Algebraic Reductions of Knowledge

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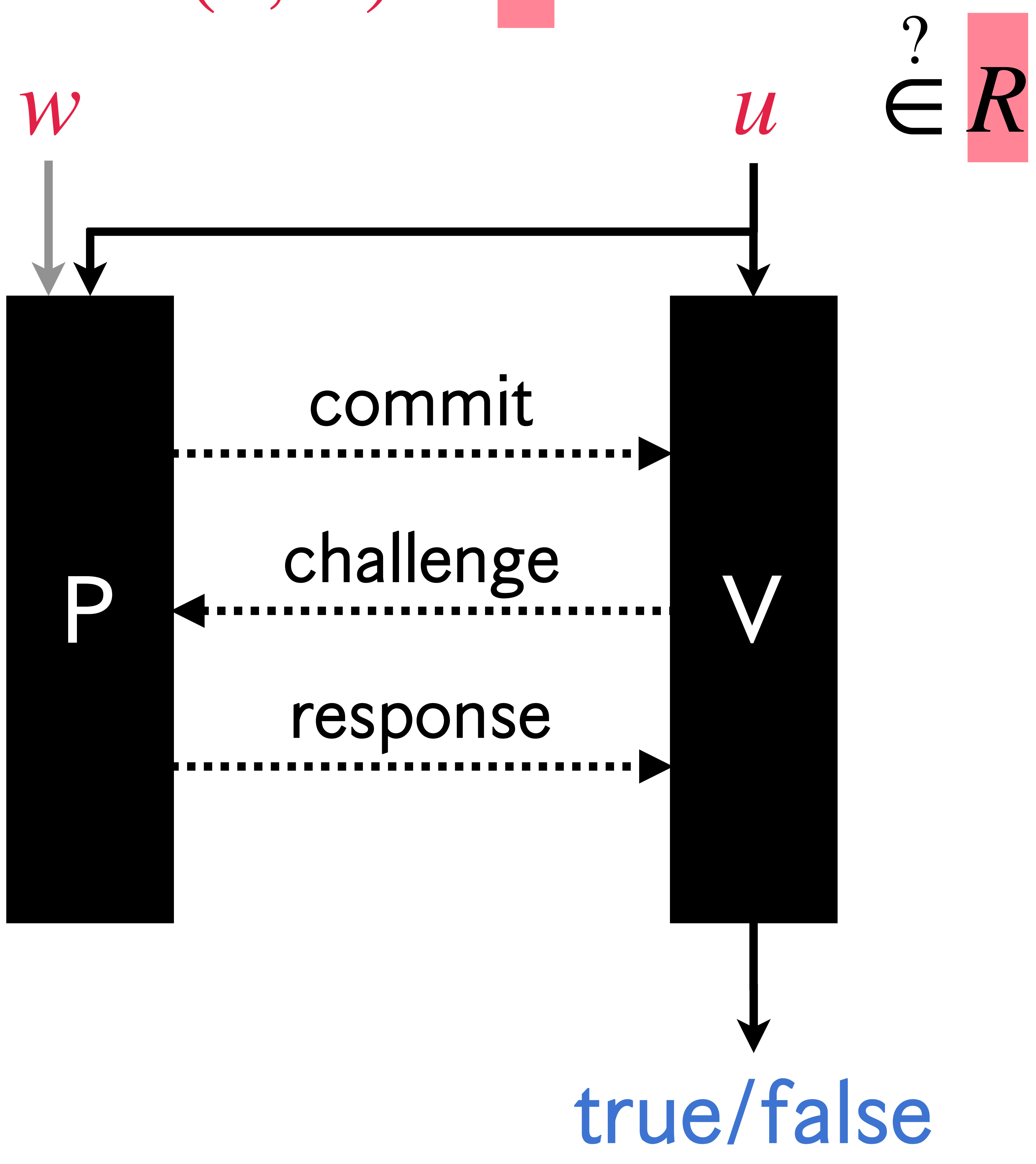
ia.cr/2022/009

Arguments of Knowledge [GMR85]

An argument of knowledge allows a prover to interactively show to a verifier that it *knows* witness w such that $(u, w) \in R$.



A Shift in Perspective

[BDFG21], [RZ21], [ACR21],
[**K**ST22], [BBBPWM18], [BC23],
[BCLMS21], [**K**S23], [CBBZ22],
[BCHO22], [Set20], [Bay13],
[BZ12], [BGH19], [CNRZZ22],
[BCS21], [BMMTV21], [AC20],
[LFKN92], [GKR15], [Lee21],
[Val08], [RZ22], [BCCGP16],
+

Emerging paradigm: The verifier does not fully resolve the prover’s statement, but rather reduces it to a simpler statement to be checked.

Recursive Inner-Product Argument

"The basic step in our inner product argument is a 2-move reduction to a smaller statement."

- Bootle, Cerulli, Chaidos, Groth, and Petit,

Eurocrypt 2016

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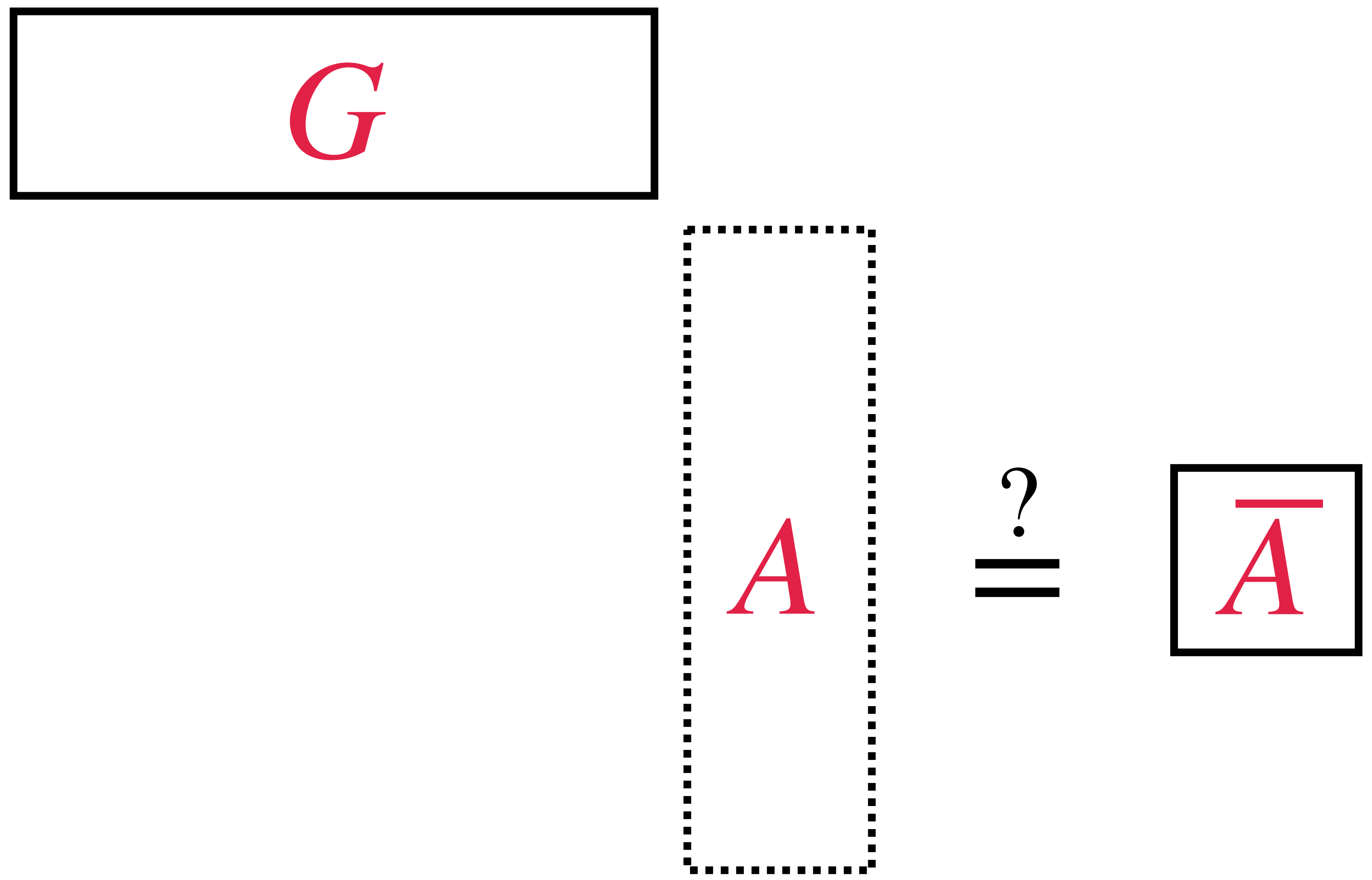

Length n inner-product

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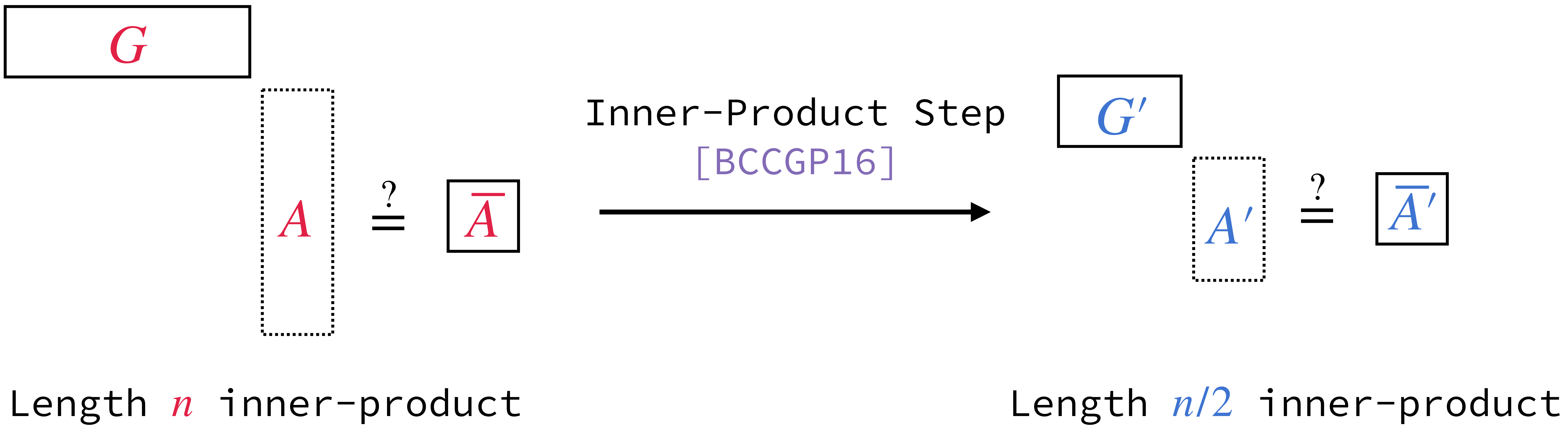

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Polynomial Aggregation

"If the prover has a witness for $(\overline{P}, x, y)$, then it must have witnesses for $(\overline{P}, x_1, y_1), \ldots, (\overline{P}, x_n, y_n)$."

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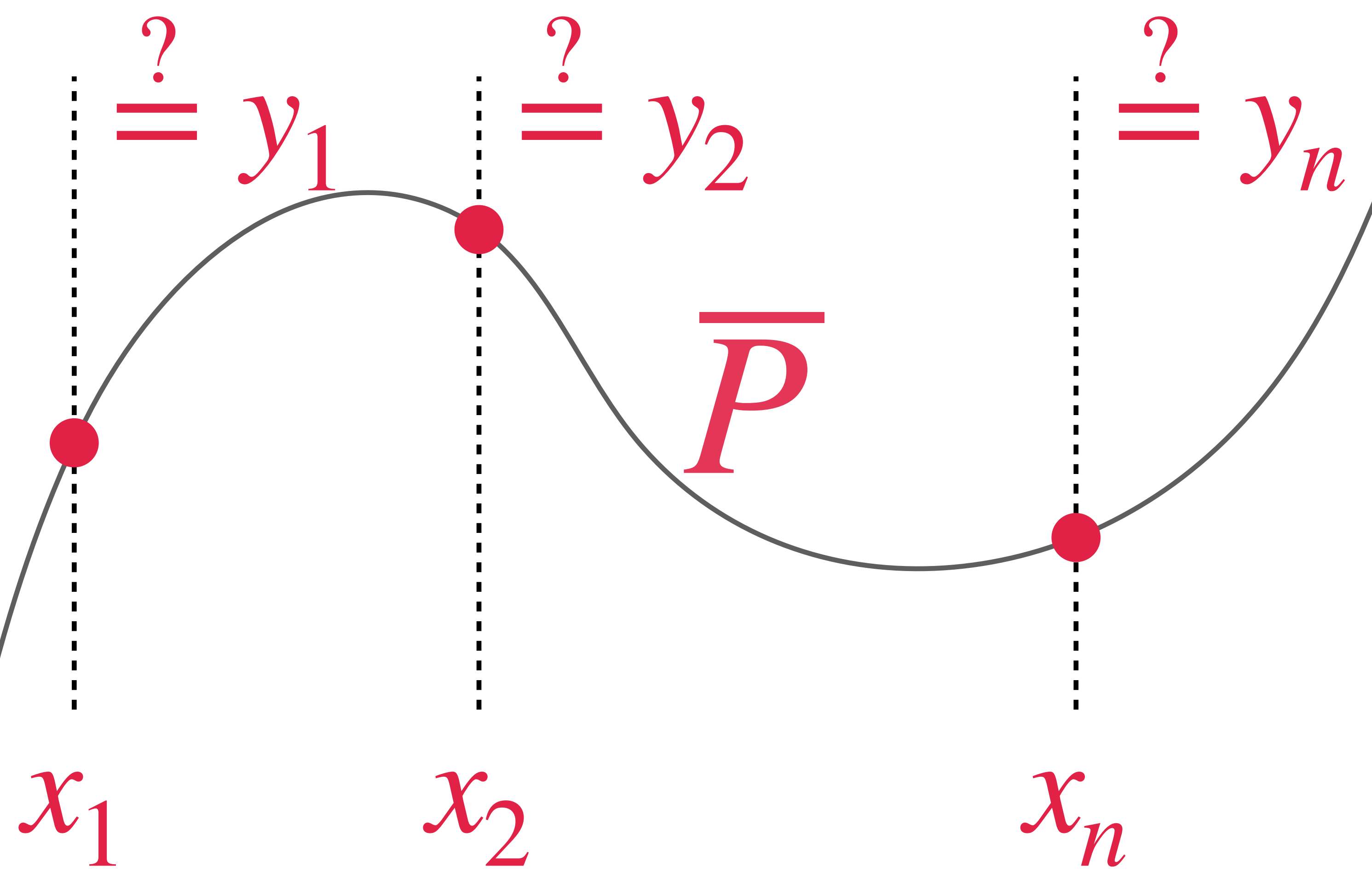

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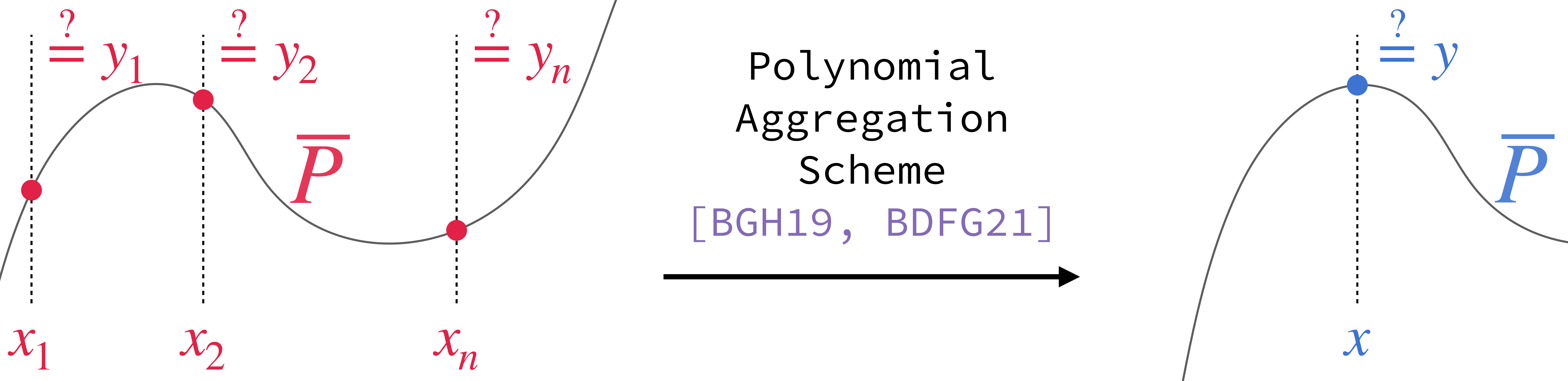

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Folding Schemes

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Joint work with Setty and Tzialla,

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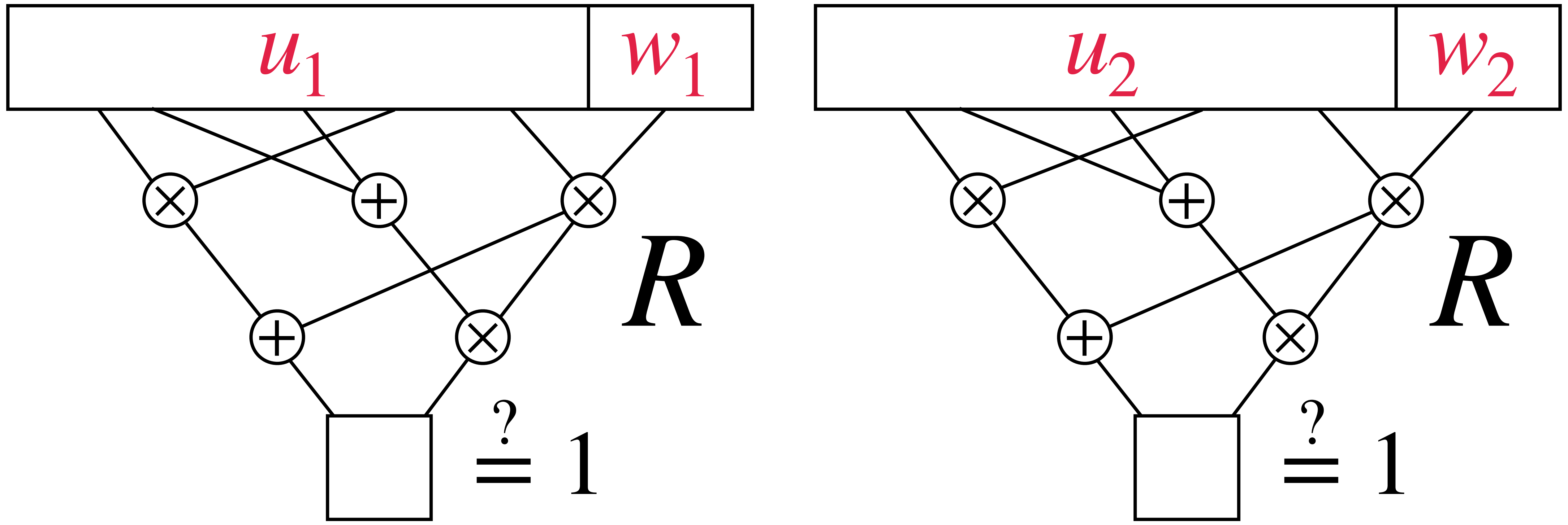

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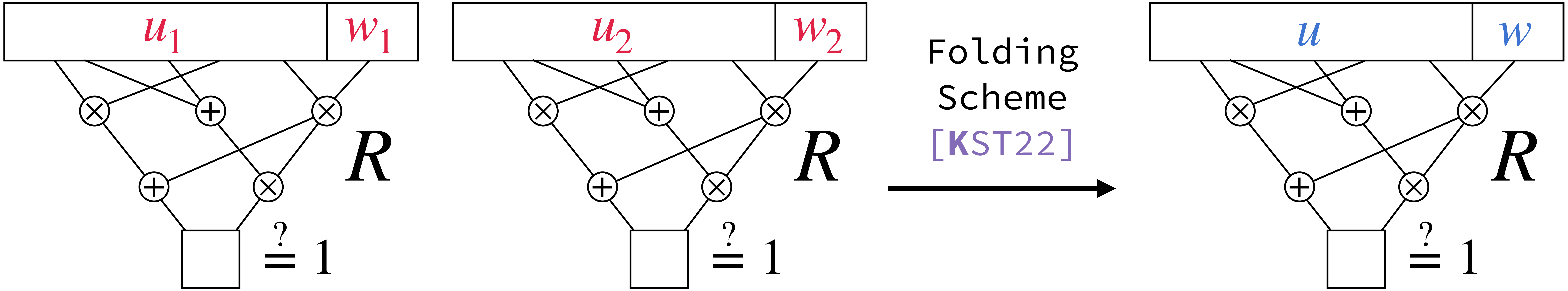

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Algebraic Arguments for NP

"We reduce R1CS constraint systems to three algebraic relations"

- Ràfols and Zapico,

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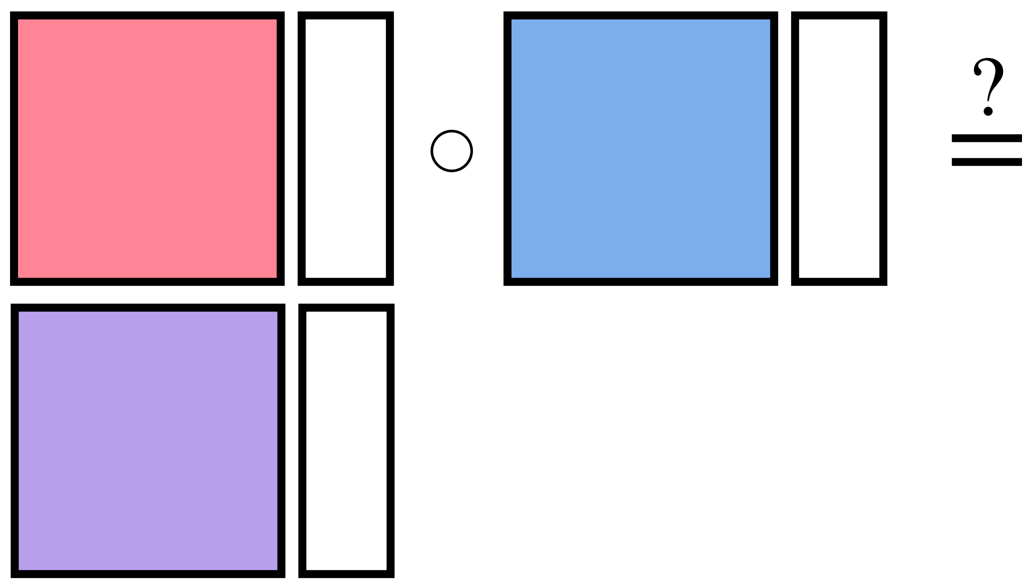

R1CS

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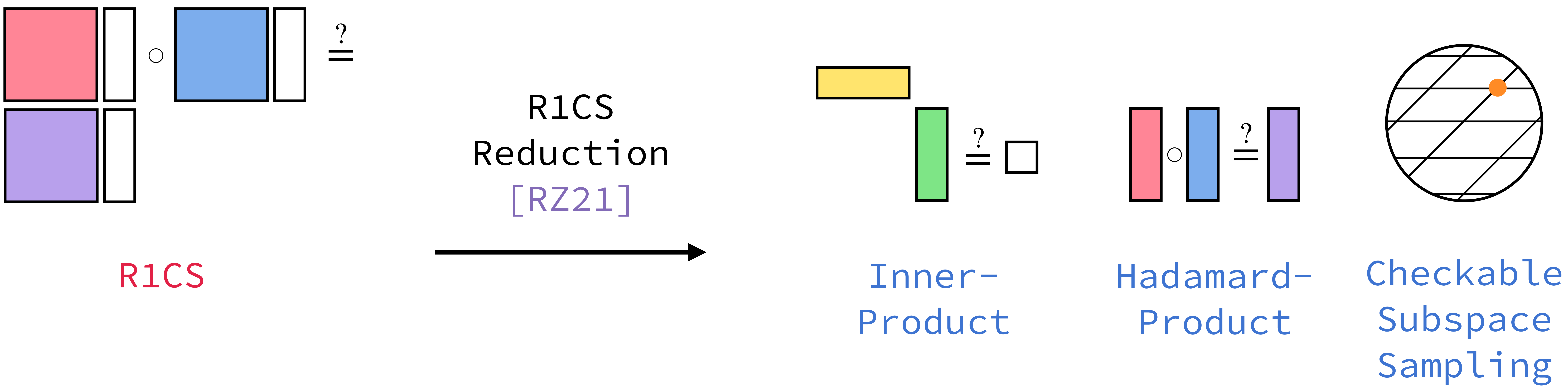

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Modern Arguments are Reductions

Split-accumulation schemes reduce the task of checking n instances and accumulators into the task of checking single accumulator. [BCLMS21]

Aggregation schemes for polynomial commitments reduce the task of checking several openings to the task of checking a single opening. [BDFG21]

The ZeroCheck protocol reduces the task of checking that a polynomial vanishes on a set to a Sumcheck. [BTVW14, Set20, CBBZ22]

The tensor-product protocol reduces the task of checking an inner-product with a structured vector to the task of checking several univariate polynomial evaluations. [BCHO22]

The Hadamard-product protocol reduces the task of checking a Hadamard product to the task of checking an inner-product. [Bay13]

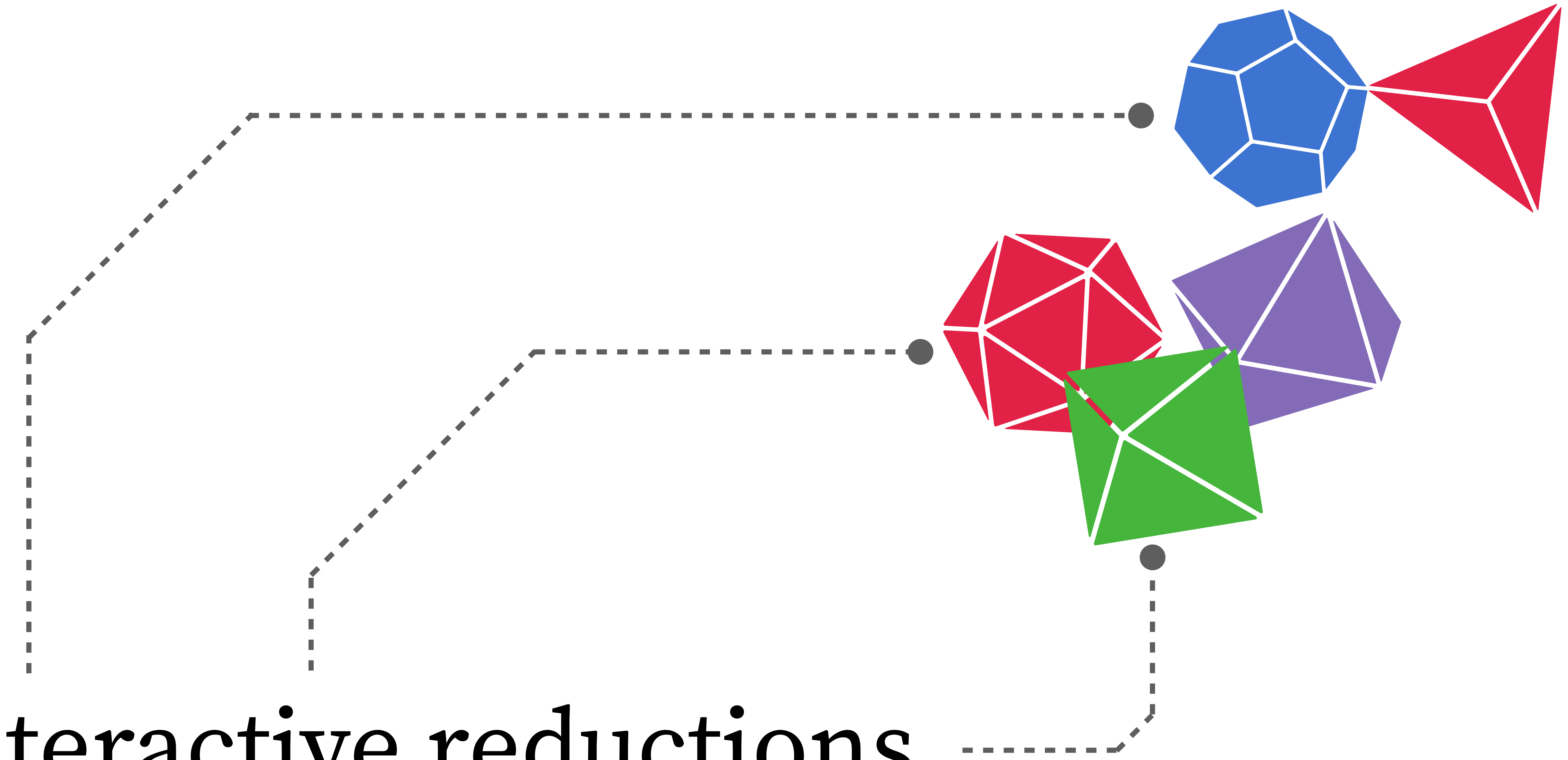
Inner-product arguments reduce the the task of checking the inner-product of size n vectors to checking the inner-product of size $n/2$ vectors. [BCCGP16, BBBPWM18, BMMTV21, Lee21]

Checkable subspace sampling reduces the task of checking matrix evaluations to the task of checking vector evaluations. [RZ21]

Incrementally verifiable computation reduces the task of checking a succinct proof of n applications of function F and a succinct proof of m subsequent applications of F to the task of checking a succinct proof of $n + m$ applications of F. [Val08]

The zero-knowledge HPI argument reduces the task of checking a pre-image of a homomorphism y to the task of checking a pre-image of a randomized homomorphism y'. [BDFG21]

Problem: Need a Unifying Theory



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making it difficult to compose compatible techniques hidden under incompatible abstractions.

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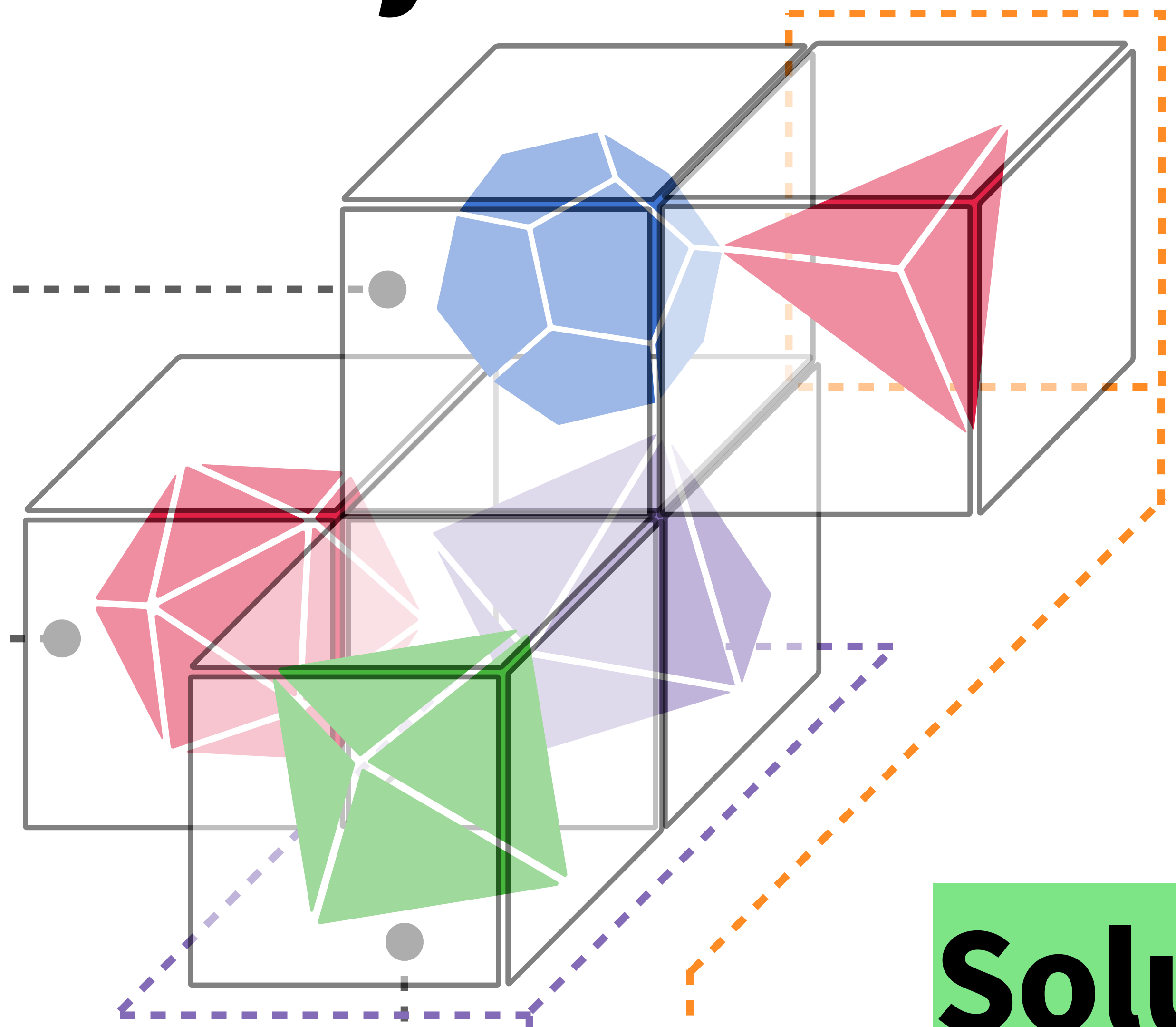


Interactive reductions are universal; their definitions are not

making it difficult to compose compatible techniques hidden under incompatible abstractions.

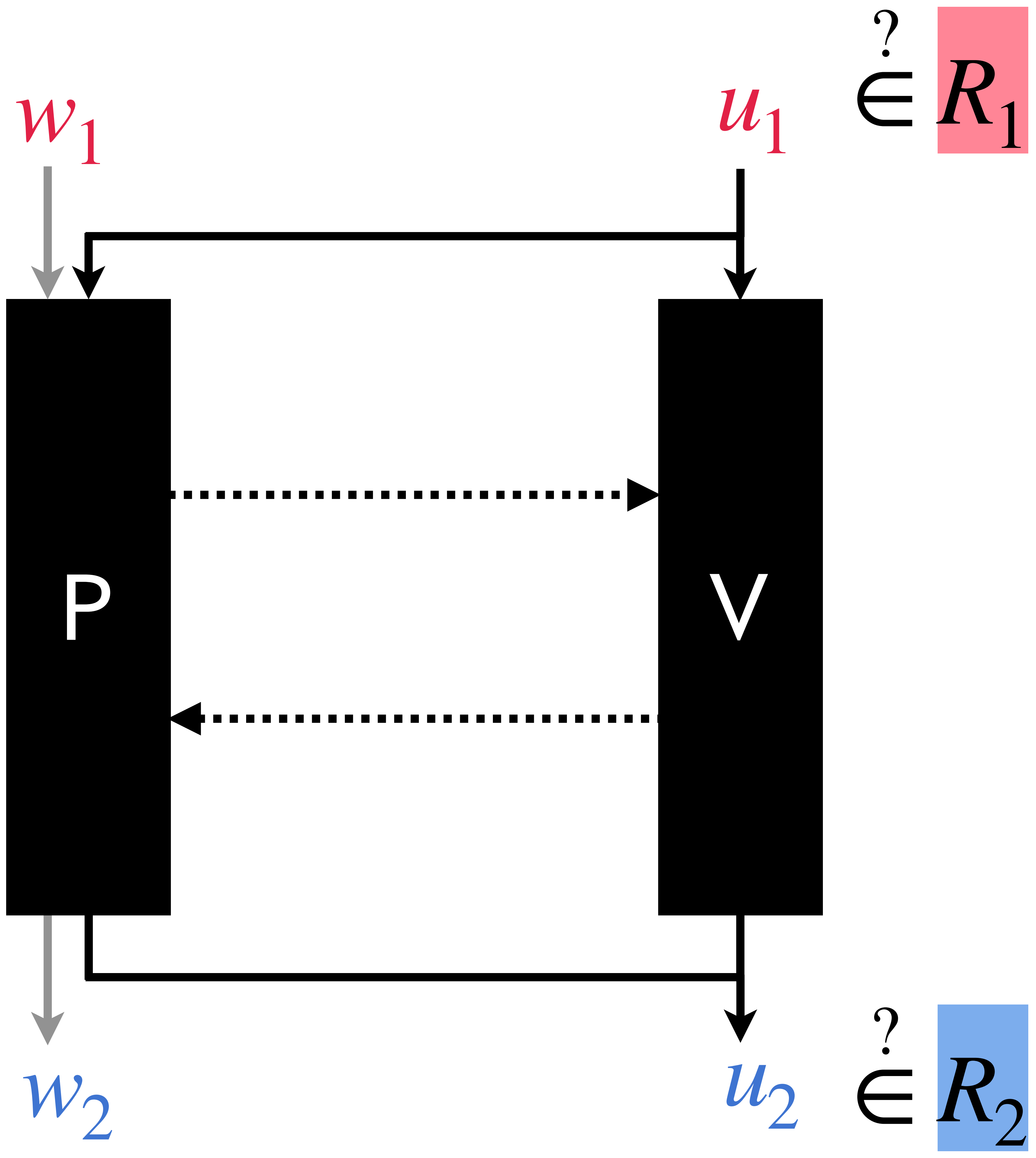
Solution

We formalize **reductions of knowledge** as a common language

which serve as both a unifying abstraction and a compositional framework.

Reductions of Knowledge: A Unifying Language

A reduction of knowledge interactively reduces the claim $(u_1, w_1) \in R_1$ to a claim $(u_2, w_2) \in R_2$



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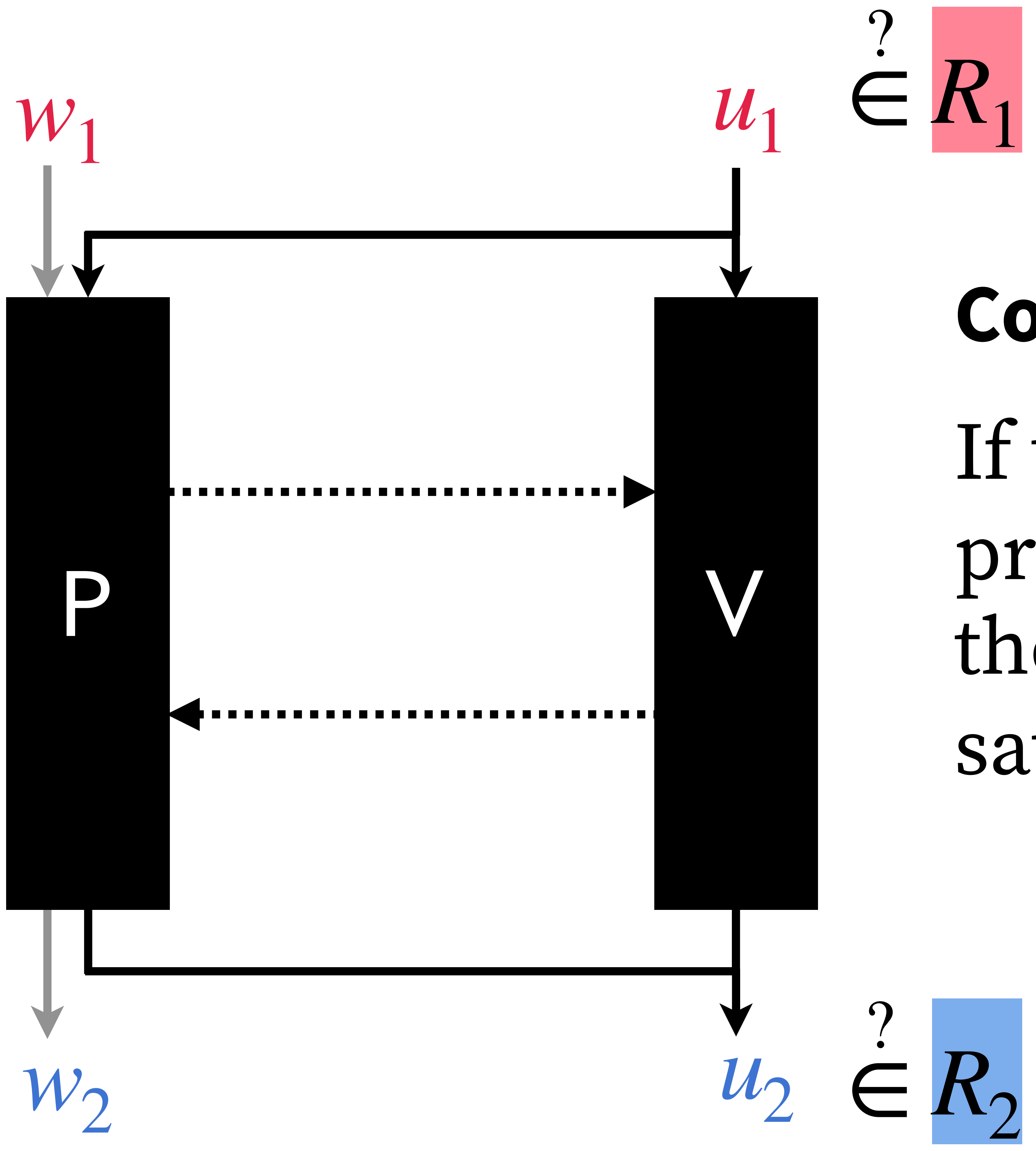

Completeness

If the prover is provided satisfying w_1 then it must output a satisfying w_2

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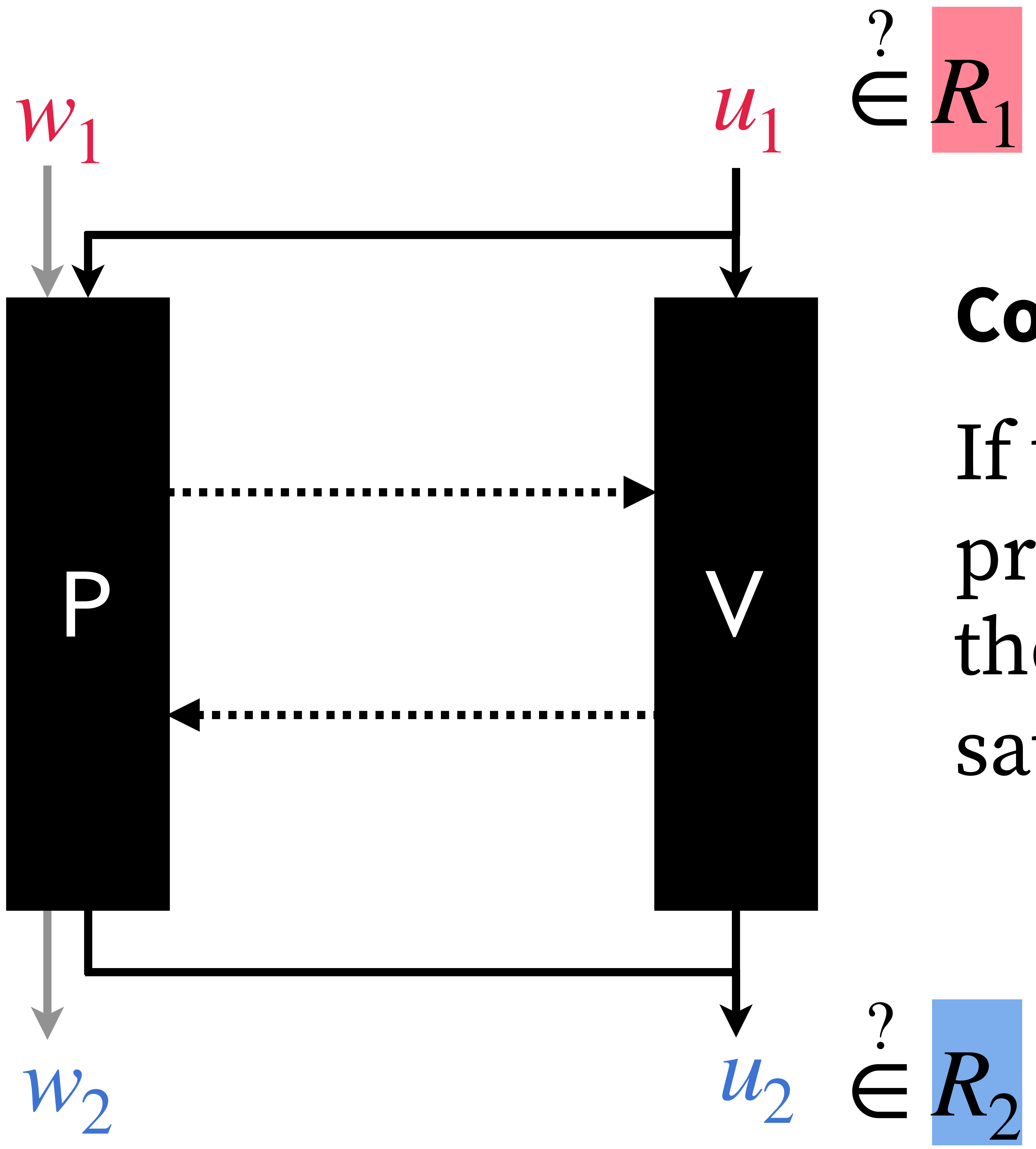


Knowledge Soundness

If prover outputs satisfying w_2 then it must almost certainly *know* a satisfying w_1

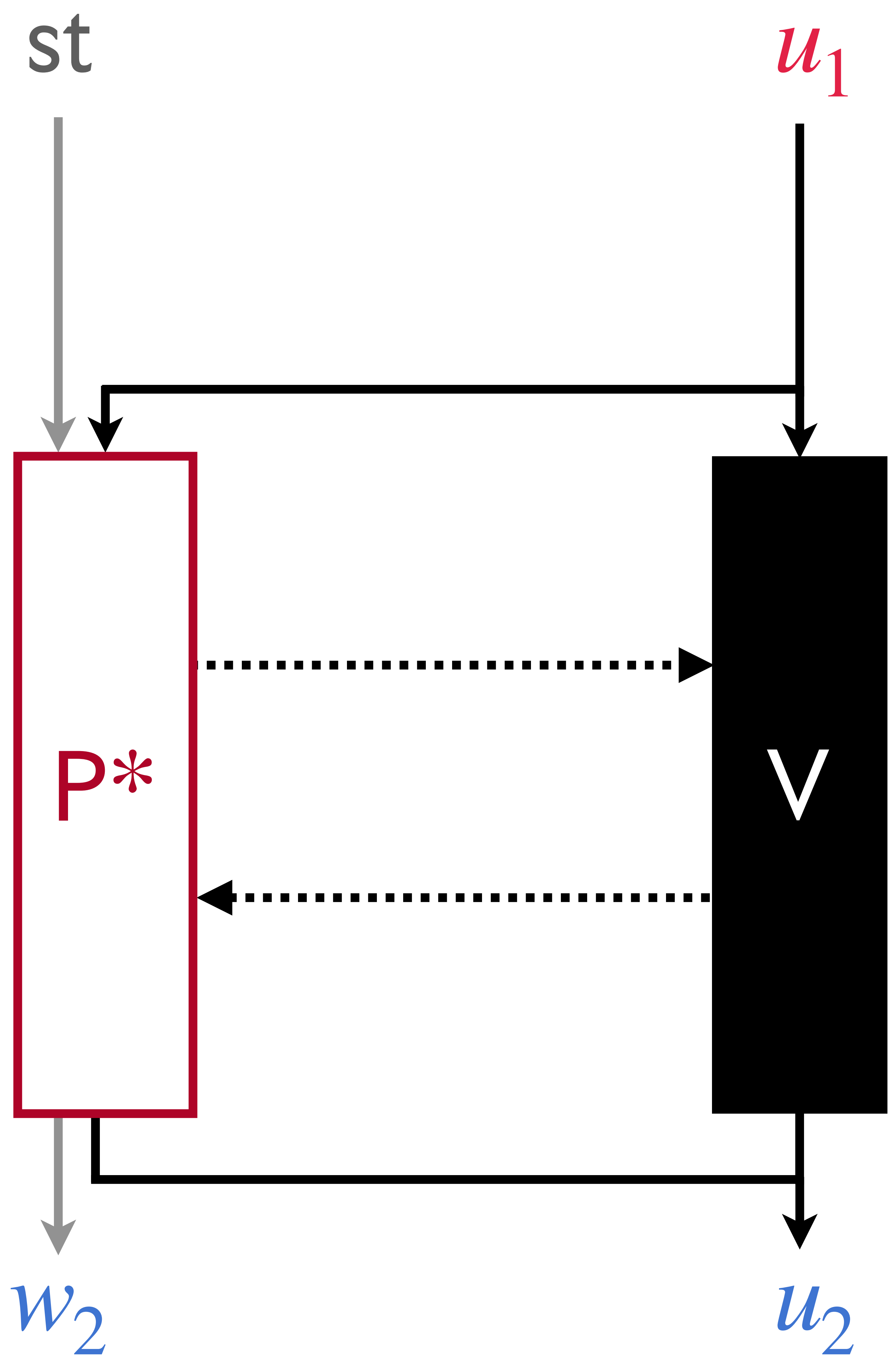
Completeness

If the prover is provided satisfying w_1 then it must output a satisfying w_2

Knowledge Soundness

Consider P^* s.t. for (u_1, st)

$$\Pr[\langle P^*, V\rangle(u_1, \mathrm{st}) \in R_2] = \varepsilon$$



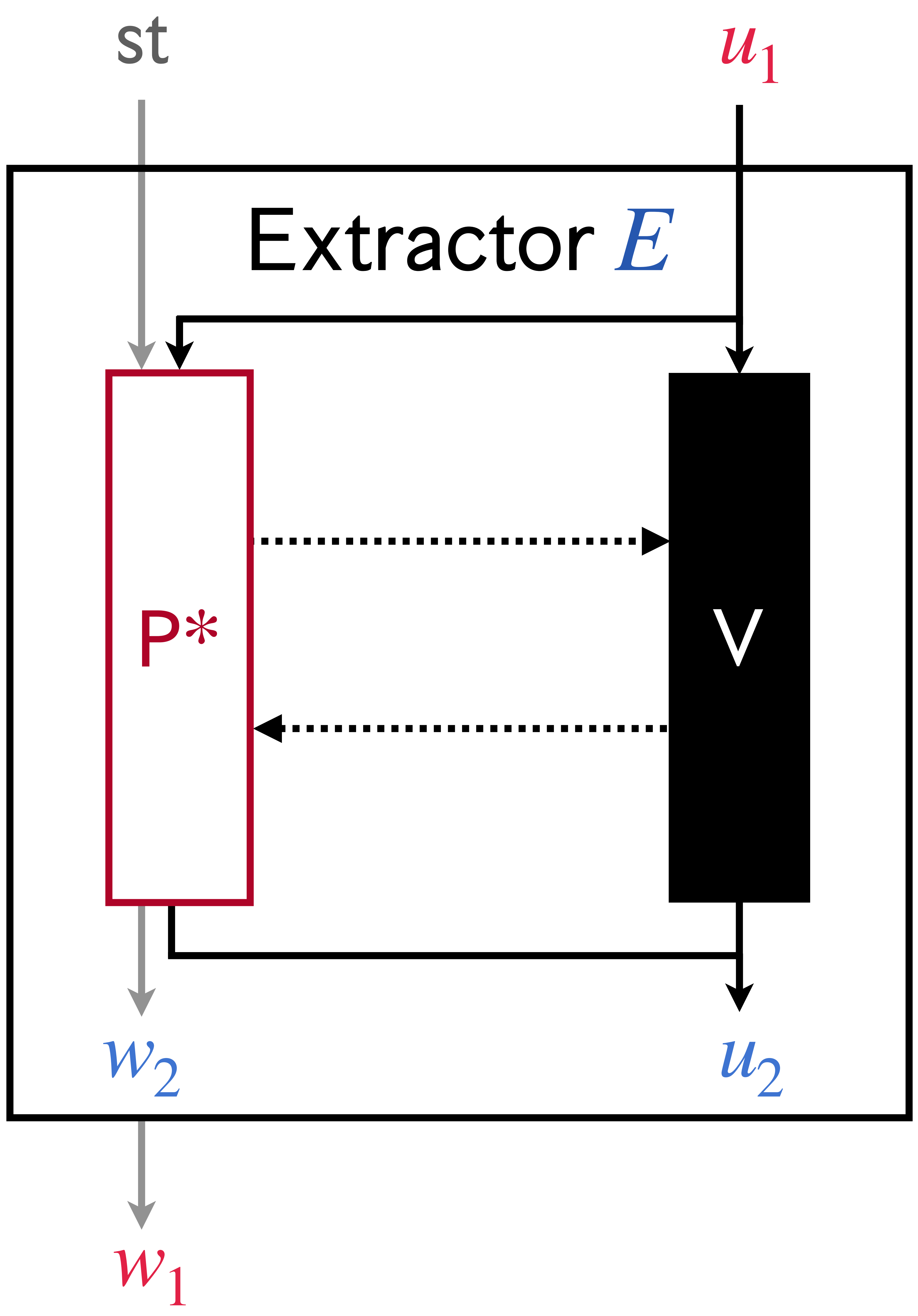
Knowledge Soundness

Consider P^* s.t. for (u_1, st)

$$\Pr[\langle P^*, V\rangle(u_1, \mathsf{st}) \in R_2] = \varepsilon$$

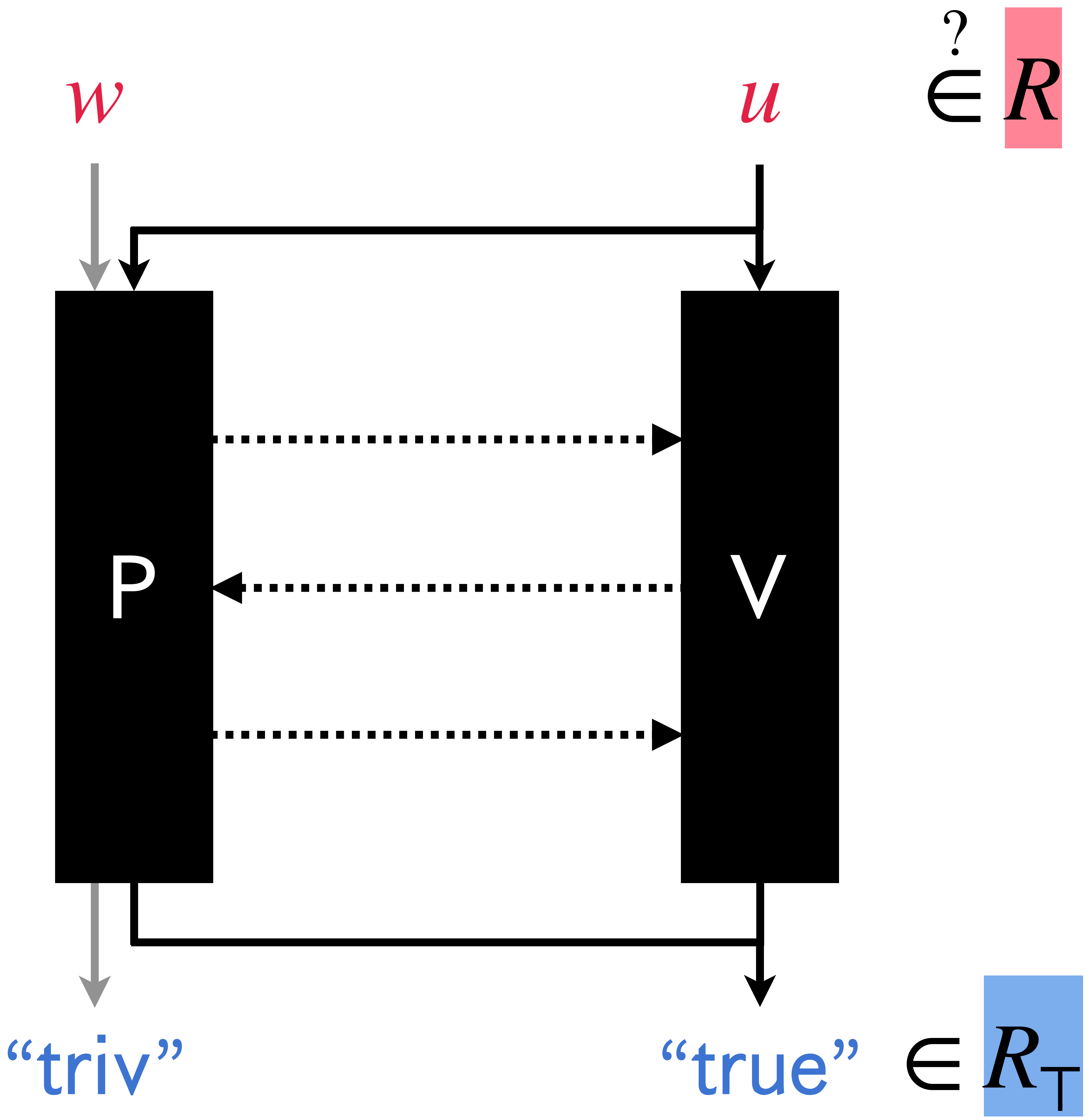
Then there exists an extractor E s.t.

$$\Pr[(u_1, E(u_1, \mathsf{st})) \in R_1] \approx \varepsilon$$



Reconciling Reductions with Arguments

An **argument of knowledge** is a reduction of knowledge from R to $R_\top = \{(\text{"true"}, \text{"triv"})\}$.



First Example: Inner-Product Reduction [BCCGP16]

Define the Inner-Product Relation as

$$R_{\mathsf{IP}}(n) = \left\{ \left((G, \overline{A}), A \right) \in \left((\mathbb{G}^n, \mathbb{G}), \mathbb{F}^n \right) \middle| \left\langle G, A \right\rangle = \overline{A} \right\}$$

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Satisfying Condition

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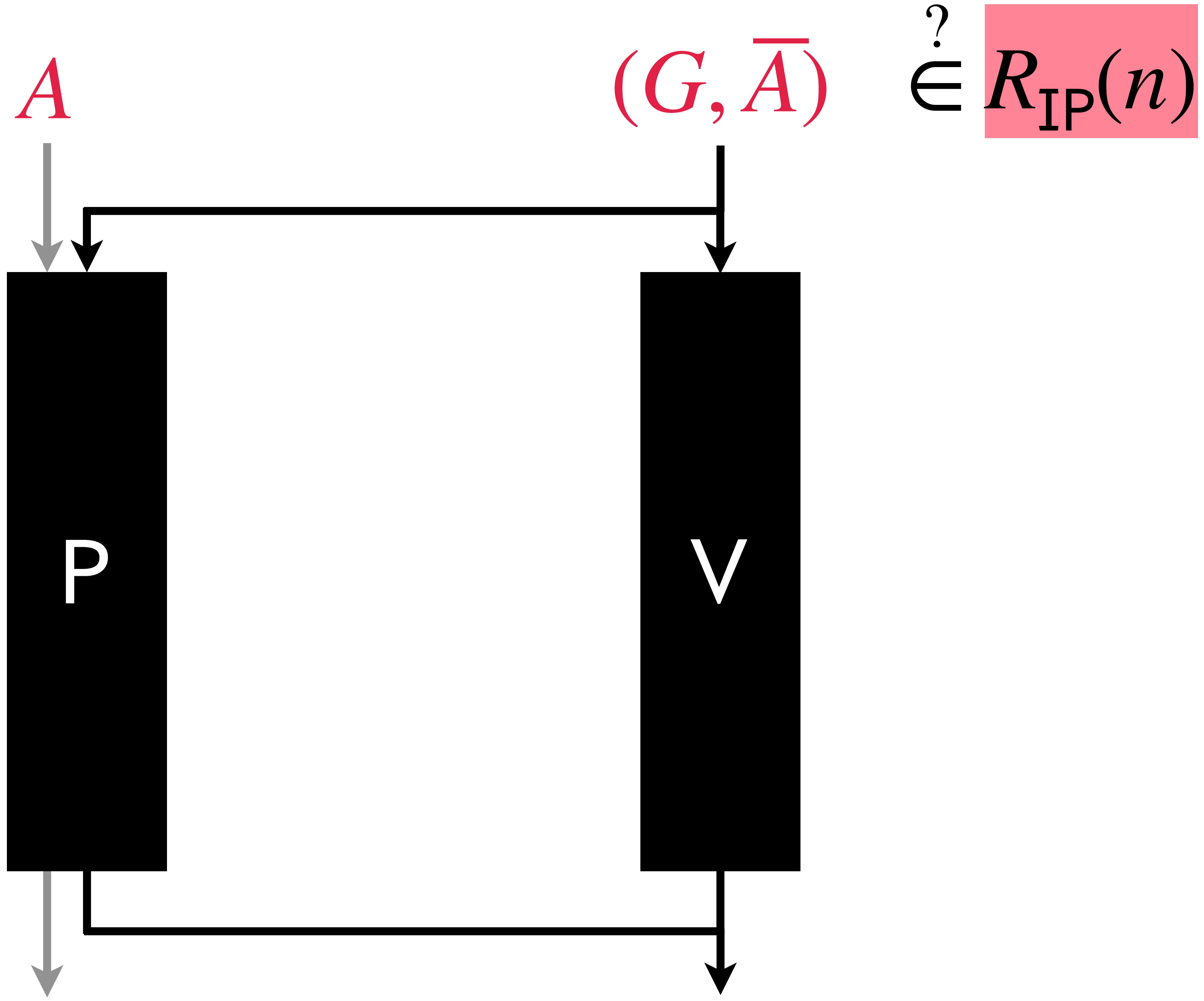
There exists a reduction of knowledge from $R_{\mathsf{IP}}(n)$ to $R_{\mathsf{IP}}(n/2)$.



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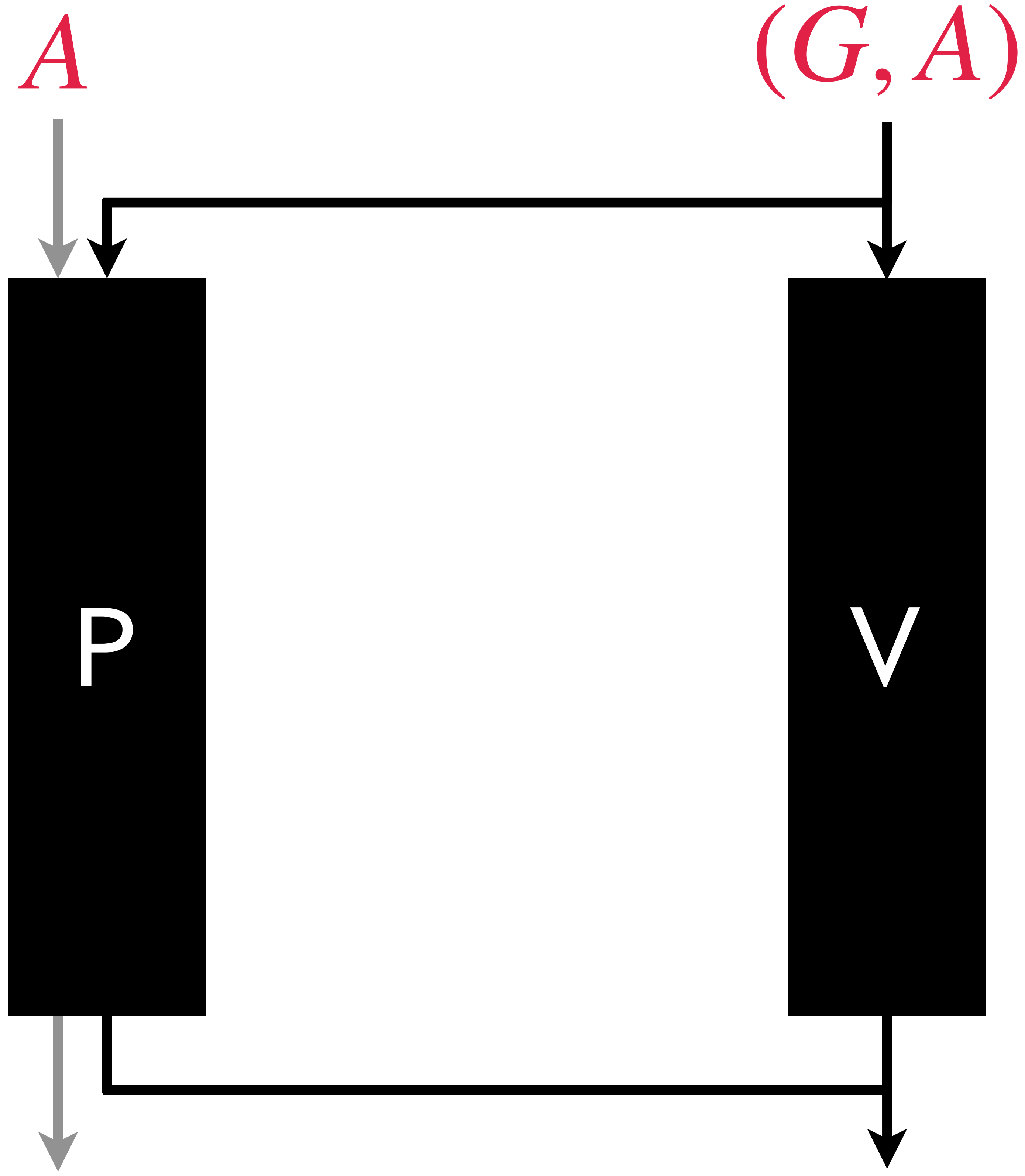
$\overline{A}_{ij} \leftarrow G_i(A_j)$ for $i, j \in \{1,2\}$



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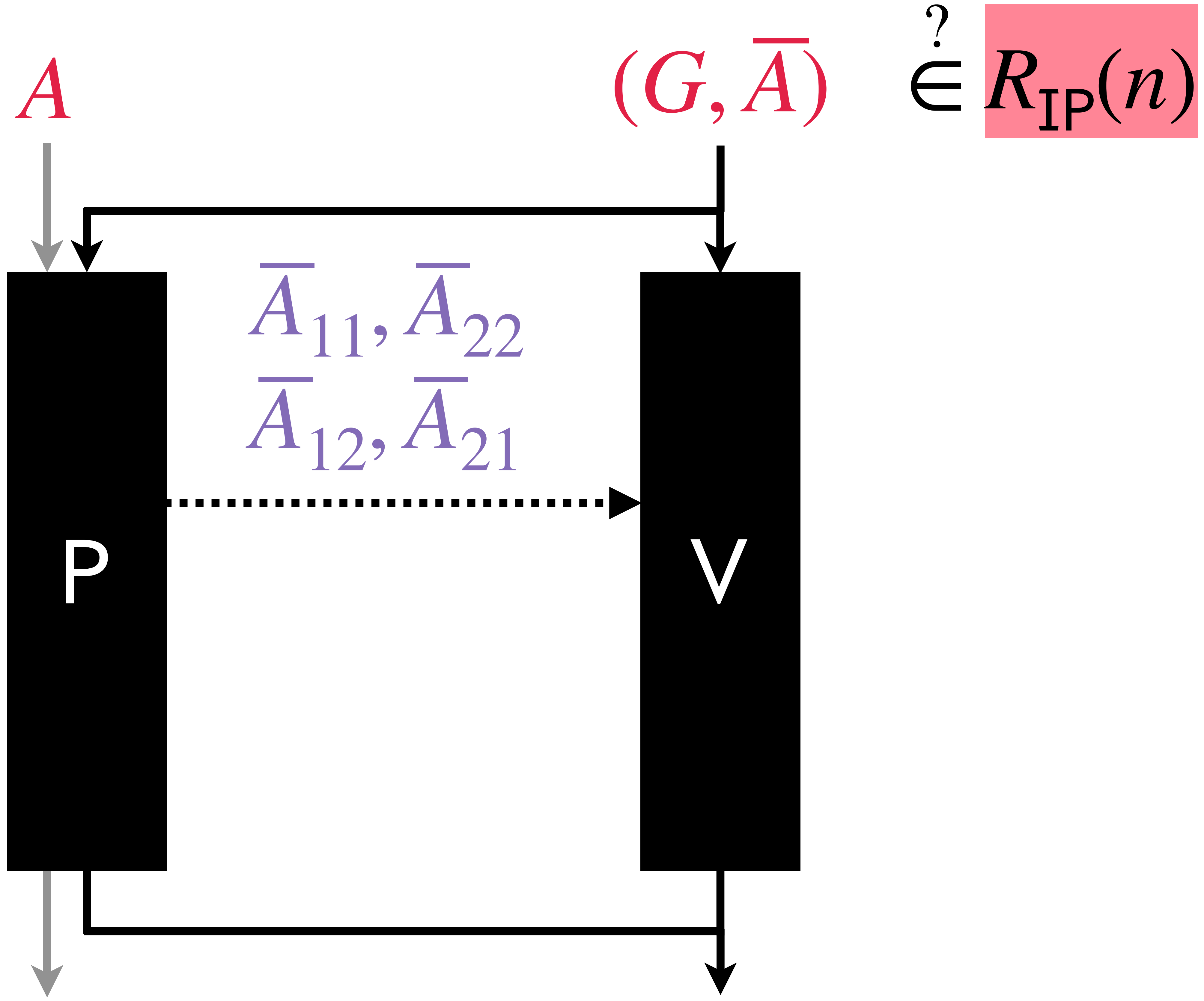

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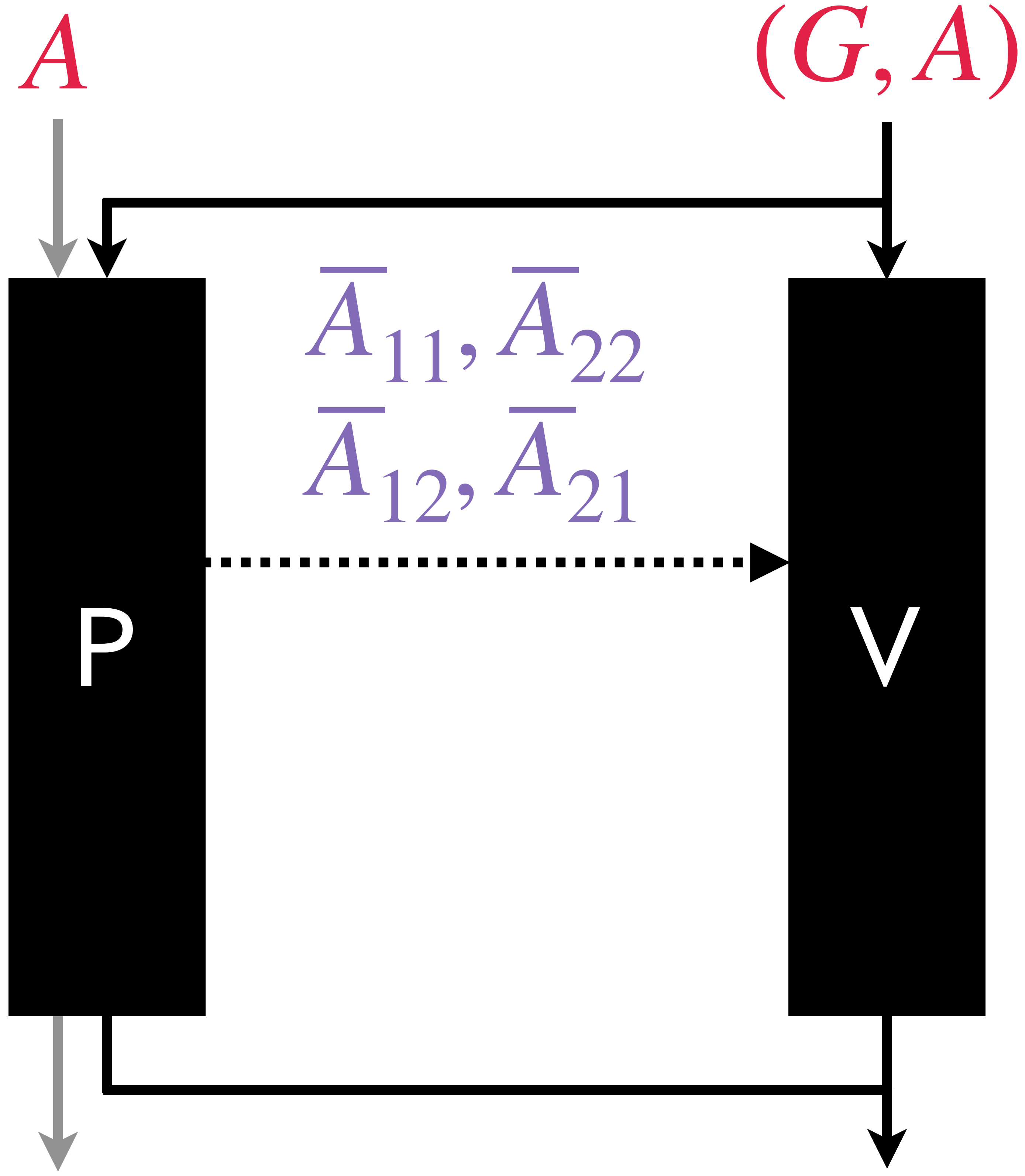

$r \xleftarrow{\$} \mathbb{F}$

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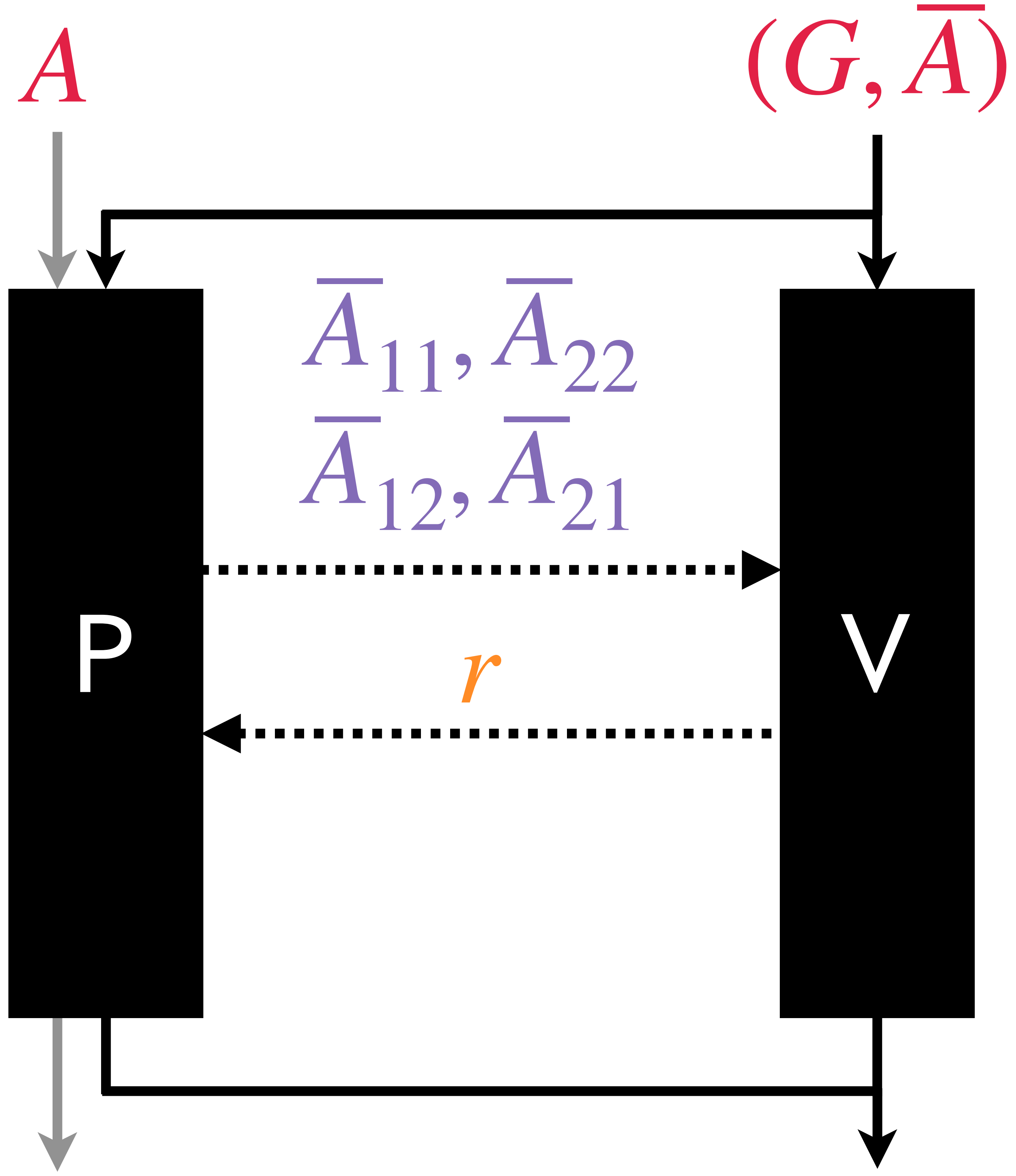

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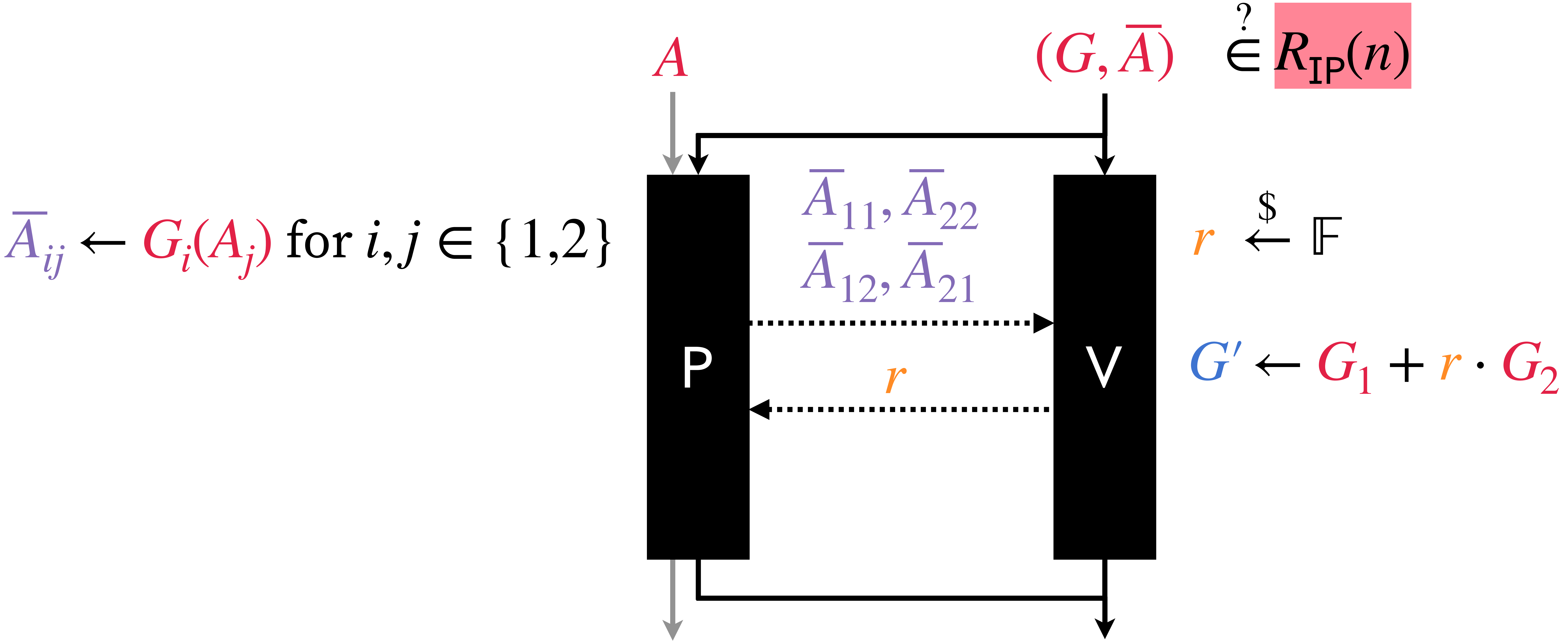

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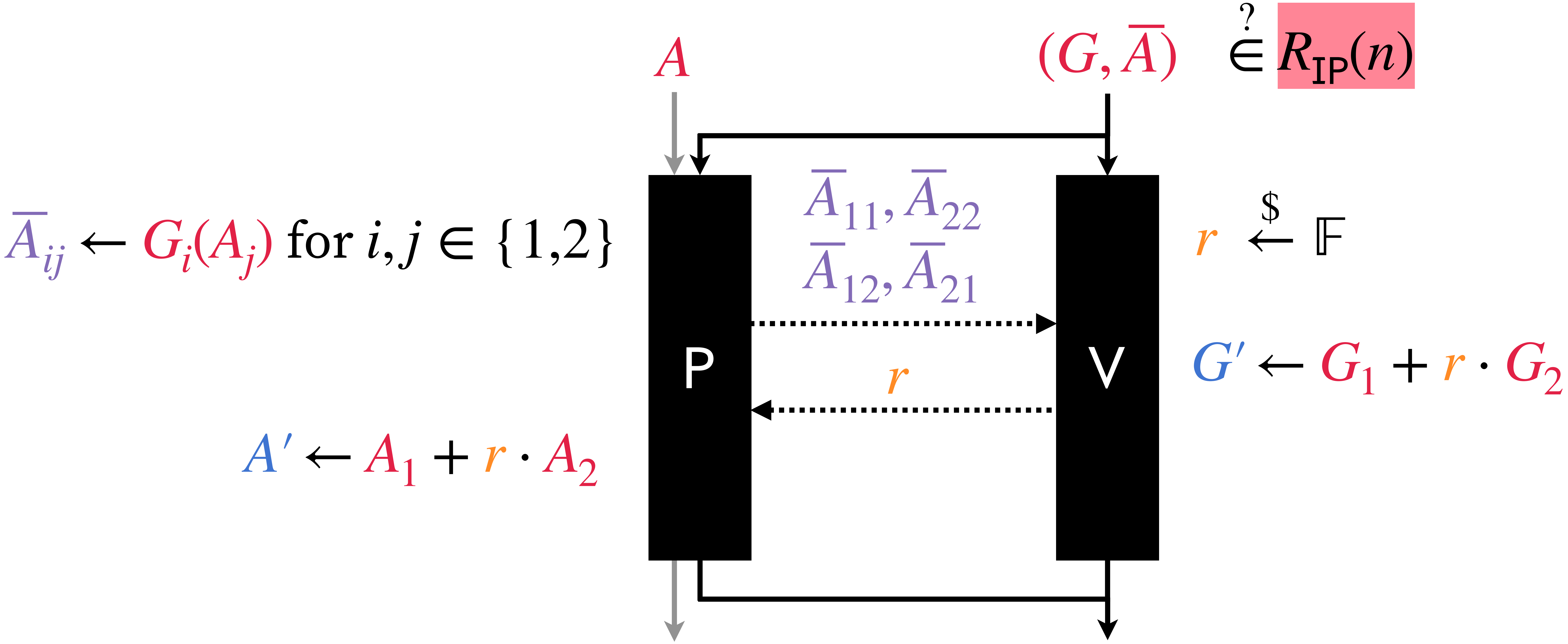

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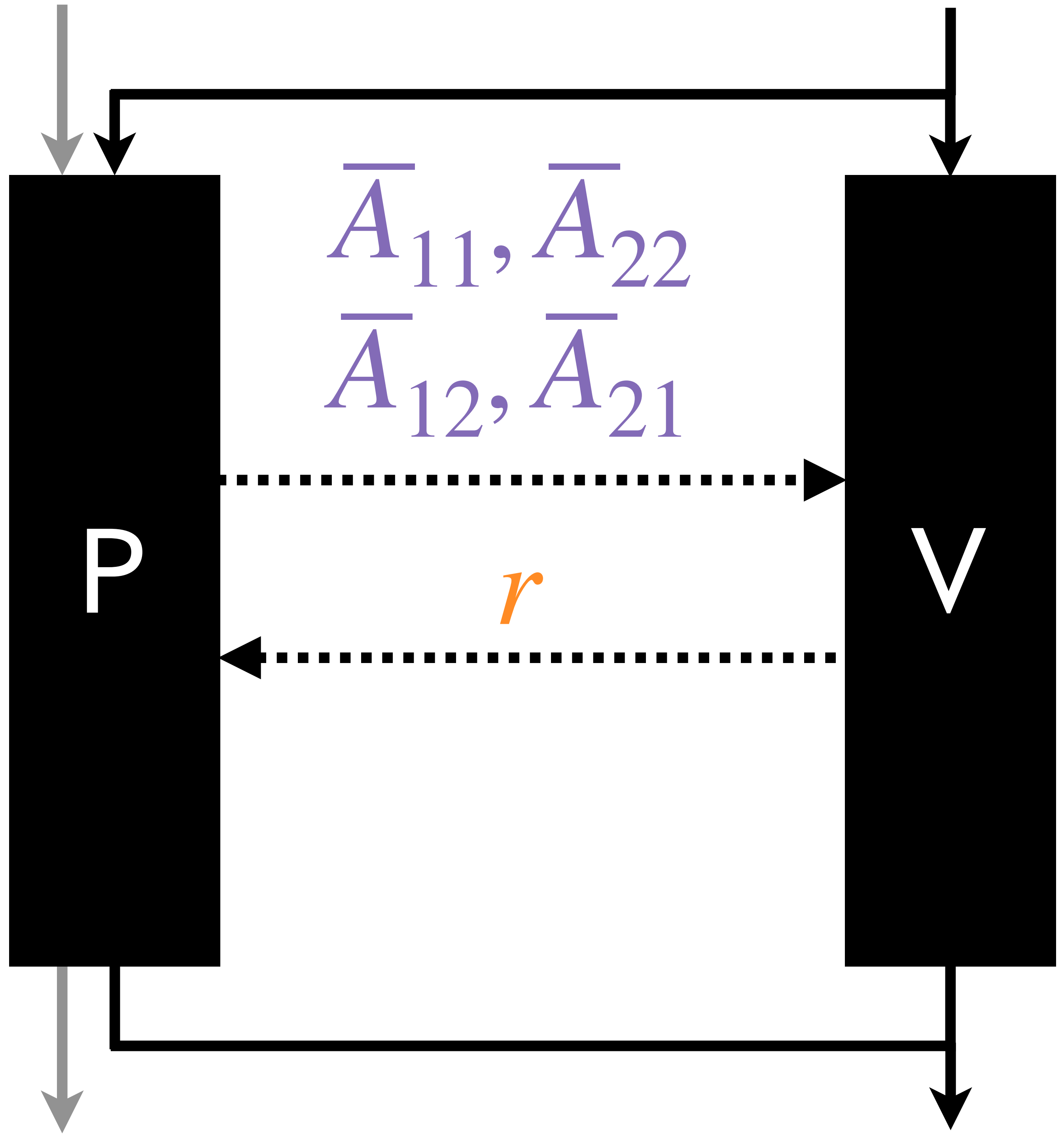

$\overline{A}_{ij} \leftarrow G_i(A_j)$ for $i, j \in \{1,2\}$

$r \xleftarrow{\$} \mathbb{F}$

$G' \leftarrow G_1 + r \cdot G_2$

$A' \leftarrow A_1 + r \cdot A_2$

$$\overline{A}' \leftarrow \overline{A}_{11} + r \cdot (\overline{A}_{12} + \overline{A}_{21}) + r^2 \cdot \overline{A}_{22}$$

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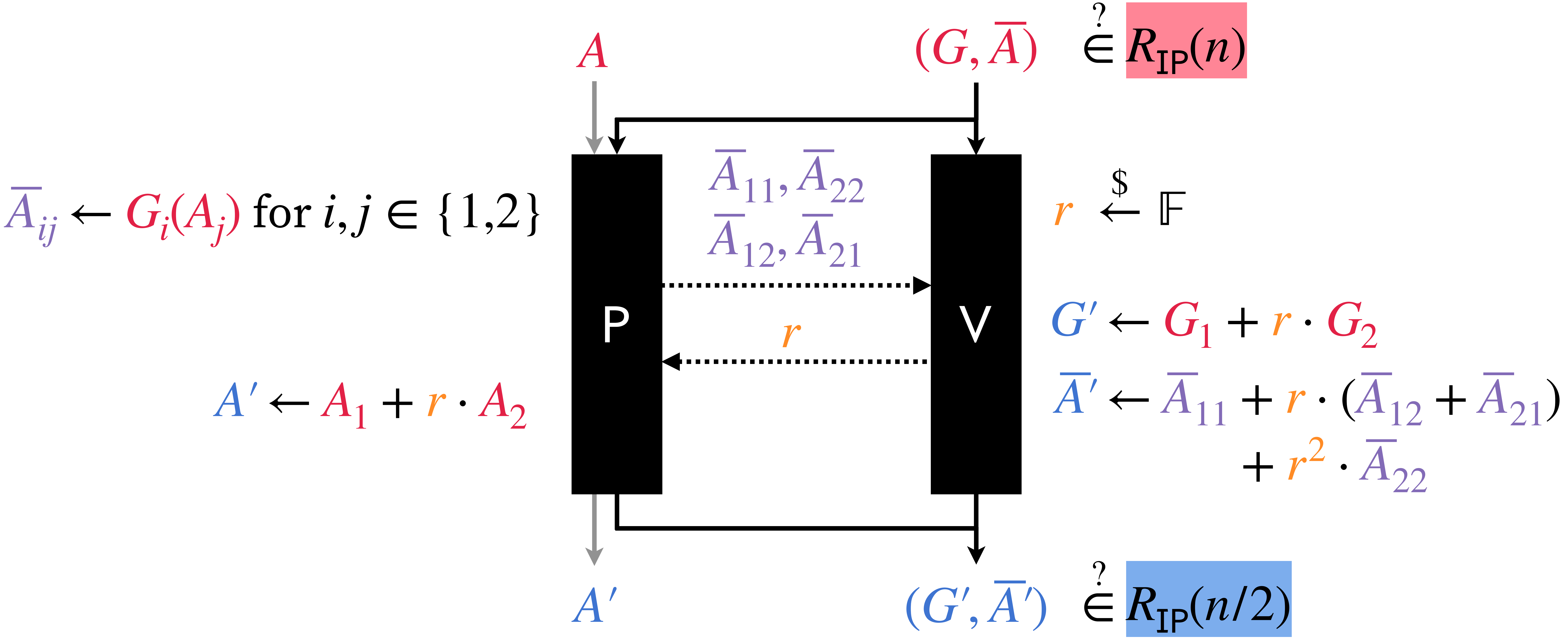
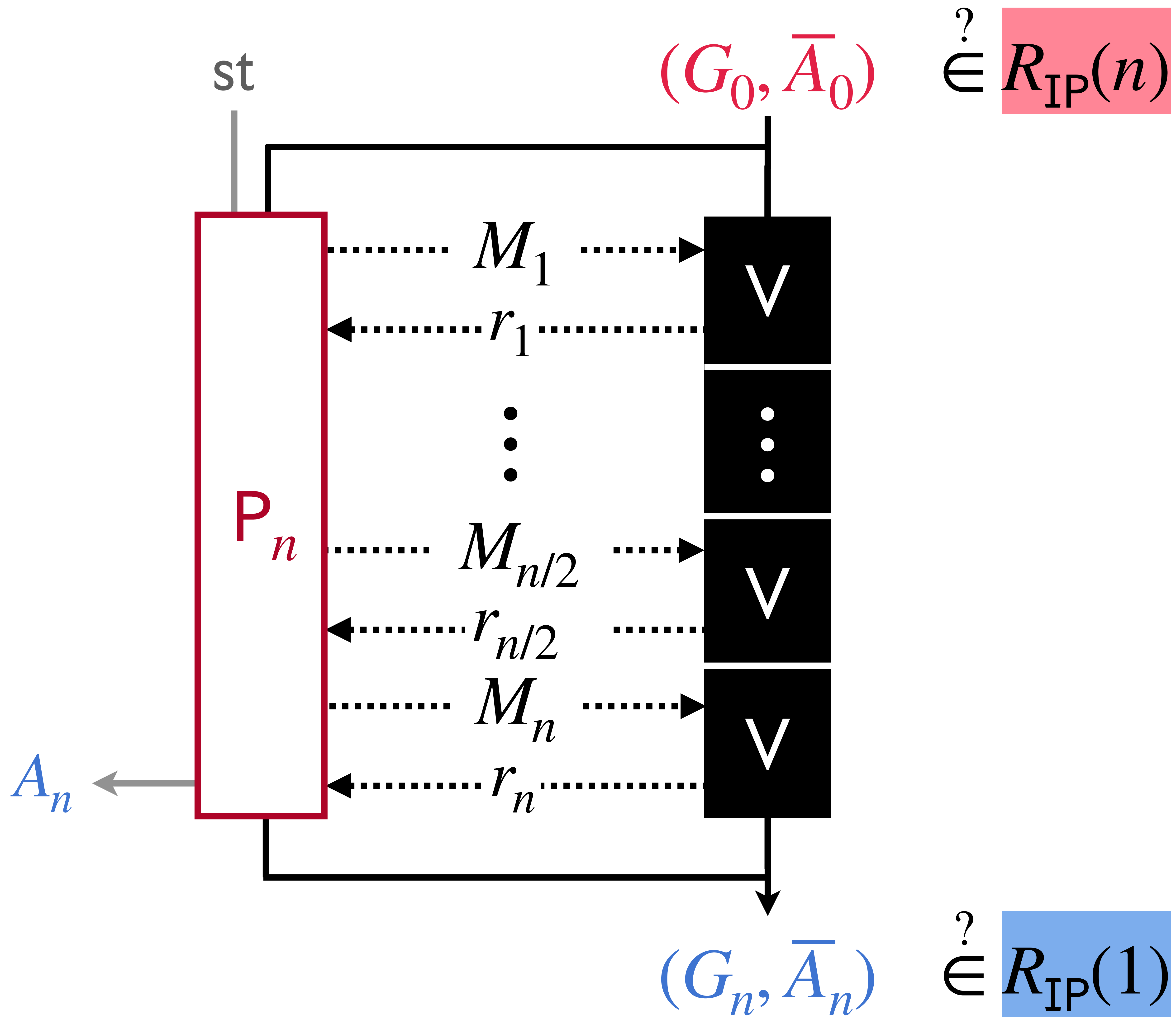


Problem: Simple Construction, Complex Proof



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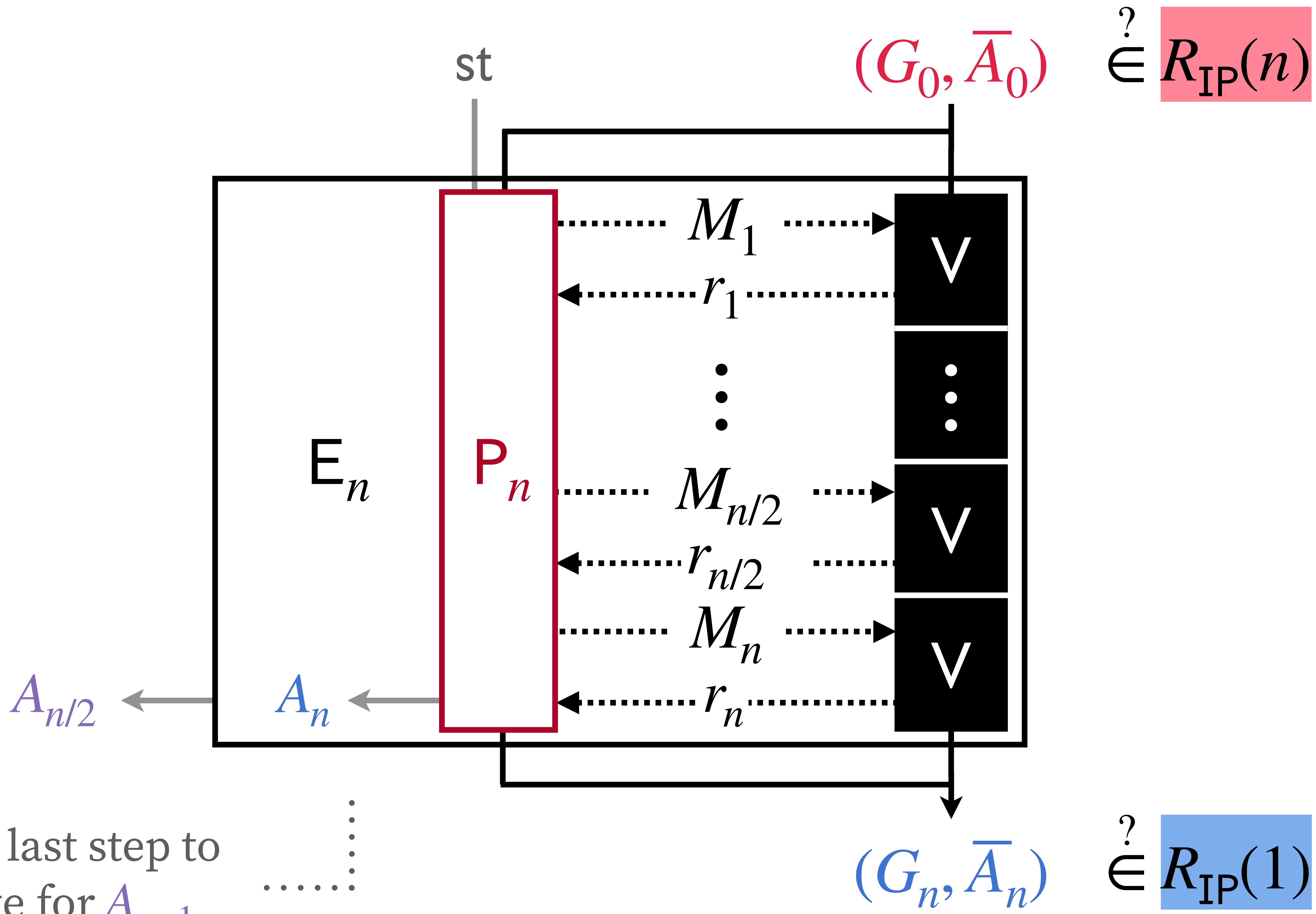

E_n rewinds last step to interpolate for A_{n-1}

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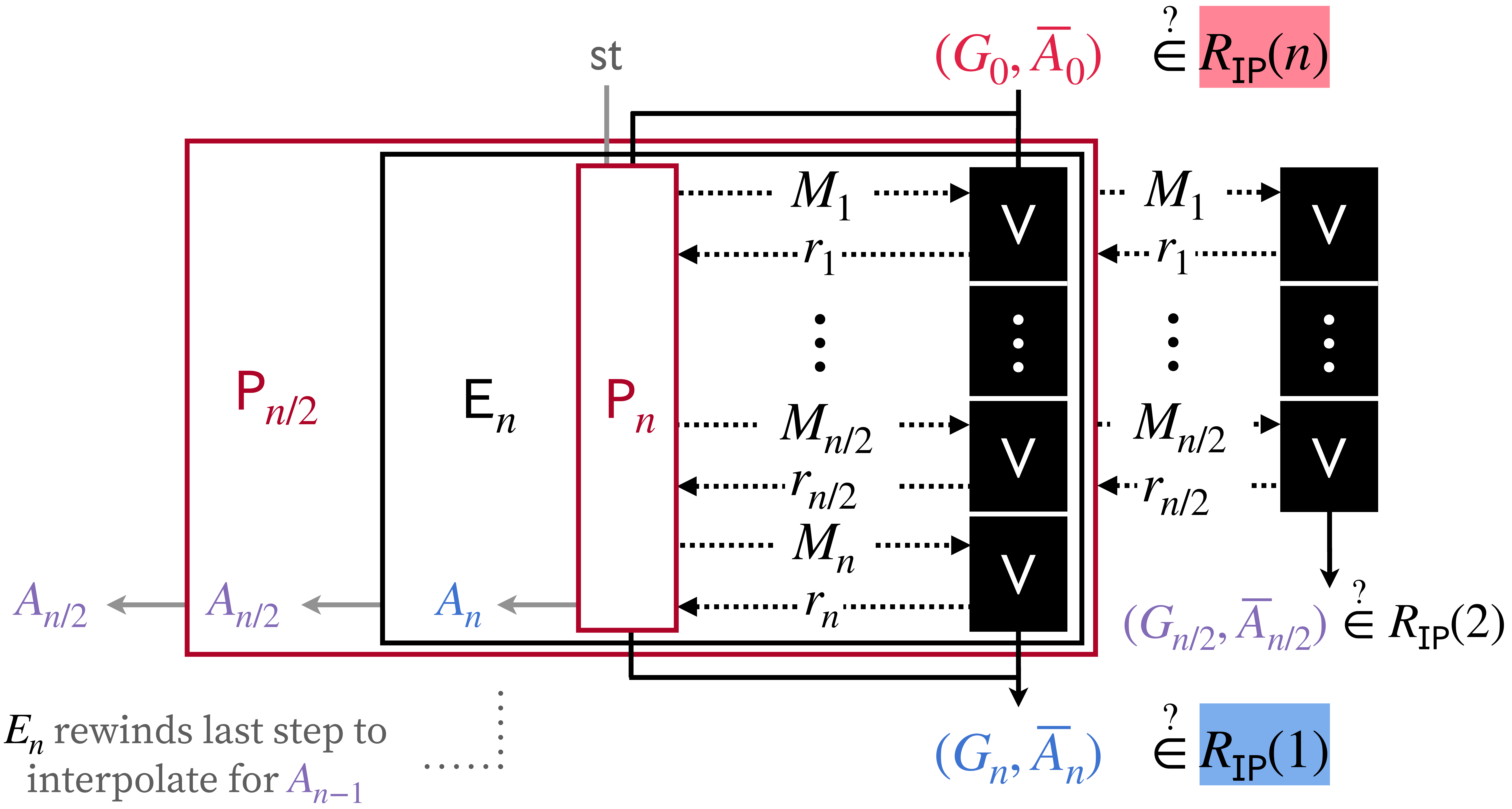

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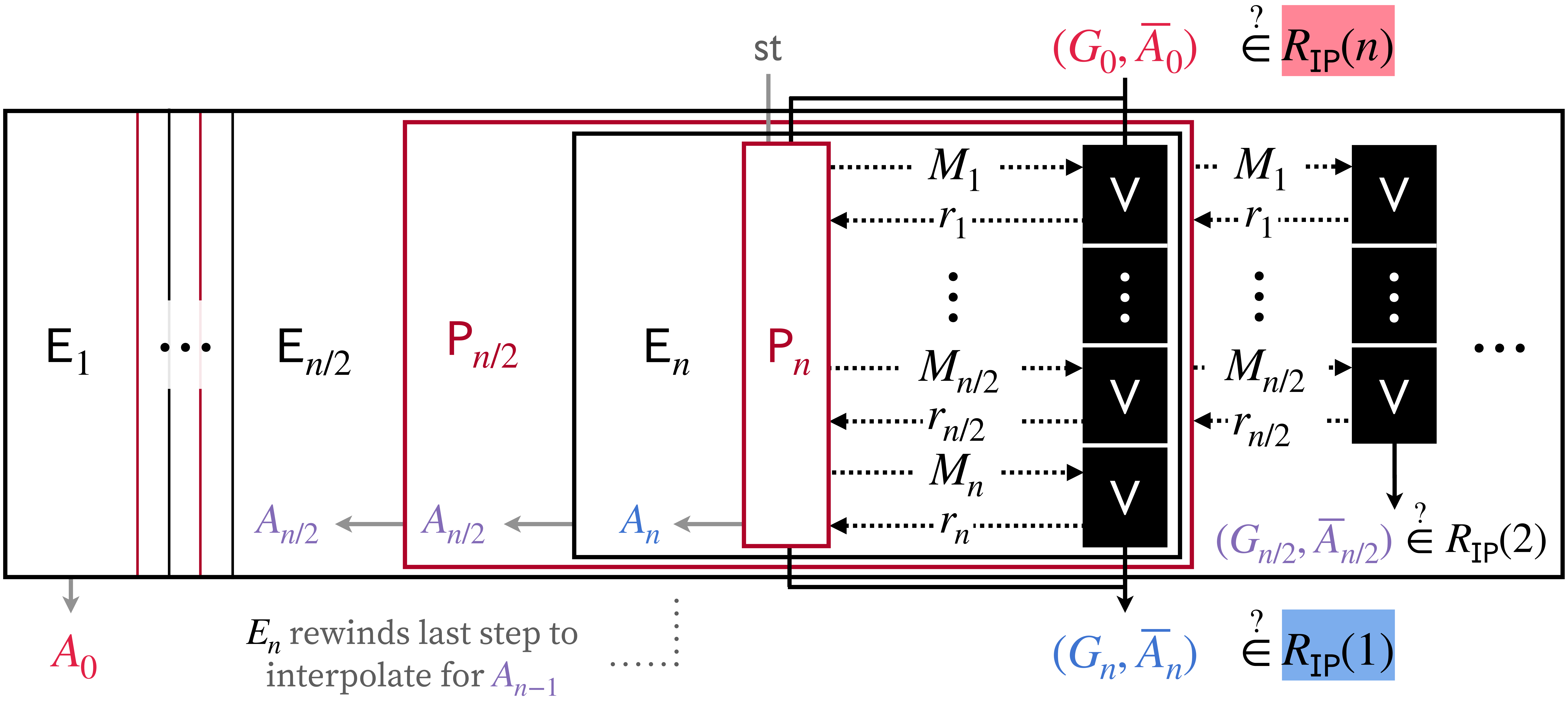

Solution: Sequential Composition Theorem

We prove that reductions are sequentially composable.

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Inner-Product Argument with a Simple Proof

Simpler soundness proof: Invoke sequential composition.

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$$\Pi_{\mathsf{step}}$$

$$R_{\mathsf{IP}(n)} \to R_{\mathsf{IP}(n/2)}$$

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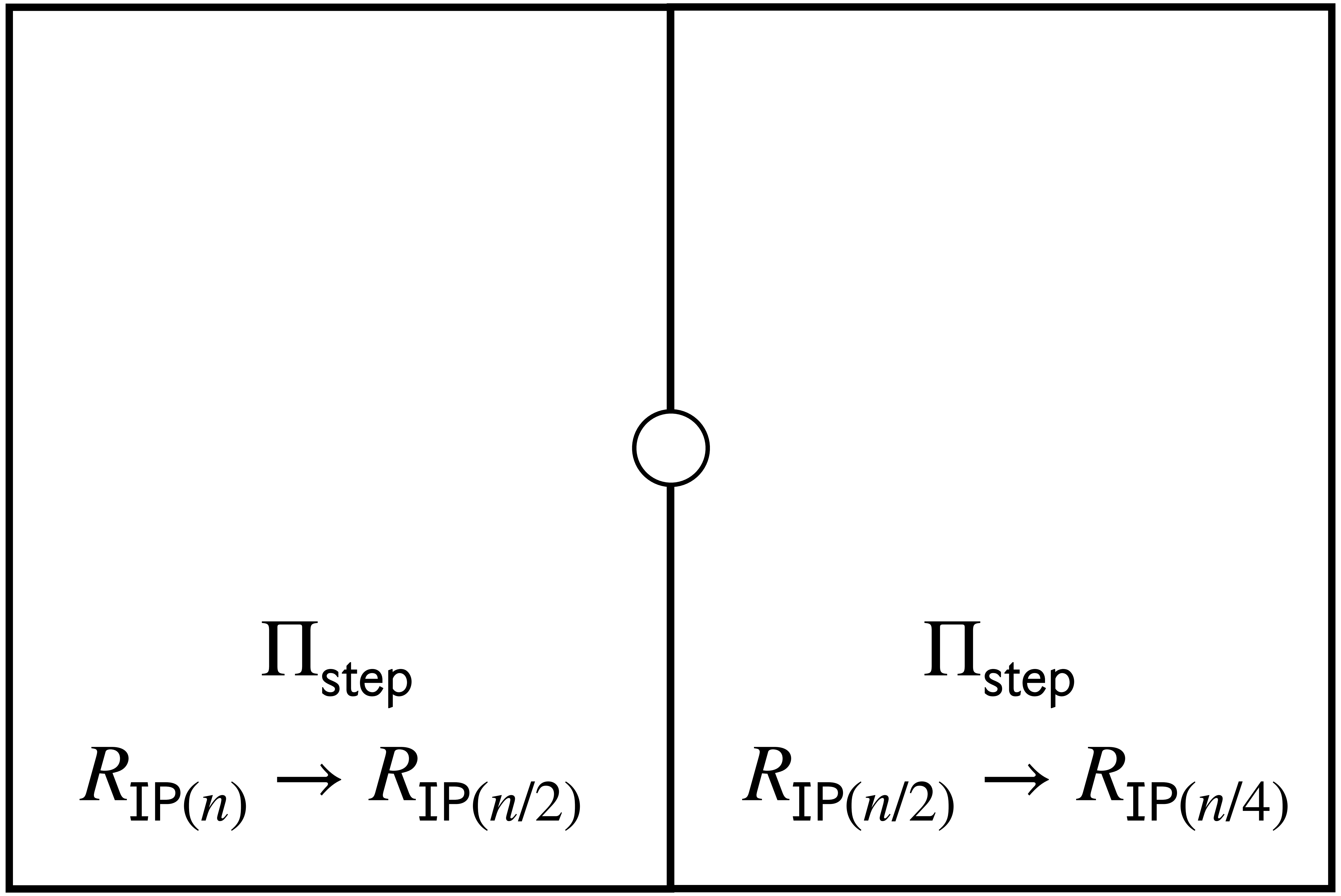

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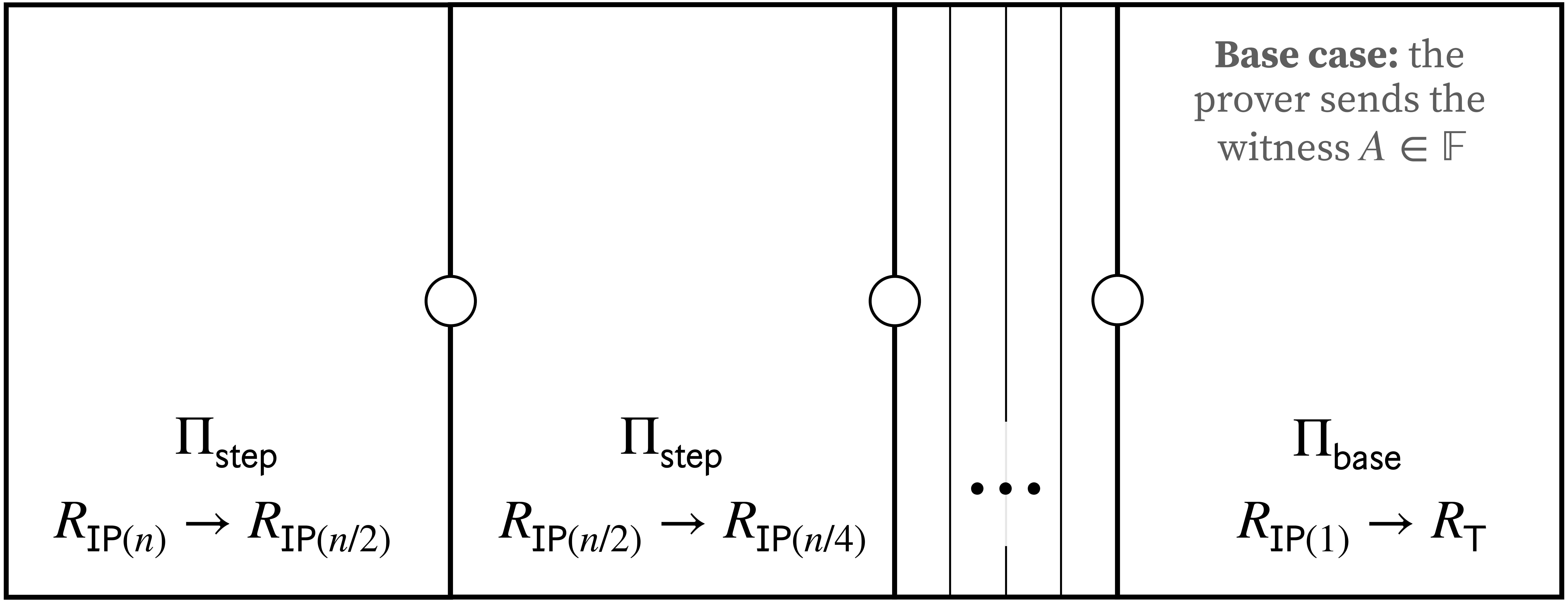

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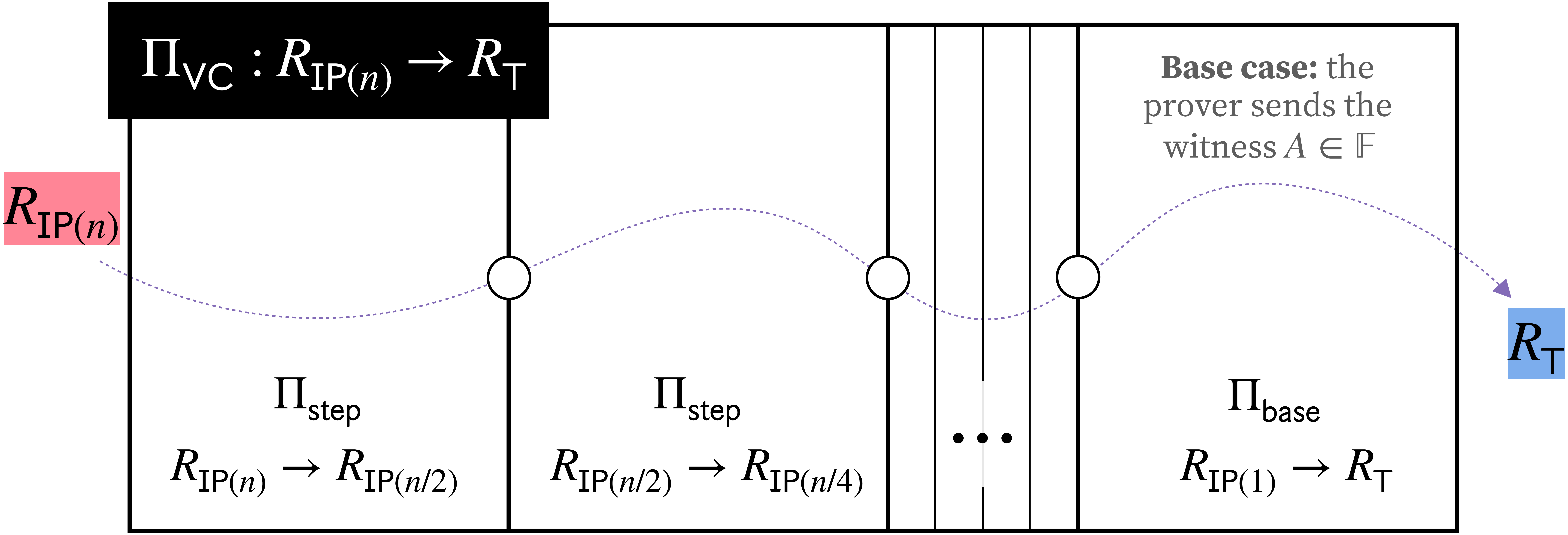
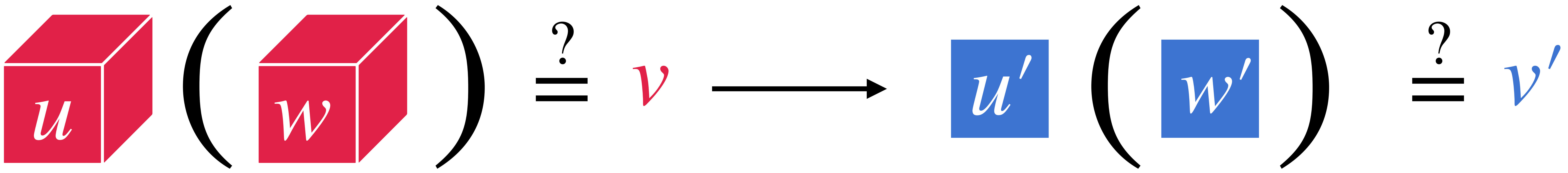


Our Generalization: Tensor Reduction of Knowledge

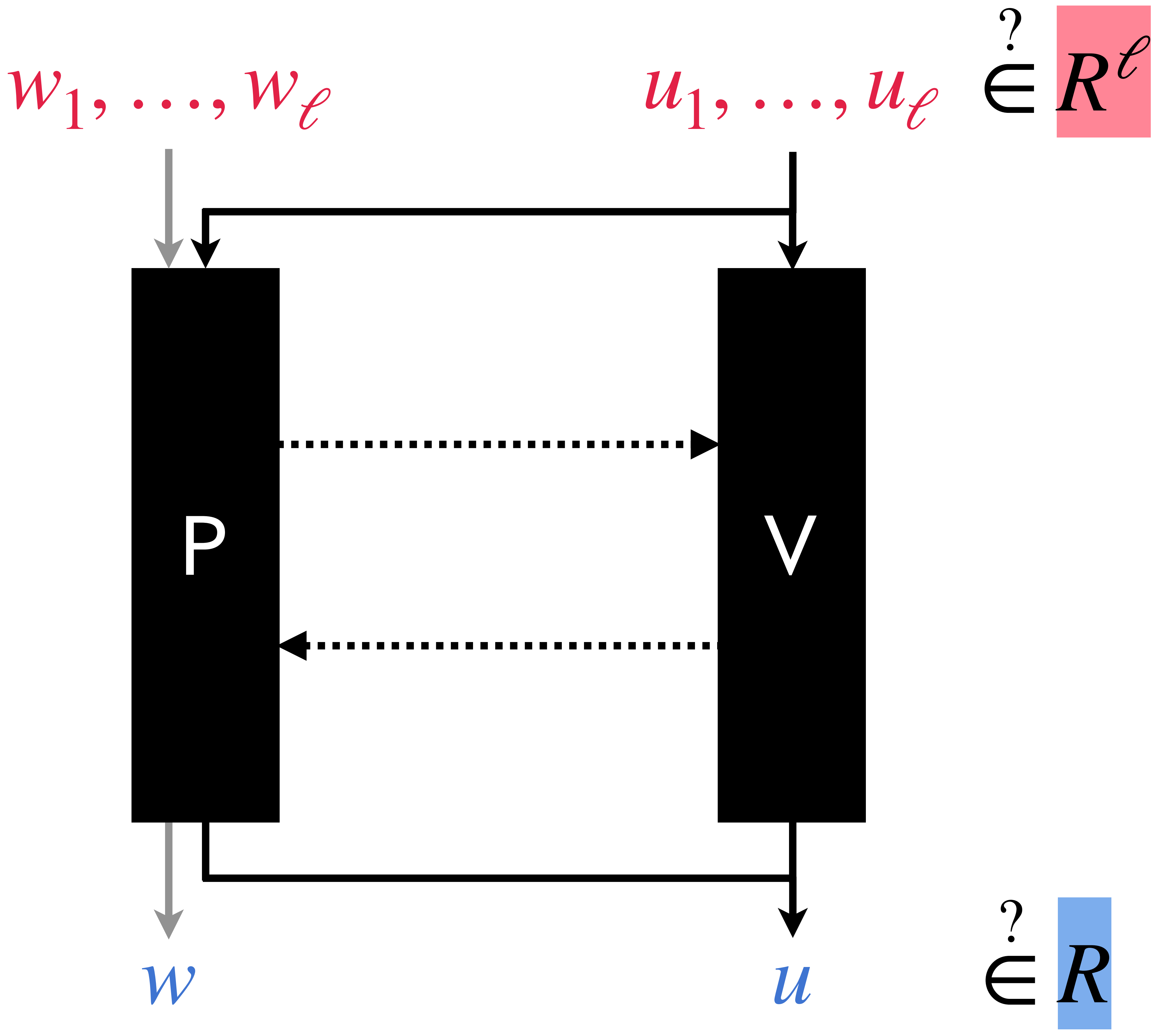
This generalizes techniques in [BCCGP16], [BBBPWM18], [BCS21], [BMMTV21], [AC20], and [ACR21]

Theorem. There exists a reduction of knowledge that reduces the task of checking knowledge of w such that $u(w) = v$ for $u \in \hom(W^n, V)$ to the task of checking knowledge of w' such that $u'(w') = v'$ for $u' \in \hom(W, V)$.



Second Example: Folding Schemes

An ℓ-**folding scheme** is a reduction of knowledge from $R^\ell = R \times \cdots \times R$ to R.



Problem: Simple Construction, Complex Proof [RZ22]

Consider a 2-folding scheme $\Pi_2 : R^2 \rightarrow R$

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R^ℓ

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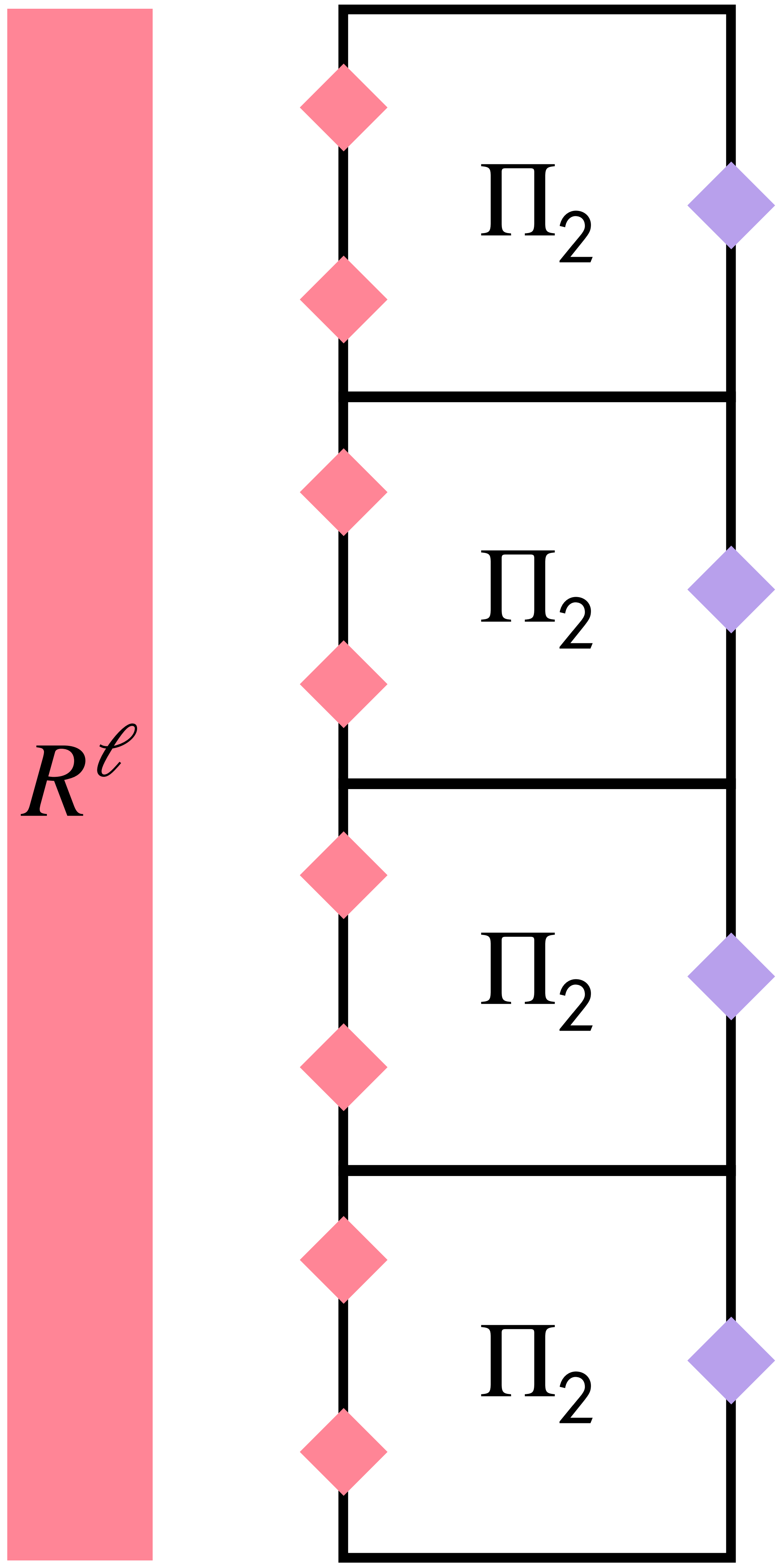

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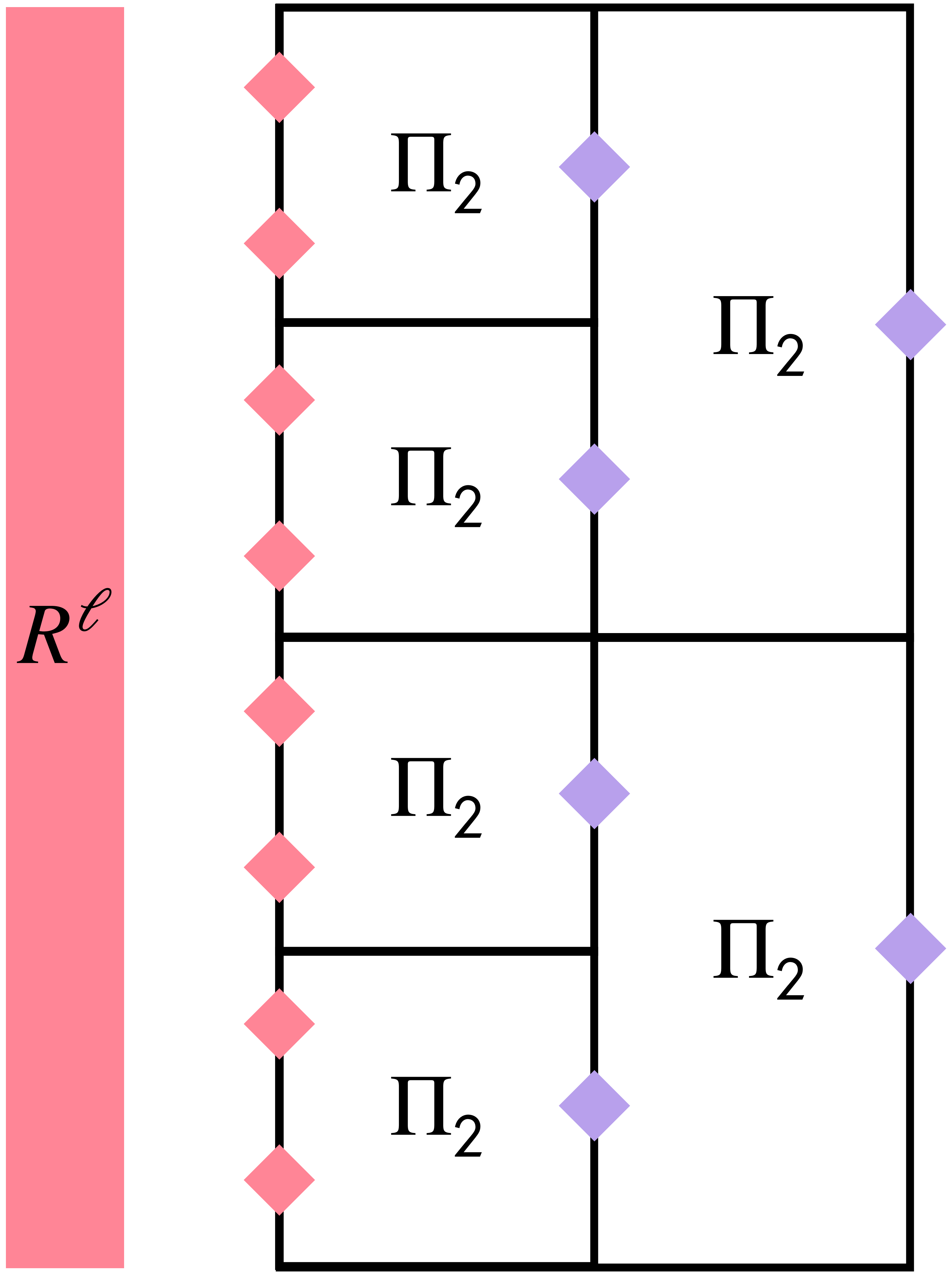

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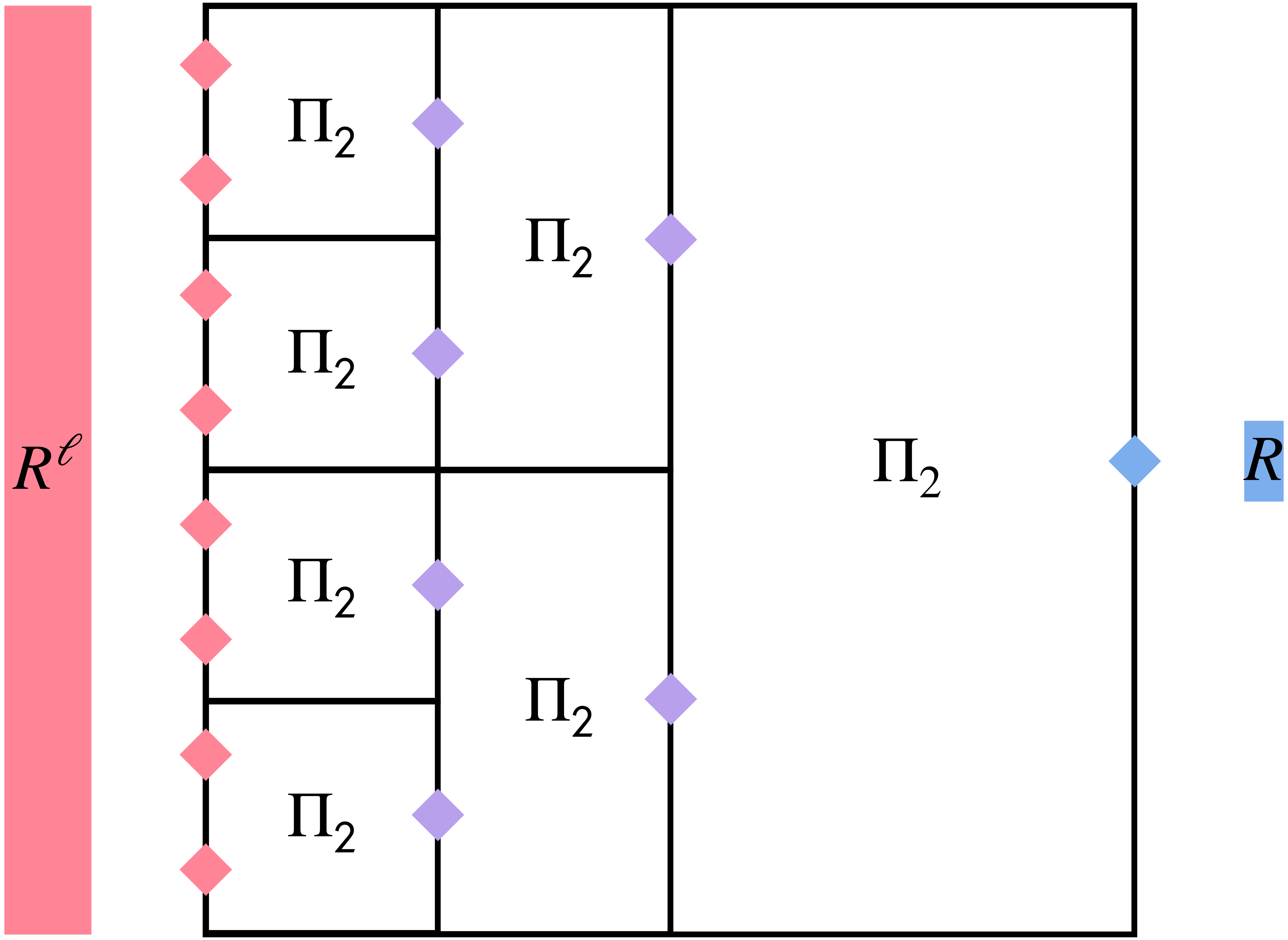

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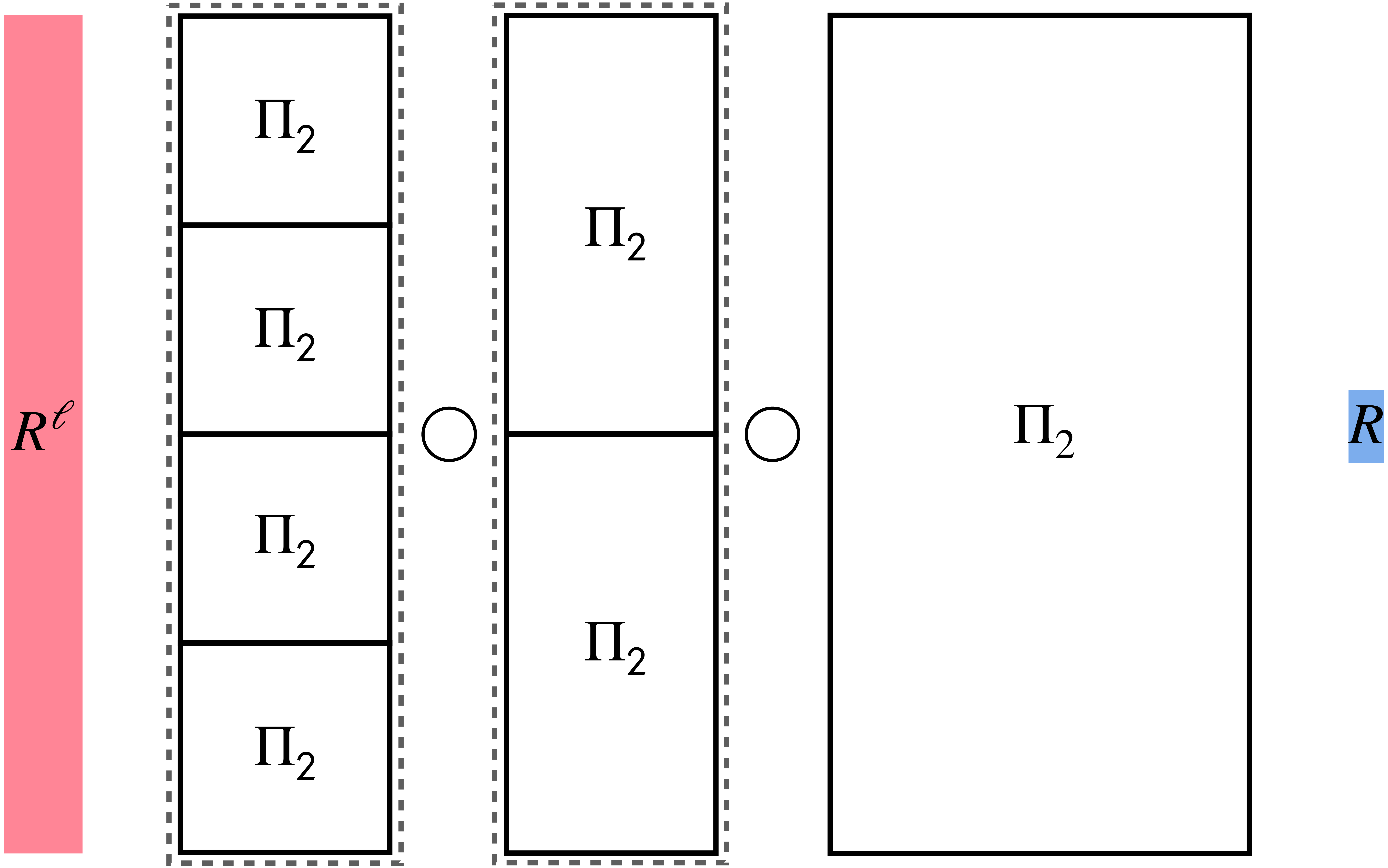

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Solution: Parallel Composition Theorem

We prove that reductions can be composed in parallel.

$$\Pi_1 : R_1 \to R_2$$

$$\Pi_2 : R_3 \to R_4$$

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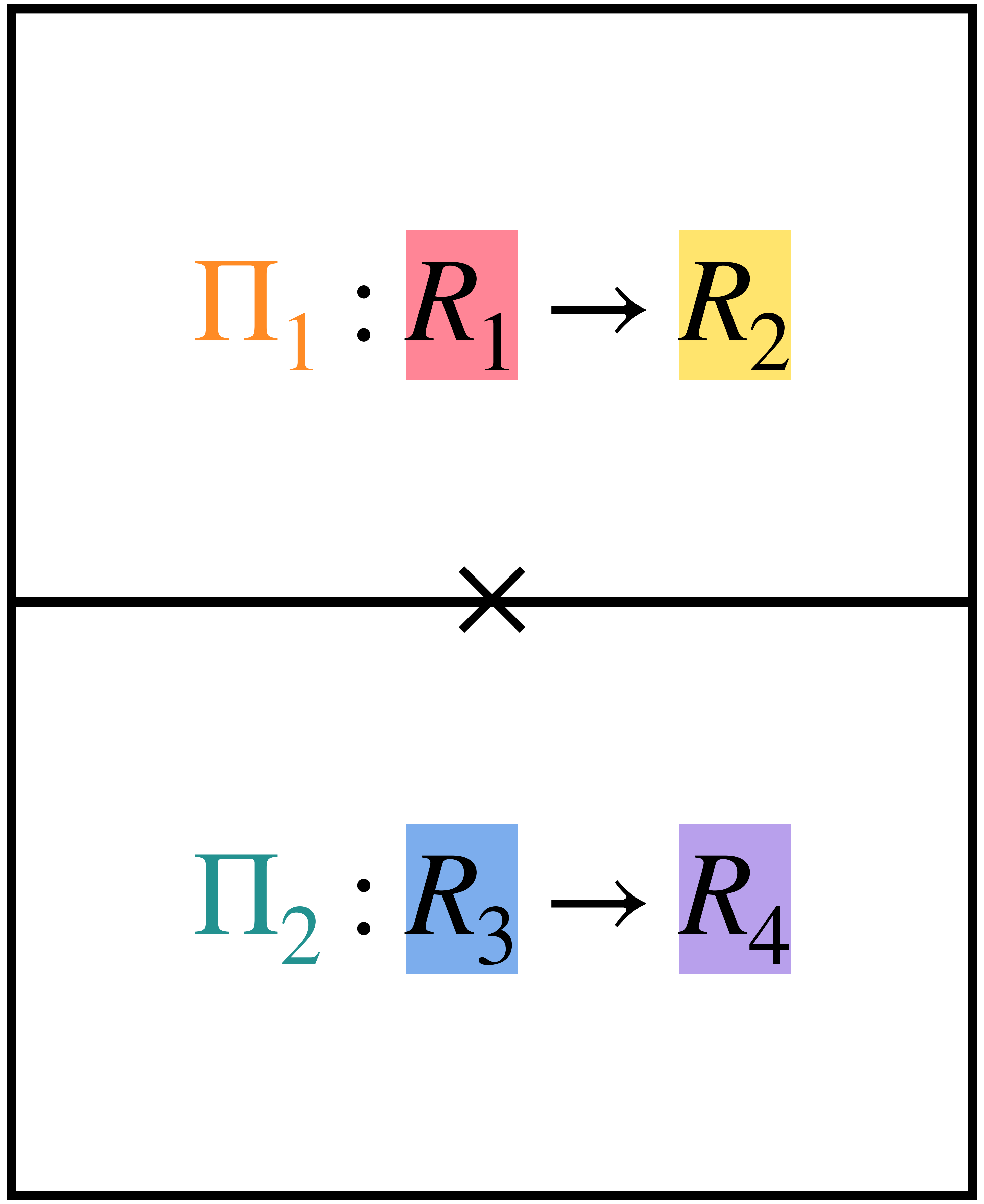

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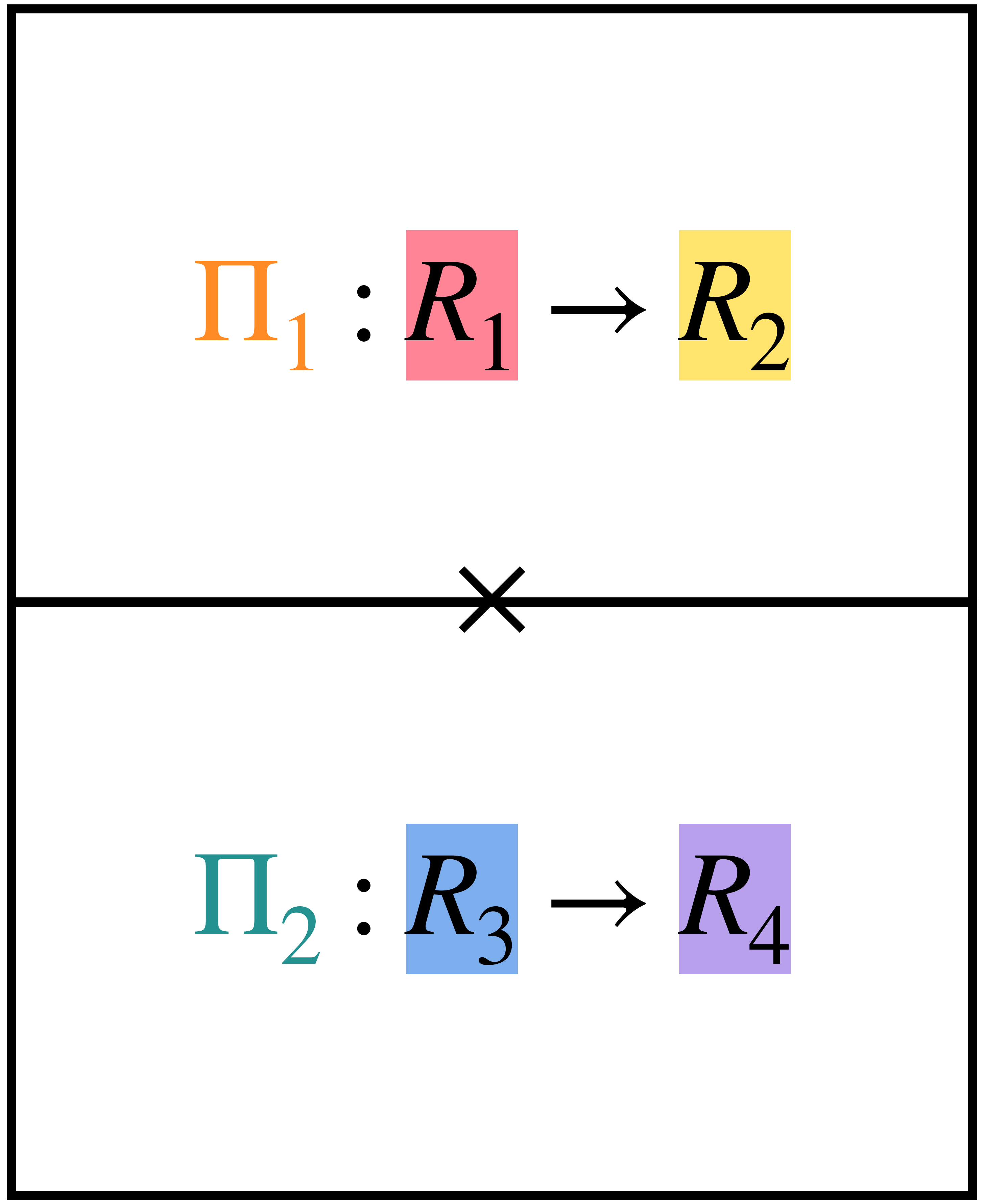

$$\left\{ (u_2, u_4), (w_2, w_4) \;\middle|\; \begin{array}{l} (u_2, w_2) \in R_2 \\ (u_4, w_4) \in R_4 \end{array} \right\}$$

$$\Pi_1 \times \Pi_2 : R_1 \times R_3 \to R_2 \times R_4$$

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Tree Folding Scheme with a Simple Proof

Given $\Pi_2 : R^2 \to R$, then $\Pi_\ell = \Pi_2 \circ (\Pi_{\ell/2} \times \Pi_{\ell/2}) : R^\ell \to R$.

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$$R^{\ell/2} \to R$$

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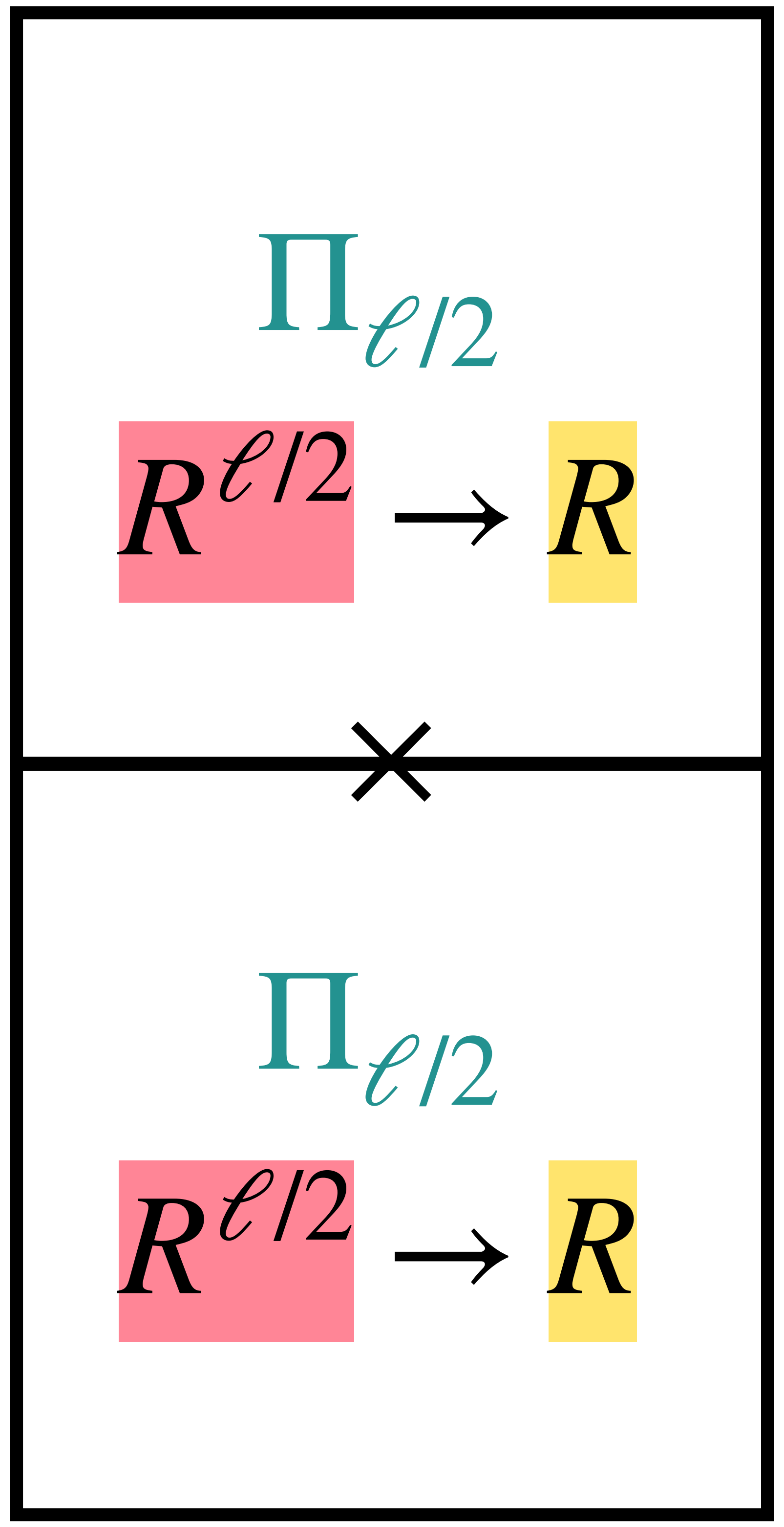

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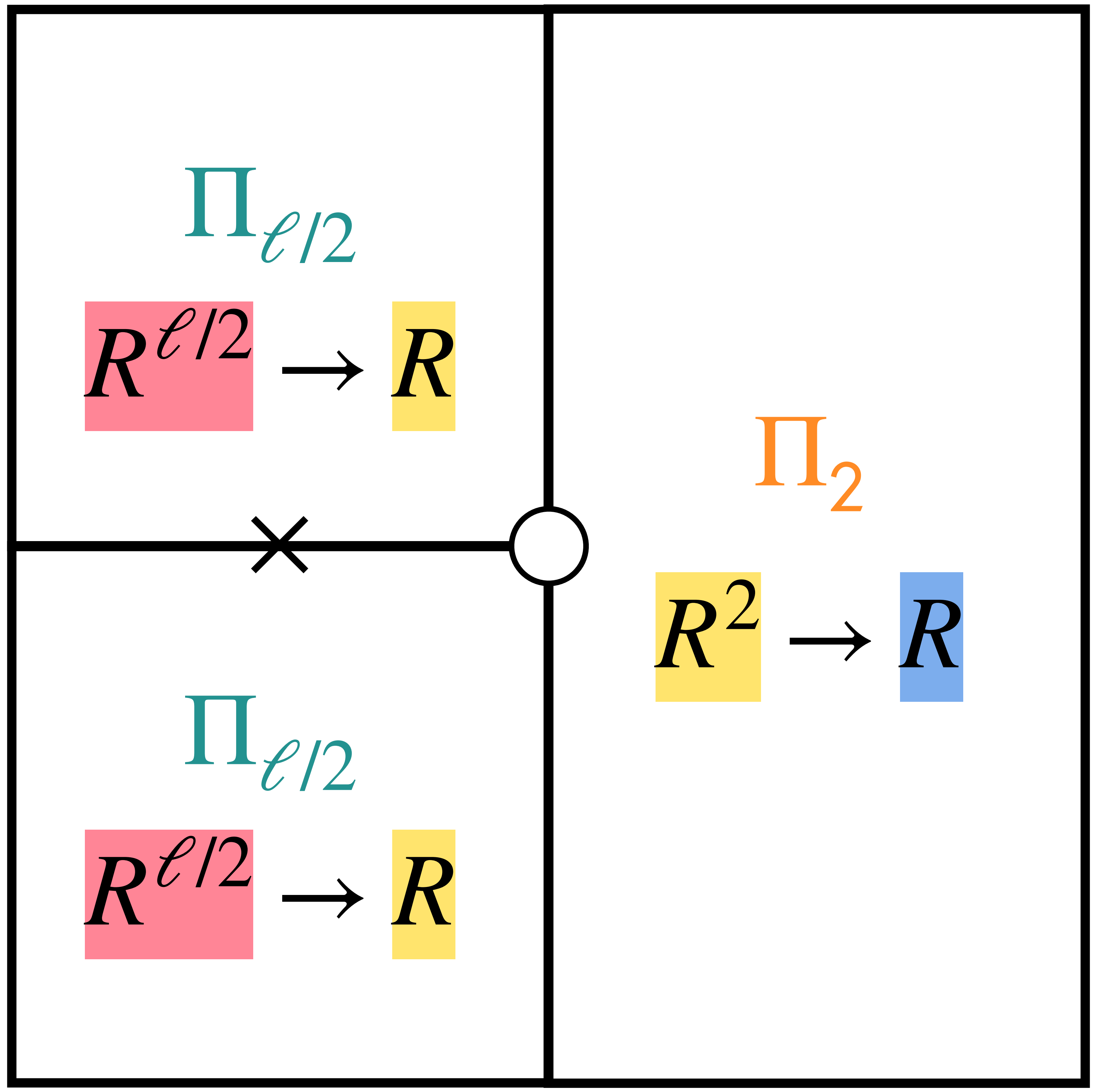

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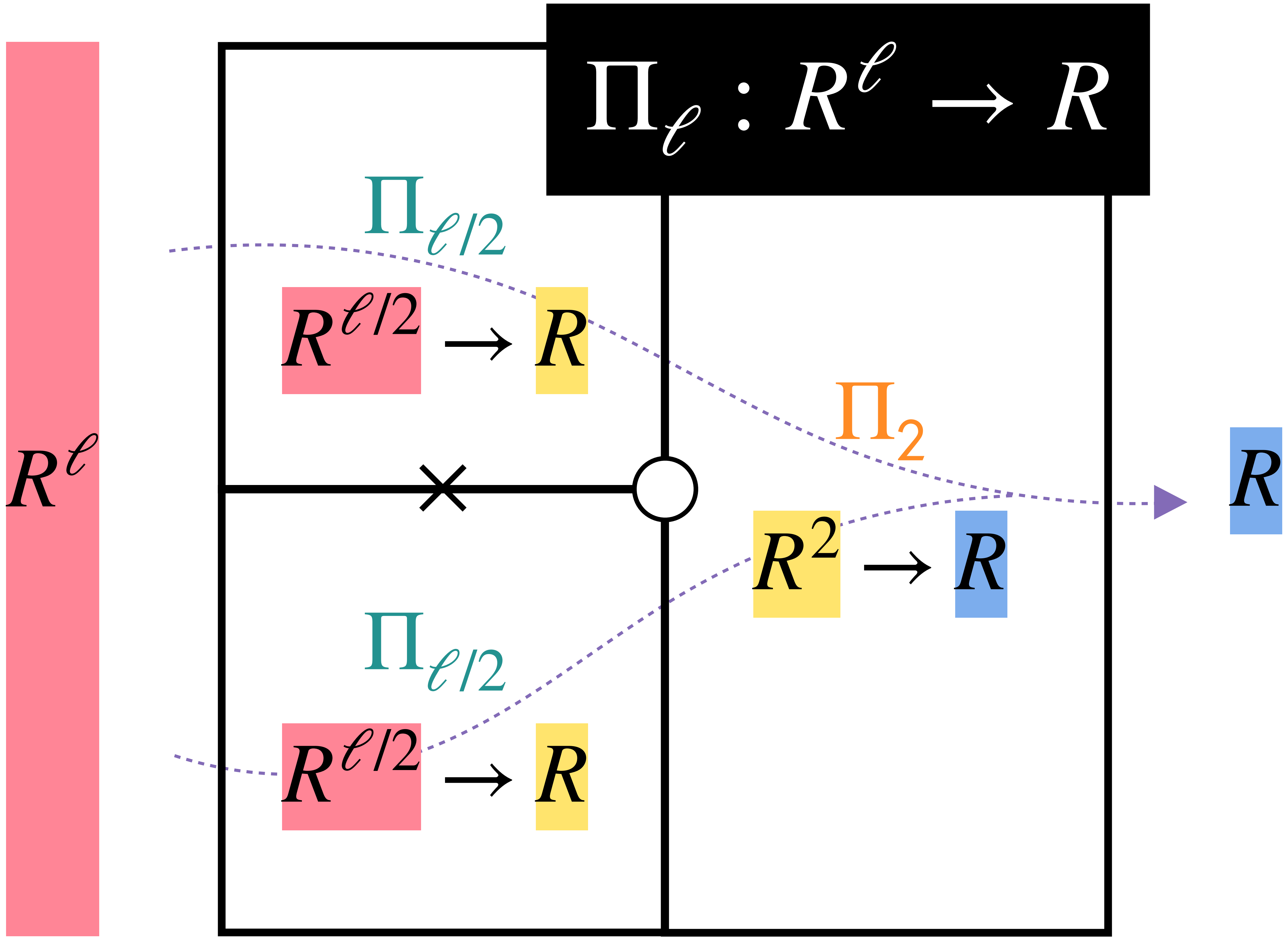
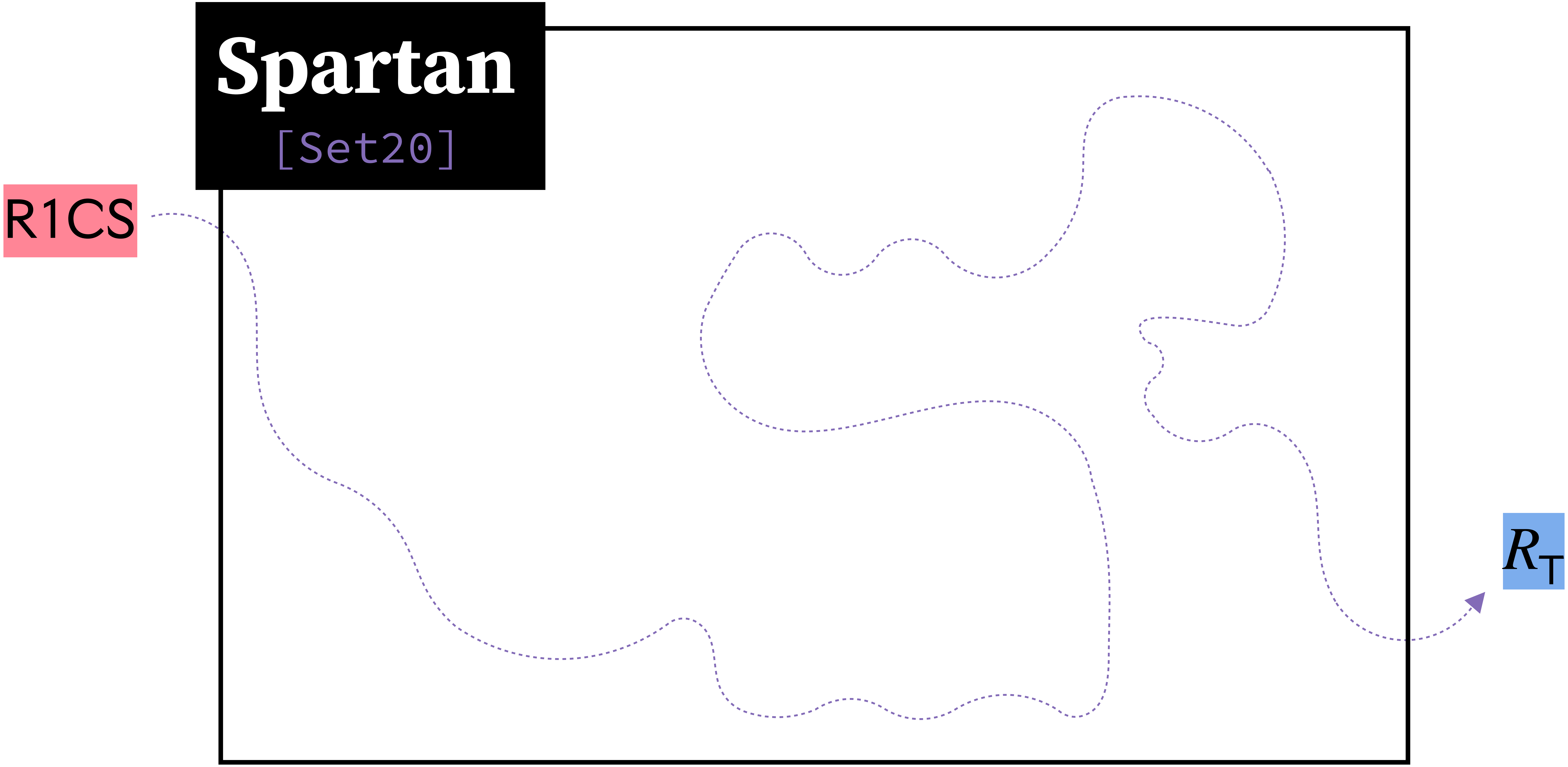


Taming the Complexity of Modern Arguments



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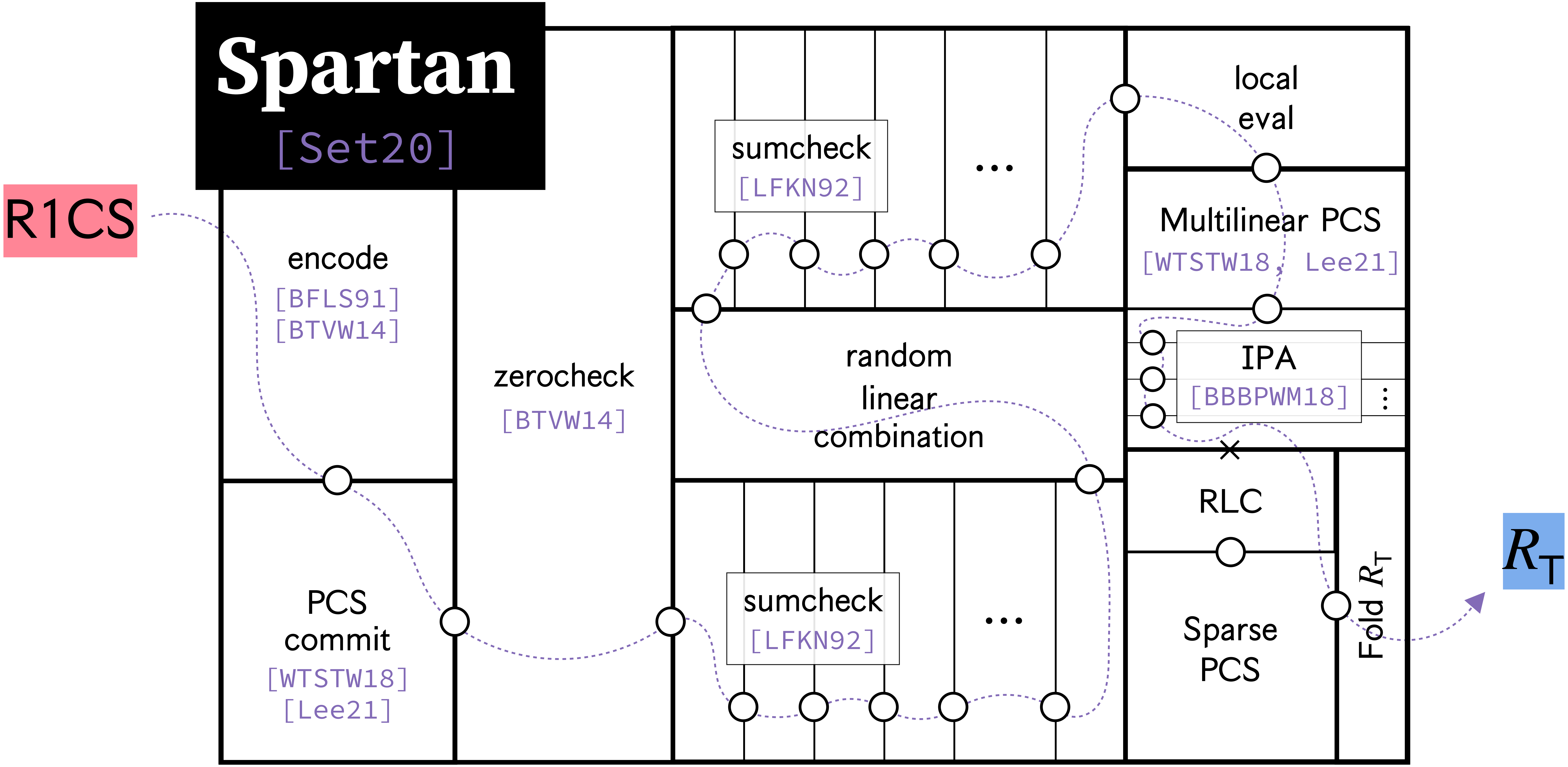

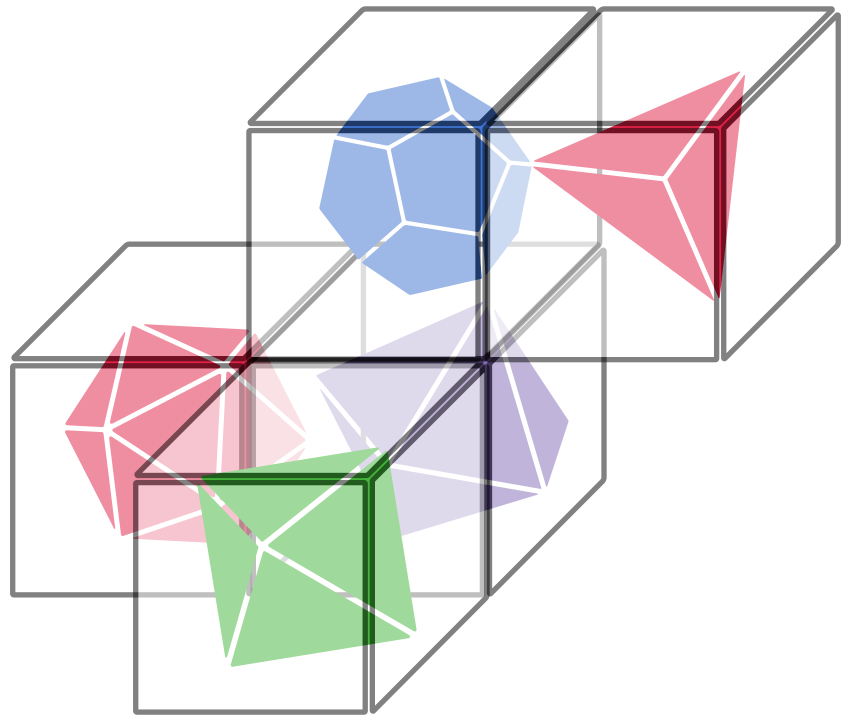

Reductions of knowledge serve as both a unifying abstraction and a compositional framework.

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