Revisiting cycles of pairing-friendly elliptic curves

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A cycle of elliptic curves

The problem:

$$E/\mathbb{F}_q \qquad \text{such that} \qquad \#E(\mathbb{F}_q) = p$$
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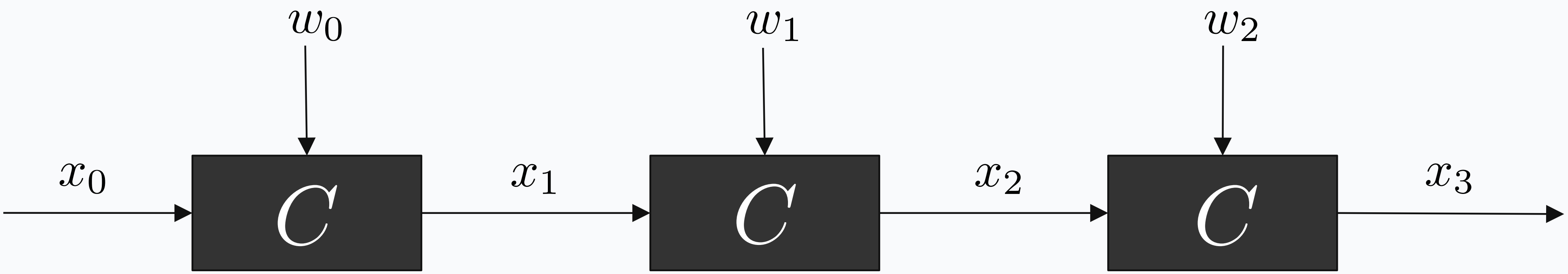
- This problem is easy.
- But if we require E, E’ to be **pairing-friendly**, the problem becomes hard.

Recursive composition of proofs



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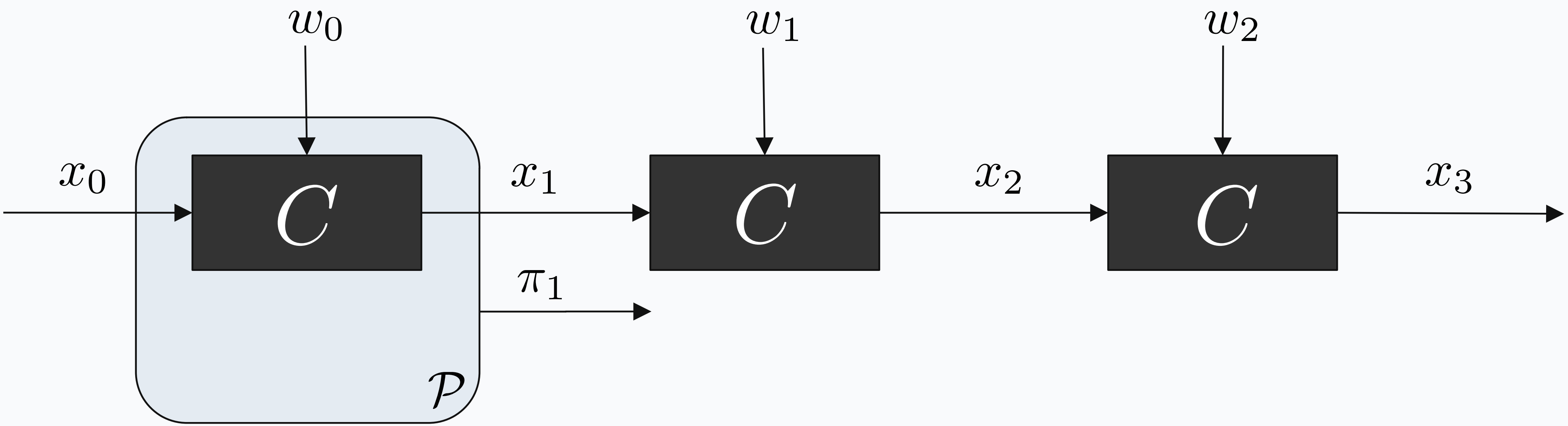

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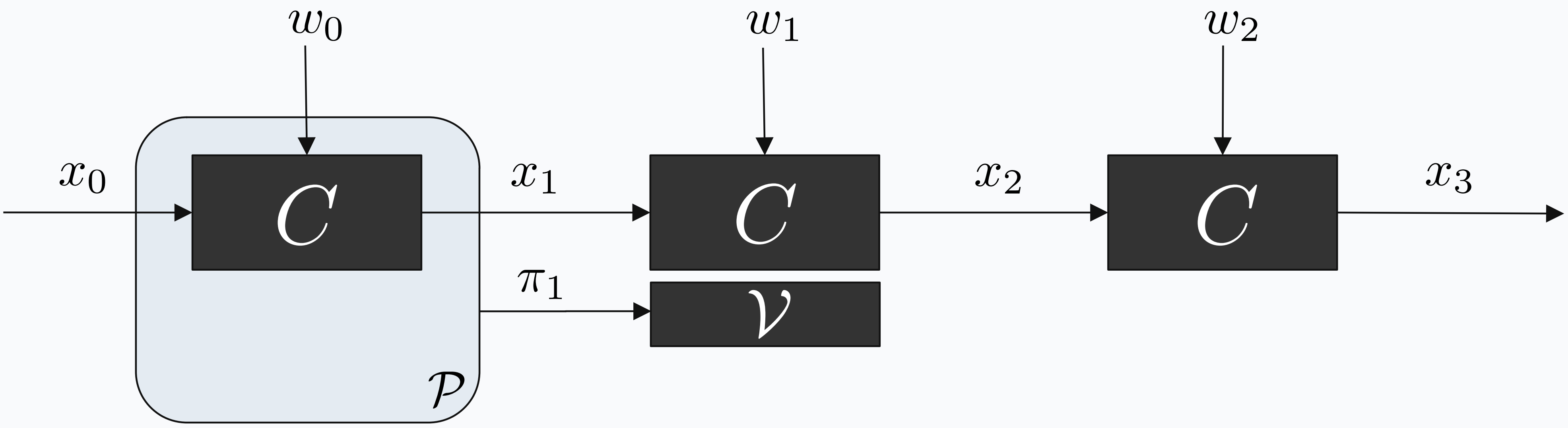

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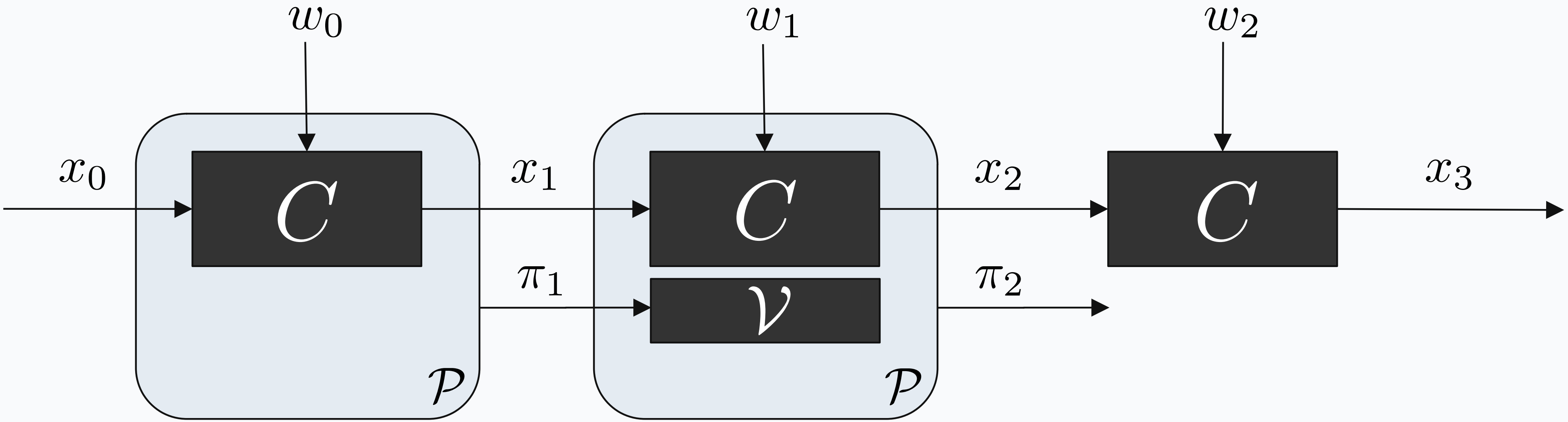

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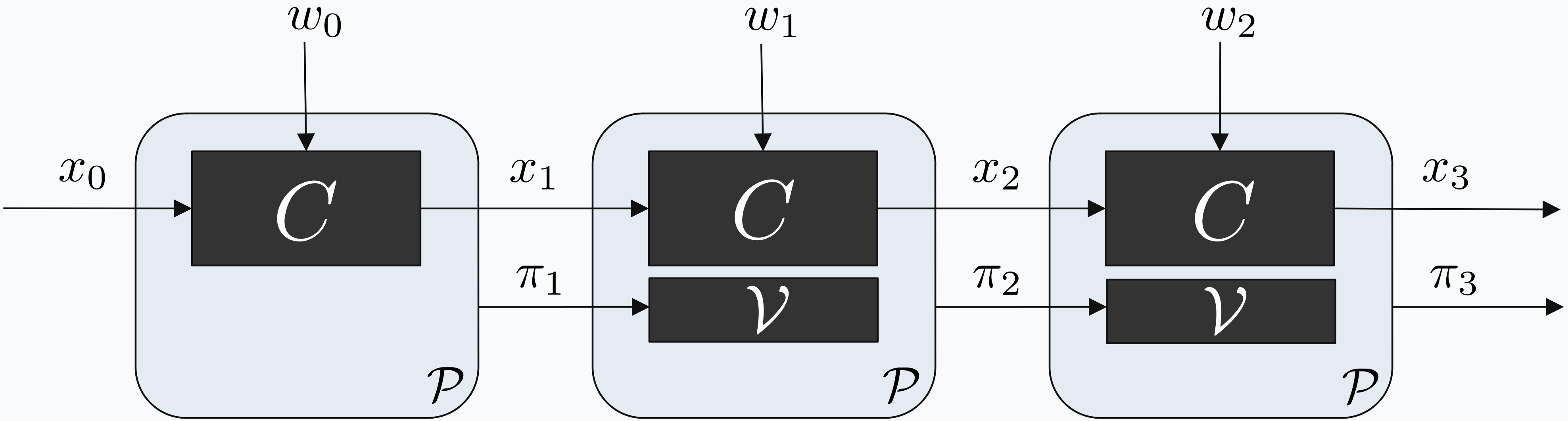

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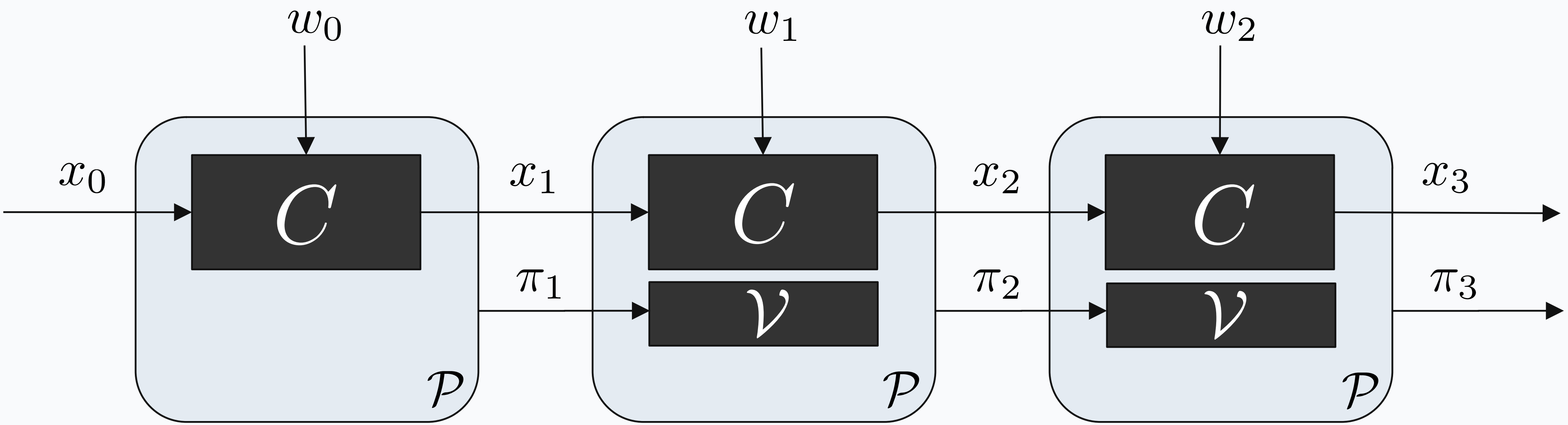

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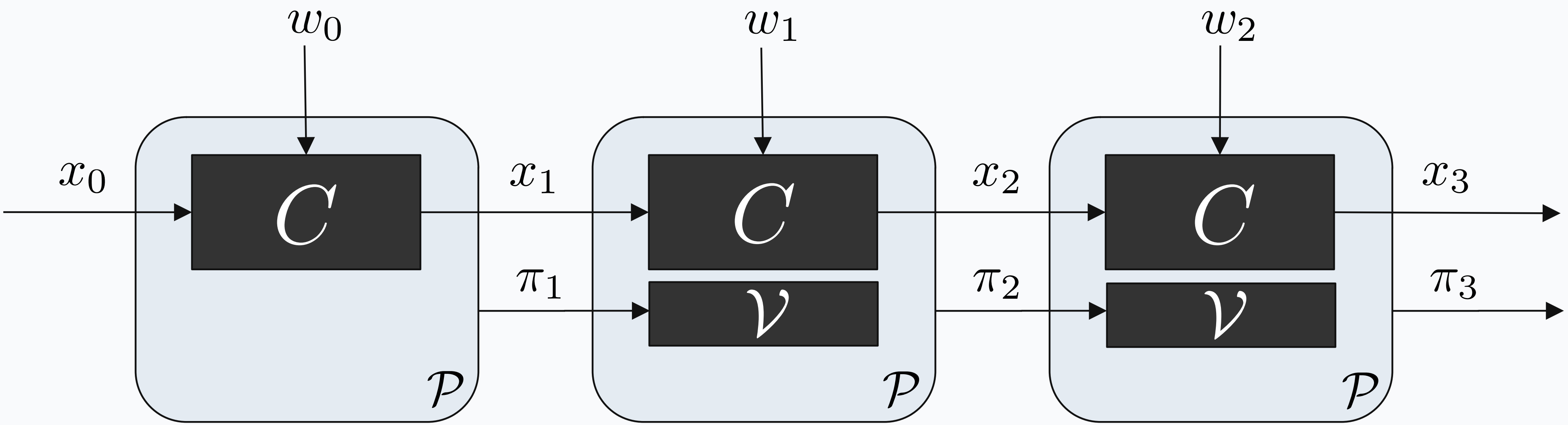

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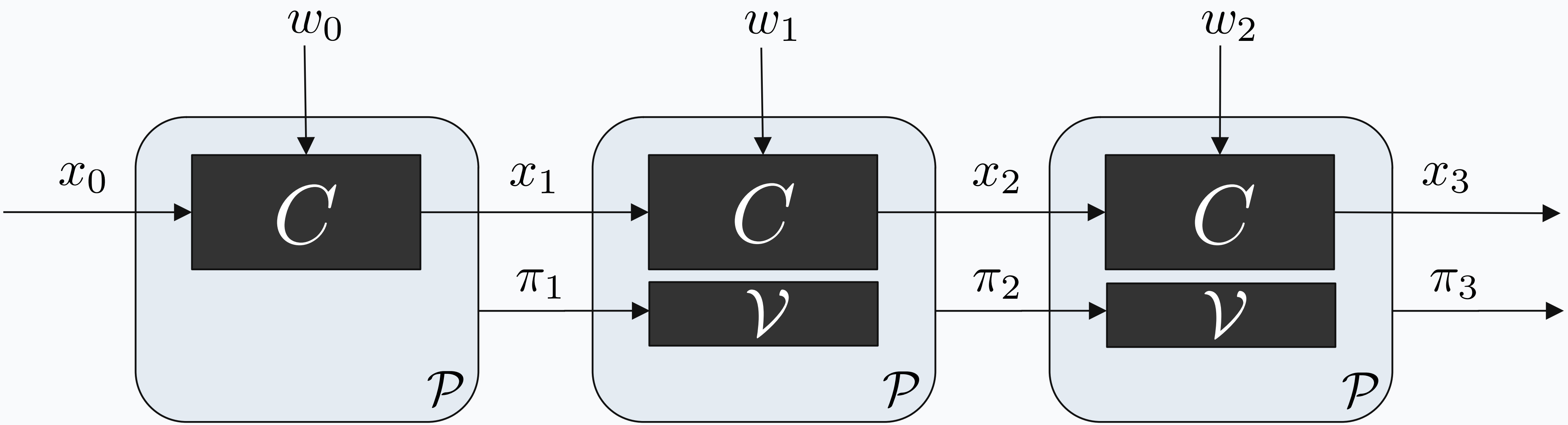

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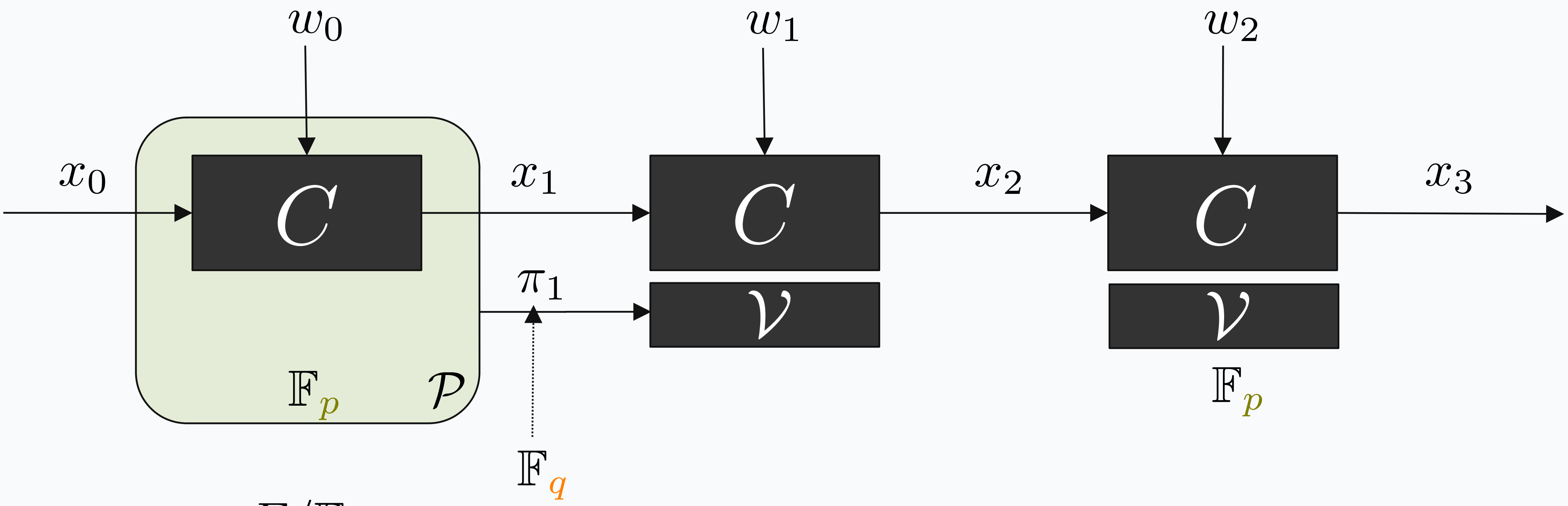
- We focus on the case of **pairing-friendly** SNARKs.
- We need to be able to write $\mathcal{V}$ in the language of the SNARK.

Recursion with cycles



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$\#E(\mathbb{F}_q) = p$

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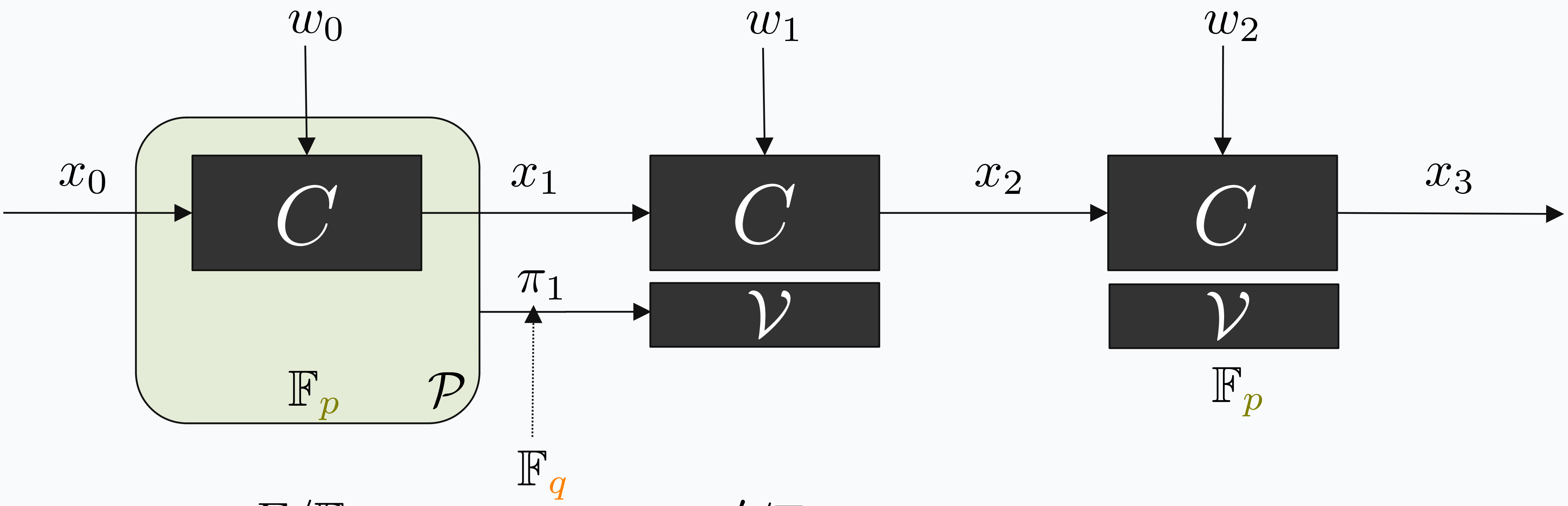

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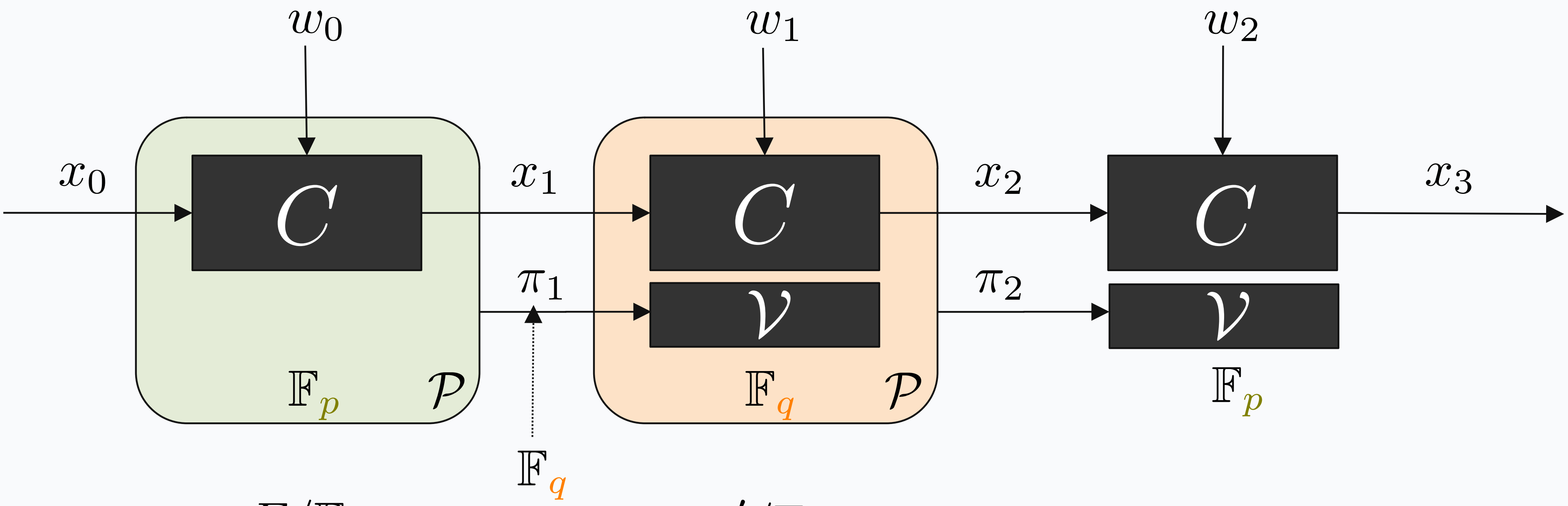

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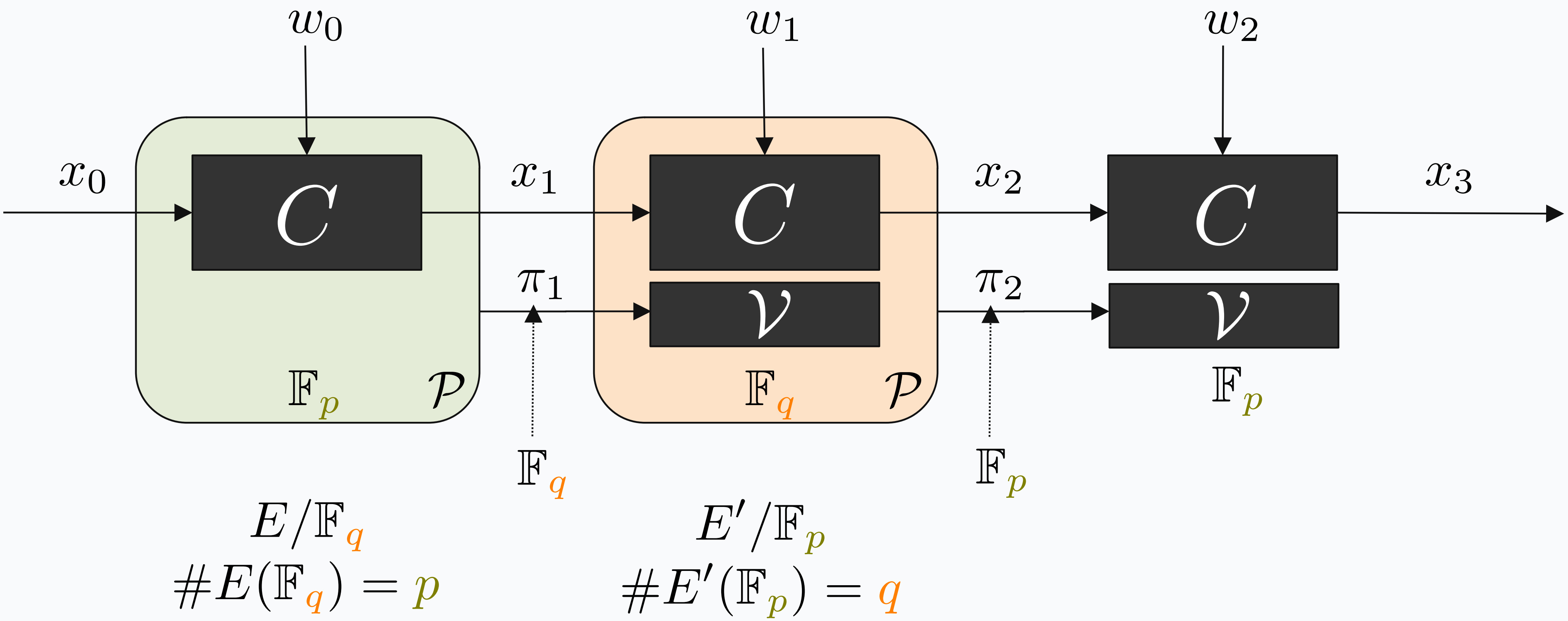

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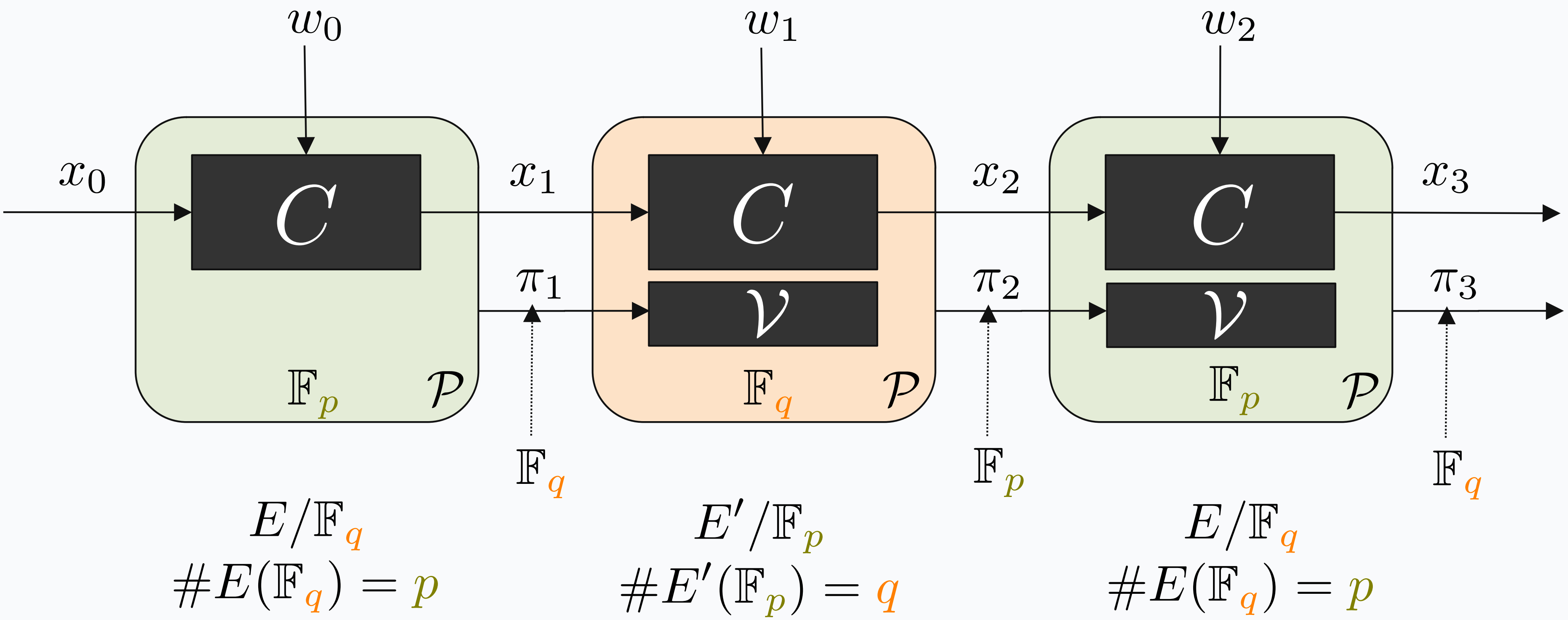

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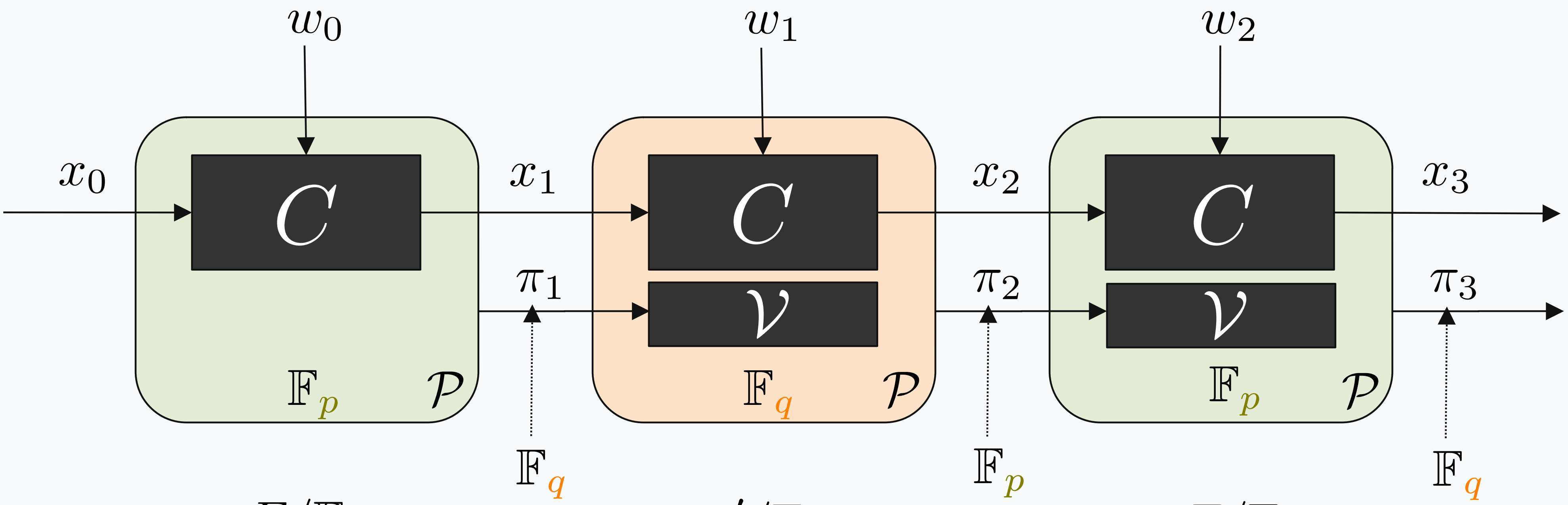

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$\#E(\mathbb{F}_q) = p$ $\quad$ $\#E'(\mathbb{F}_p) = q$ $\quad$ $\#E(\mathbb{F}_q) = p$

Only **two curves** needed in total.

A cycle of elliptic curves

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We want E, E' to be **pairing-friendly**.

$$e : E(\mathbb{F}_q) \times E(\mathbb{F}_q) \to \mu_p \subset \mathbb{F}_{q^k}$$

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← **embedding degree** of E

↑ roots of unity of order p

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k: **embedding degree** of E

μ_p: roots of unity of order p

$p \mid q^k - 1$ for **small** k, ℓ (to ensure efficient pairing computation),
$q \mid p^\ell - 1$ but **not too small** (to prevent security degradation).

Known facts about cycles

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✗

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Polynomial families

- $p(X)$, $q(X)$ polynomials.
- For infinitely many x, the values $p(x)$, $q(x)$ correspond to the order of the scalar field and base field of a curve, and both values are prime.

Polynomial families with prime order

MNT3 $(k = 3)$

$$p(X) = 12X^2 - 6X + 1$$

$$q(X) = 12X^2 - 1$$

MNT4 $(k = 4)$

$$p(X) = X^2 + 2X + 2$$

$$q(X) = X^2 + X + 1$$

MNT6 $(k = 6)$

$$p(X) = 4X^2 - 2X + 1$$

$$q(X) = 4X^2 + 1$$

Freeman $(k = 10)$

$$p(X) = 25X^4 + 25X^3 + 15X^2 + 5X + 1$$

$$q(X) = 25X^4 + 25X^3 + 25X^2 + 10X + 3$$

BN $(k = 12)$

$$p(X) = 36X^4 + 36X^3 + 18X^2 + 6X + 1$$

$$q(X) = 36X^4 + 46X^3 + 24X^2 + 6X + 1$$

Cycles from known families

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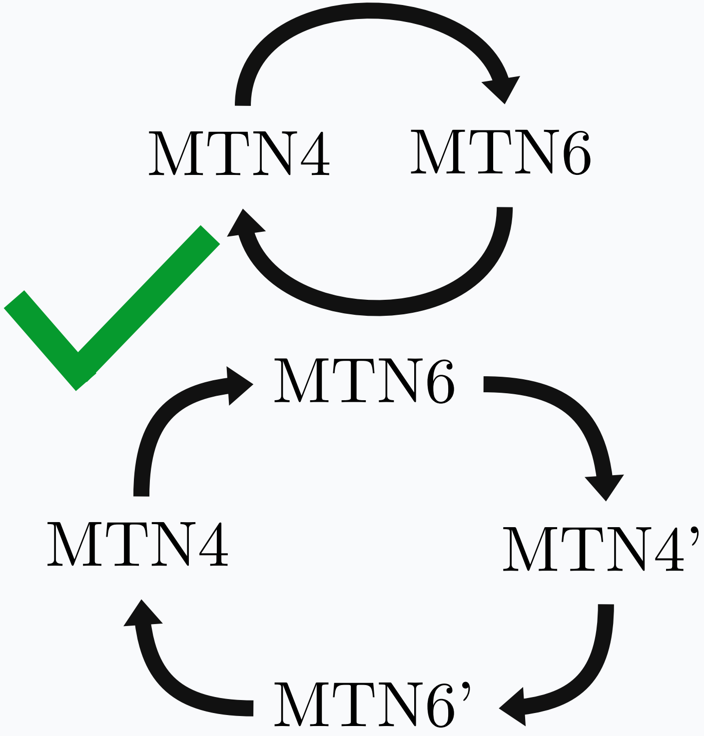

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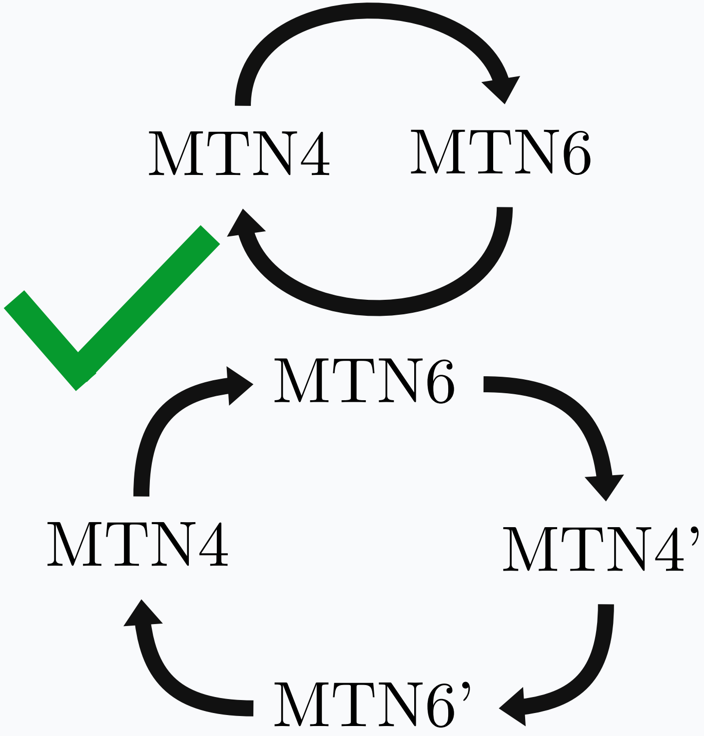

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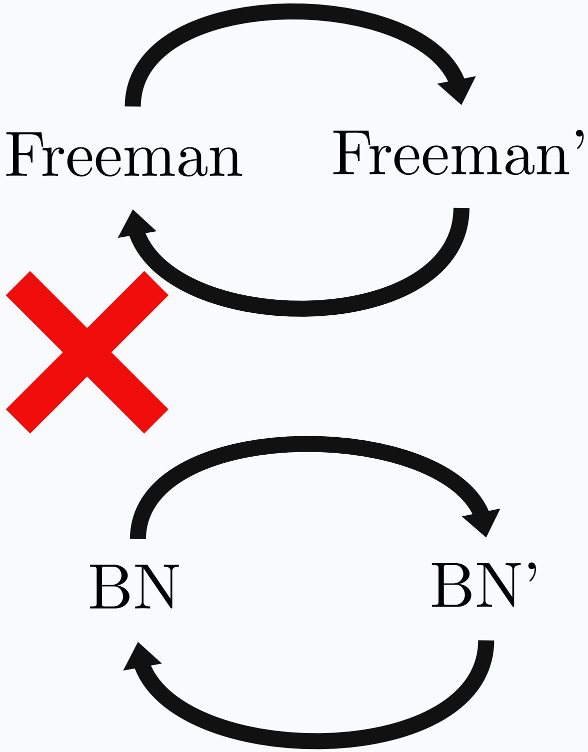

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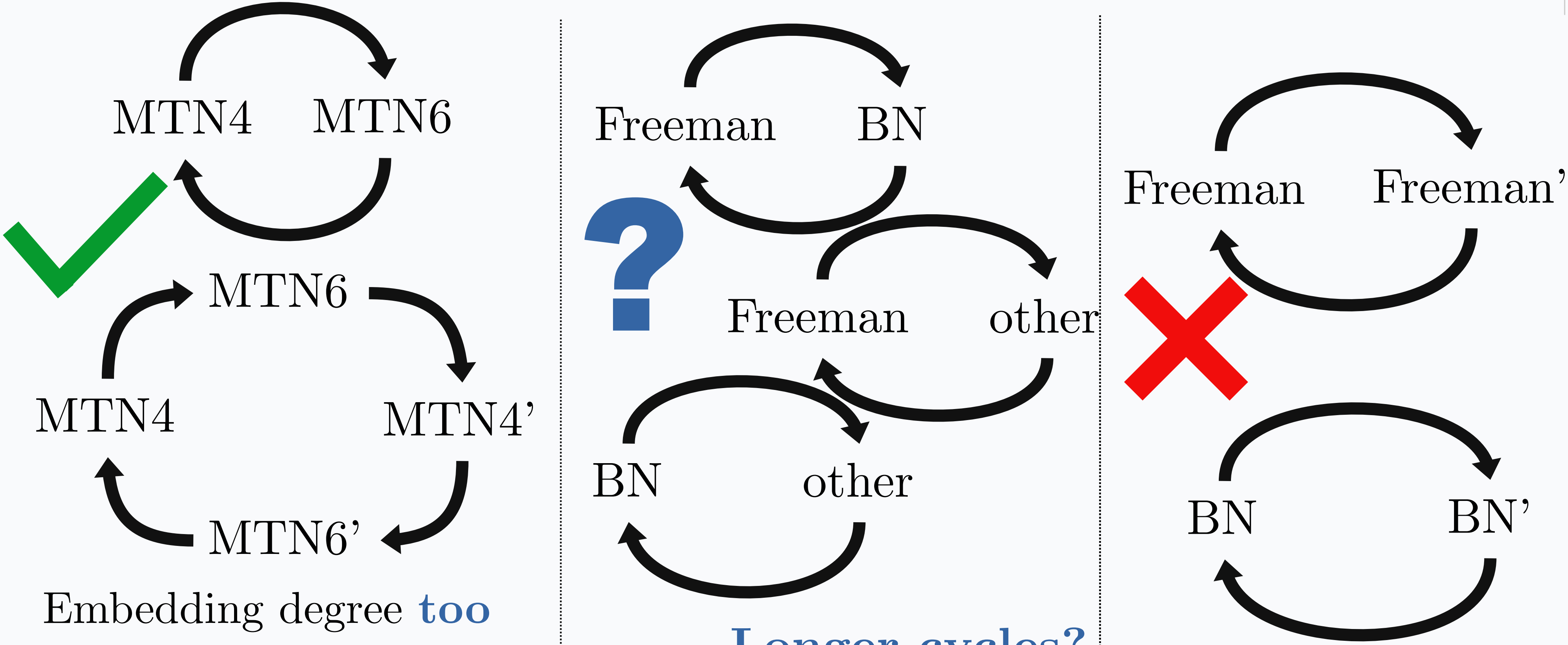

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Longer cycles?

Our results

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We can find explicit bounds on x for different embedding degrees.

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We could go higher, but the bounds on x grow quite fast.

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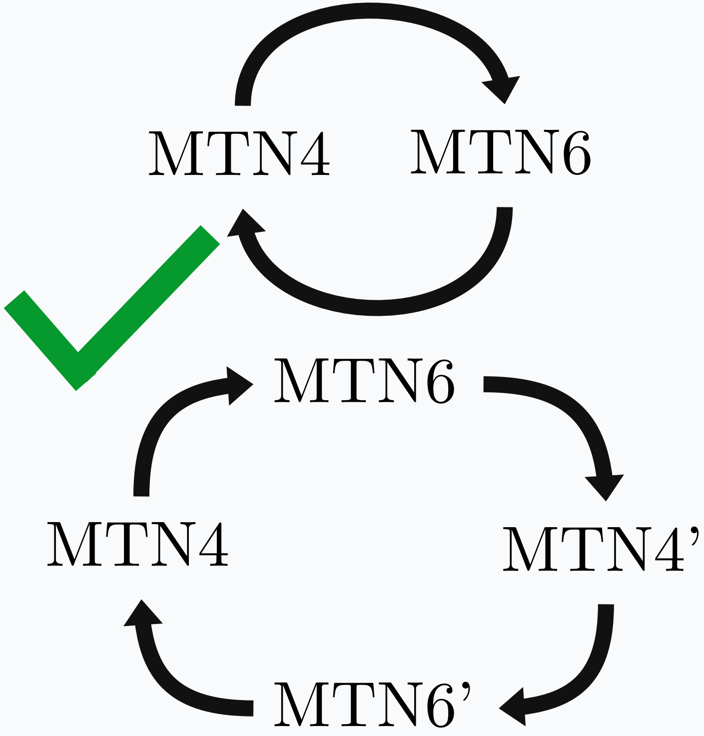

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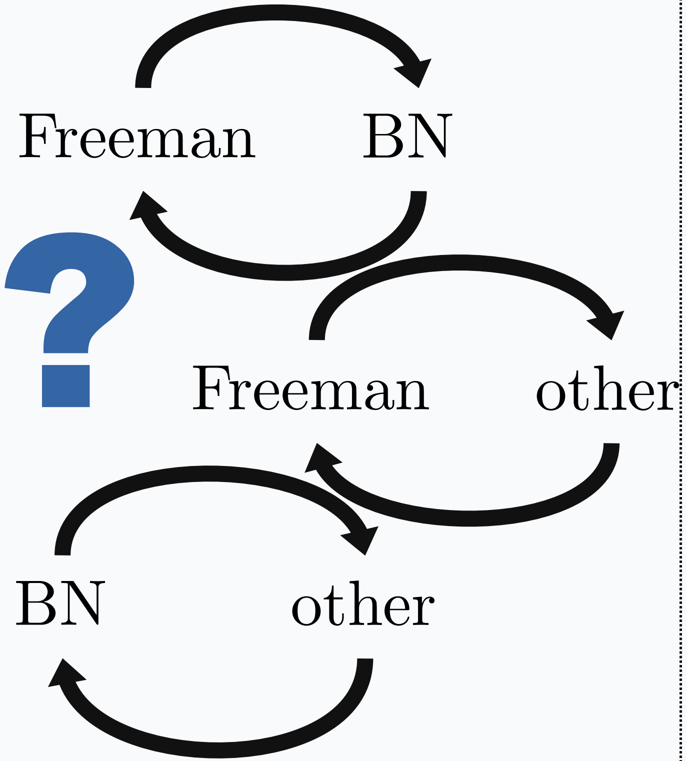
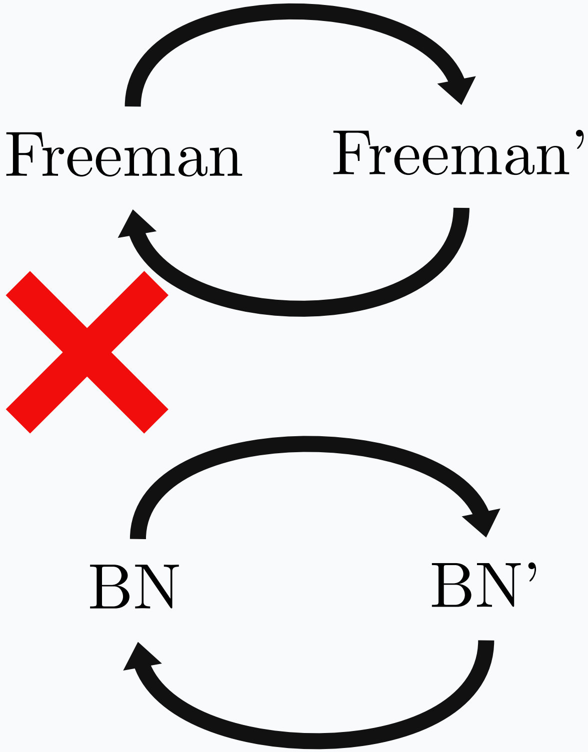
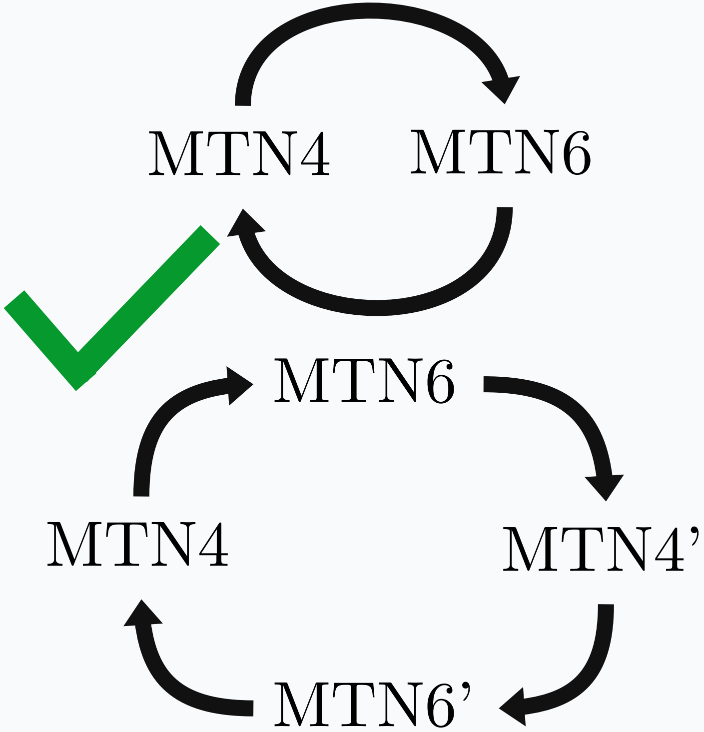


Longer cycles?



New impossibility results



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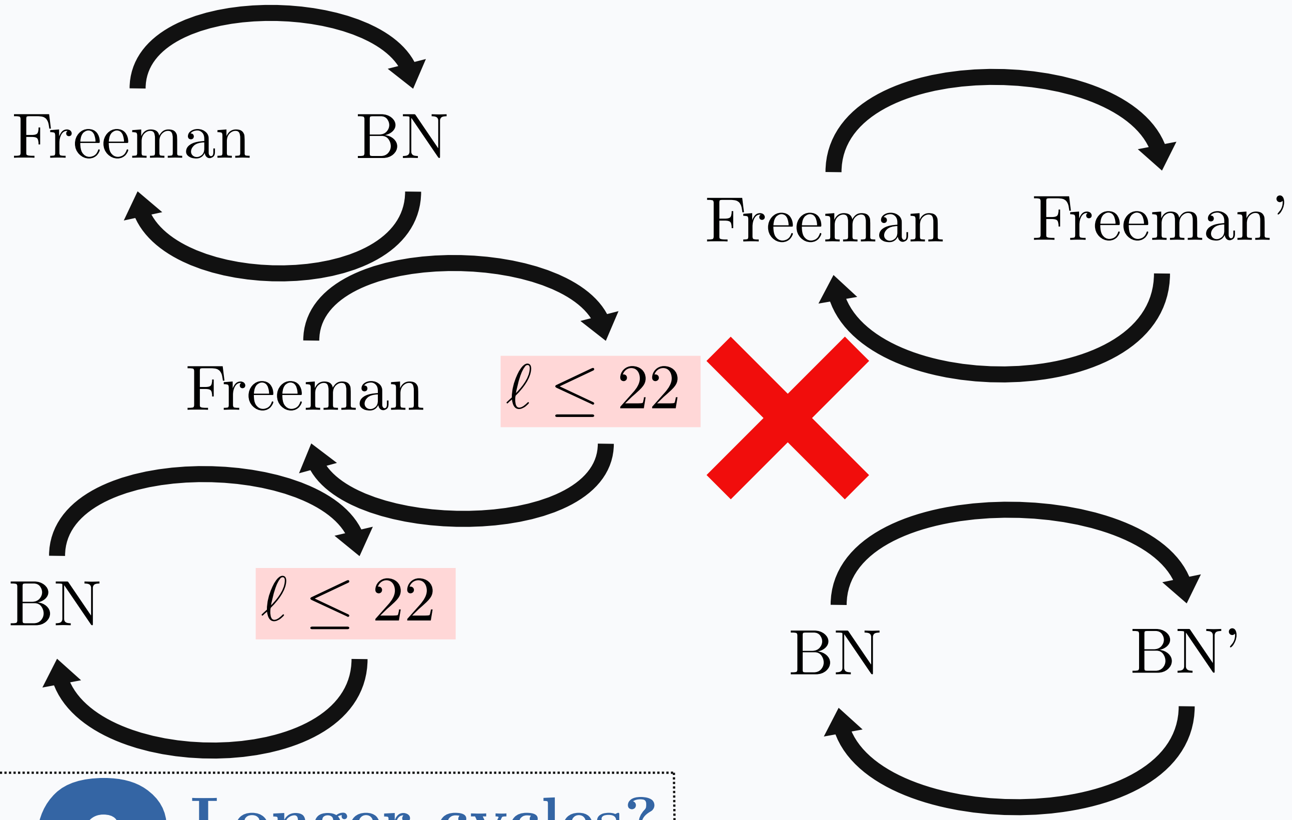

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- We also provide density estimates of pairing-friendly cycles among all cycles.

Thank you!

Paper

Code

Questions?