \mathsf{ct}_d)$
  3. broadcast $\mathsf{ct}_{z_i} \leftarrow [\![x]\!]_i \cdot \mathsf{ct}_y$ with PoCM, and accumulate $\mathsf{ct}_z \leftarrow \sum_i \mathsf{ct}_{z_i}$

Setting

Security model

- active security
- static corruptions
- honest majority ($t < N/2$)
- broadcast available

Goals

- guaranteed output delivery
- transparent setup

Zero-Knowledge

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Over the integers?

- Binary challenges $\rightsquigarrow$ repetitions
- Strong Root / Low Order assumptions $\rightsquigarrow$ additional setup and complications
- Sometimes normal, set-membership soundness ($\exists x \,.\, h = g^x$) is enough!

New Assumption

Definition (C-Rough Order Assumption (informal))

Let $C \in \mathbb{N}$. The following are computationally indistinguishable:

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Justified?

- Cohen-Lenstra heuristic [CL84] $\rightsquigarrow$ class group orders roughly "behave like random integers"
  $\implies$ there are significantly many C-rough-order class groups
- Efficient distinguisher would be great!

Building Threshold Encryption

Our Goal: $\mathcal{F}_{\mathsf{TE}}$ Ideal Functionality

$\mathcal{F}_{\mathsf{TE}}$

Key Generation

- Run $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{KeyGen}(\mathsf{pp})$
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reconstruction in the exponent of unknown order group element

Pedersen-style Distributed Key Generation

1. All parties P_i
   1.1 sample contribution α_i
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Shamir's Secret Sharing over $\mathbb{Z}$

Sharing

Just do it over the integers: To share $\alpha_i \in [0, 2^\ell)$,

- sample random $f(X) := \alpha_i + \sum_{k=1}^{t} r_k \cdot X^k$ with large enough r_k
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$\implies$ See 90's papers for threshold RSA [DF92; FGMY97; Rab98]

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Define $\Delta := N!$

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Just do it over the integers: To share $\alpha_i \in [0, 2^\ell)$,

- sample random $f(X) := \alpha_i \cdot \Delta + \sum_{k=1}^{t} r_k \cdot X^k$ with large enough r_k
- give $y_j := f(j)$ to P_j

Define $\Delta := N!$

⚠ $f(j) \bmod j$ leaks $\alpha_i \bmod j$

share $\alpha_i \cdot \Delta$ instead ✓

$\Longrightarrow$ See 90's papers for threshold RSA [DF92; FGMY97; Rab98]

Shamir's Secret Sharing over $\mathbb{Z}$

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Reconstruction (in the Exponent)

Lagrange interpolation: Given $\geq t+1$ shares $(x_j = j, y_j = f(j))$, compute

$$f(X) = \sum_i y_i \cdot \prod_{j \neq i} \frac{x_j - X}{x_j - x_i}$$

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⚠ unknown group order $\implies$ cannot divide in the exponent

multiply by Δ, reconstruct $\alpha_i \cdot \Delta^2$ ✓

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Pedersen-style Distributed Key Generation

1. All parties P_i
   1.1 sample contribution α_i
   1.2 publish g^{α_i}
   1.3 share $\alpha_i \rightarrow \langle \alpha_i \rangle$

3. Define public key $\mathsf{pk} := \prod_{P_i} g^{\alpha_i}$
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Feldman's Verifiable Secret Sharing over $\mathbb{Z}$

Goal: Shares of α_i consistent with each other and g^{α_i}

Recall: Sharing polynomial $f(X) := \alpha_i \cdot \Delta + \sum_{k=1}^{t} r_k \cdot X^k$ and shares $(x_j = j, y_j = f(j))$

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F-Share:

- additionally publish $C_0 := g^{\alpha_i}$ and $C_k := g^{\Delta \cdot r_k}$ for $k \in [1, t]$
- prove that $C_k \in \langle g \rangle$

commit to coefficients of f

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F-Check: $P_j \neq P_i$ checks

$$g^{\Delta \cdot y_j} \stackrel{?}{=} C_0^{\Delta^2} \cdot \prod_{k=1}^{t} (C_k)^{(j^k)} = g^{\overbrace{\Delta^2 \cdot \alpha_i + \sum_{k=1}^{t} \Delta \cdot r_k \cdot j^k}^{\text{evaluate } \Delta \cdot f(j) \text{ in the exponent}}}$$

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- Issues with Rabin's VSS [Rab98]: Does not use *ORD*
  $\Longrightarrow$ Corrupt dealer knowing $\mathrm{ord}(g)$ can prevent reconstruction

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⚠ $\mathcal{A}$ can bias distribution of pk!
(Gennaro et al. [GJKR07])

Fixing the Bias

Gennaro et al. [GJKR07]: Unbiased DKG with two-stage approach + Pedersen VSS

- needs additional rounds and extra setup (but bias is ok for e.g. Schnorr signatures)

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Allowing the Adversary to bias the distribution:

Similar observations for ElGamal by Courtier et al. [CGGI13] and Stengele et al. [SRMH21].

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$\underline{\mathsf{BiasedKeyGen}^{\mathcal{A}}}$

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2. compute encryption under (biased) pk

$$\begin{aligned} \mathsf{ct}' &:= (g^r, ((\mathsf{pk}^*)^r \cdot f^m) \cdot (g^r)^{\boxed{\delta}}) \\ &= (g^r, (\mathsf{pk})^r \cdot f^m) \end{aligned}$$

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YOSO

YOSO MPC – You Only Speak Once

YOSO???

- large scale MPC for many parties
- work done by many small committees
- mechanism for passing secrets to future committees without knowing them
- each party sends only <u>one</u> round of messages

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Why is our work YOSO-friendly?

- transparent setup! – open problem in previous work [Gen+21]
- simple one-round distributed key generation and decryption protocols
- small secret state: only shared sk needs to be passed between the committees

Summary

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- First actively-secure threshold version of the HSM-CL cryptosystem
- UC-secure MPC using the CDN paradigm
- New zero-knowledge protocols for multiplicative relations of encrypted values
- Adaption to the YOSO setting and solution to the open problem of transparent setup

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Thank you!

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