Lattice-Based Timed Cryptography

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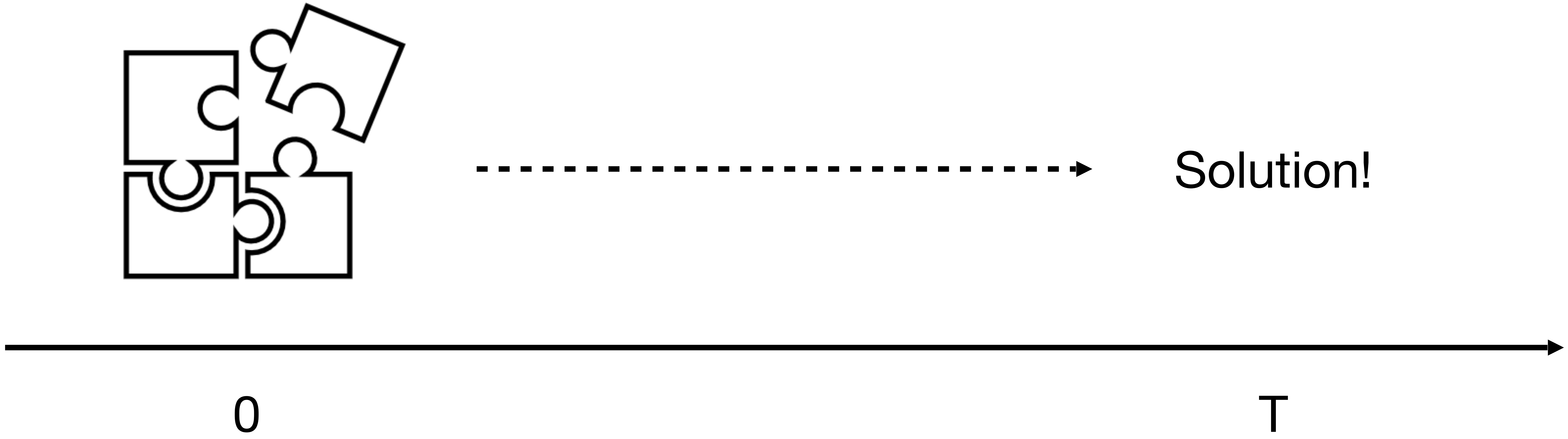
(Max Planck Institute —> Bocconi University)

Joint work with Russell Lai

Timed Cryptography



Timed Cryptography

- Time-Lock Puzzles

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- Proofs of Sequential Work

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- Verifiable Delay Functions

Applications

Seal-Bid Auctions

E-Voting

Randomness Generation

Contract Signing

More Applications

Hardness vs Fine-Grained Hardness

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<u>One-Way Problems</u>

poly-size

P $\neq$ NP

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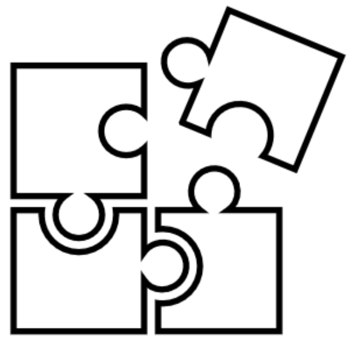
Sequential Problems



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Landscape of Sequential Problems

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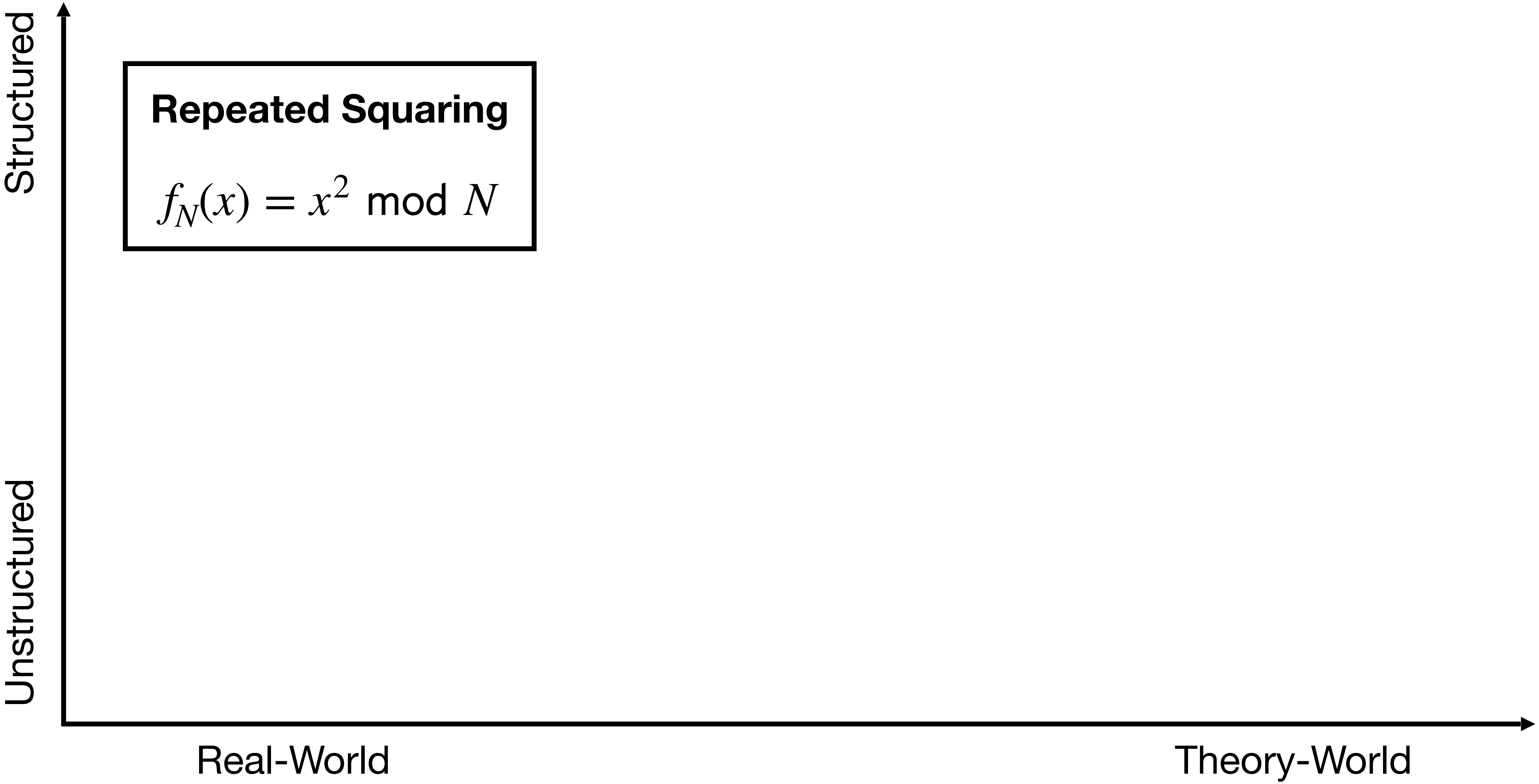


Landscape of Sequential Problems

Structured

Repeated Squaring

$f_N(x) = x^2 \text{ mod } N$

Unstructured

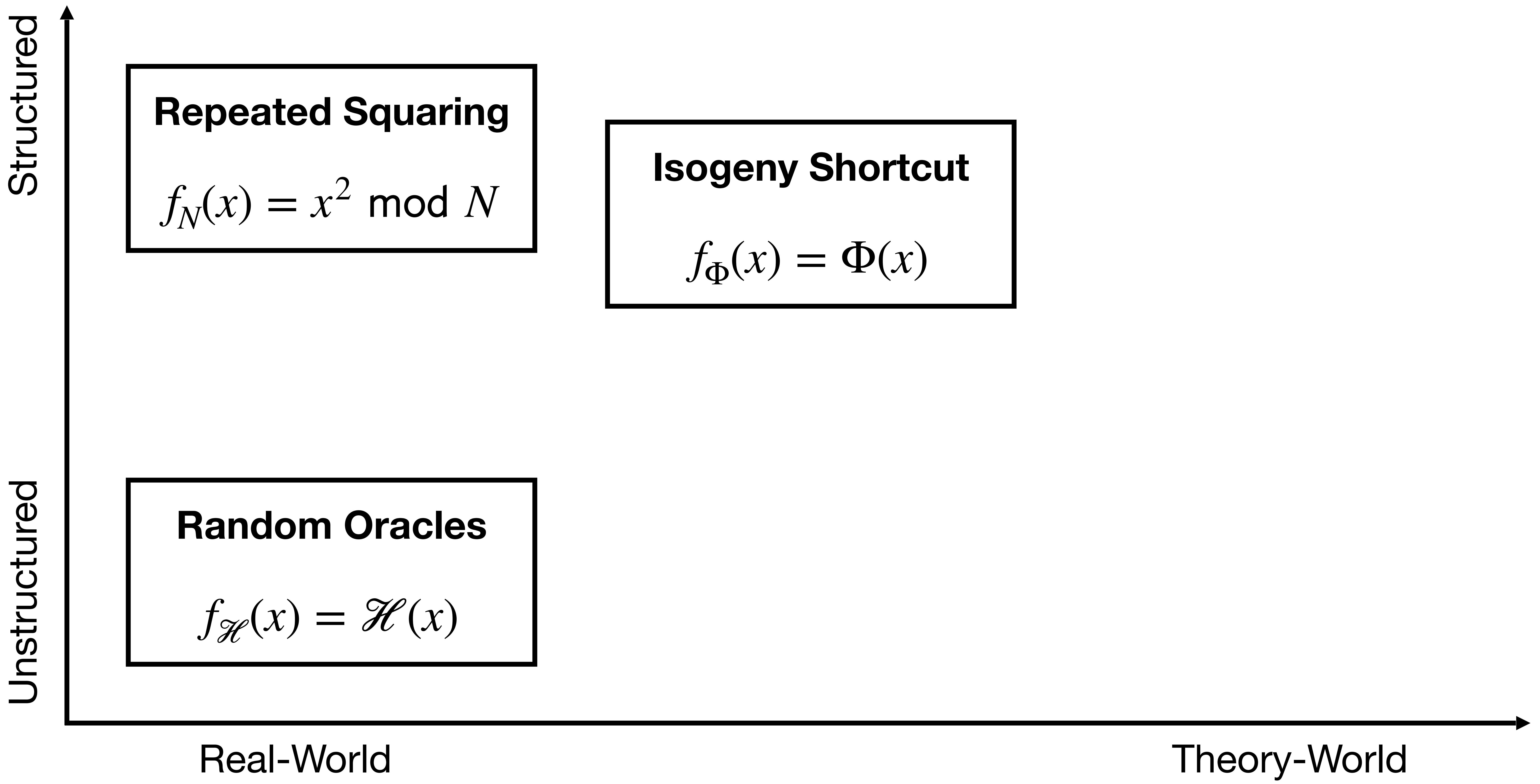
Random Oracles

$f_{\mathscr{H}}(x) = \mathscr{H}(x)$

Real-World

Theory-World

Landscape of Sequential Problems



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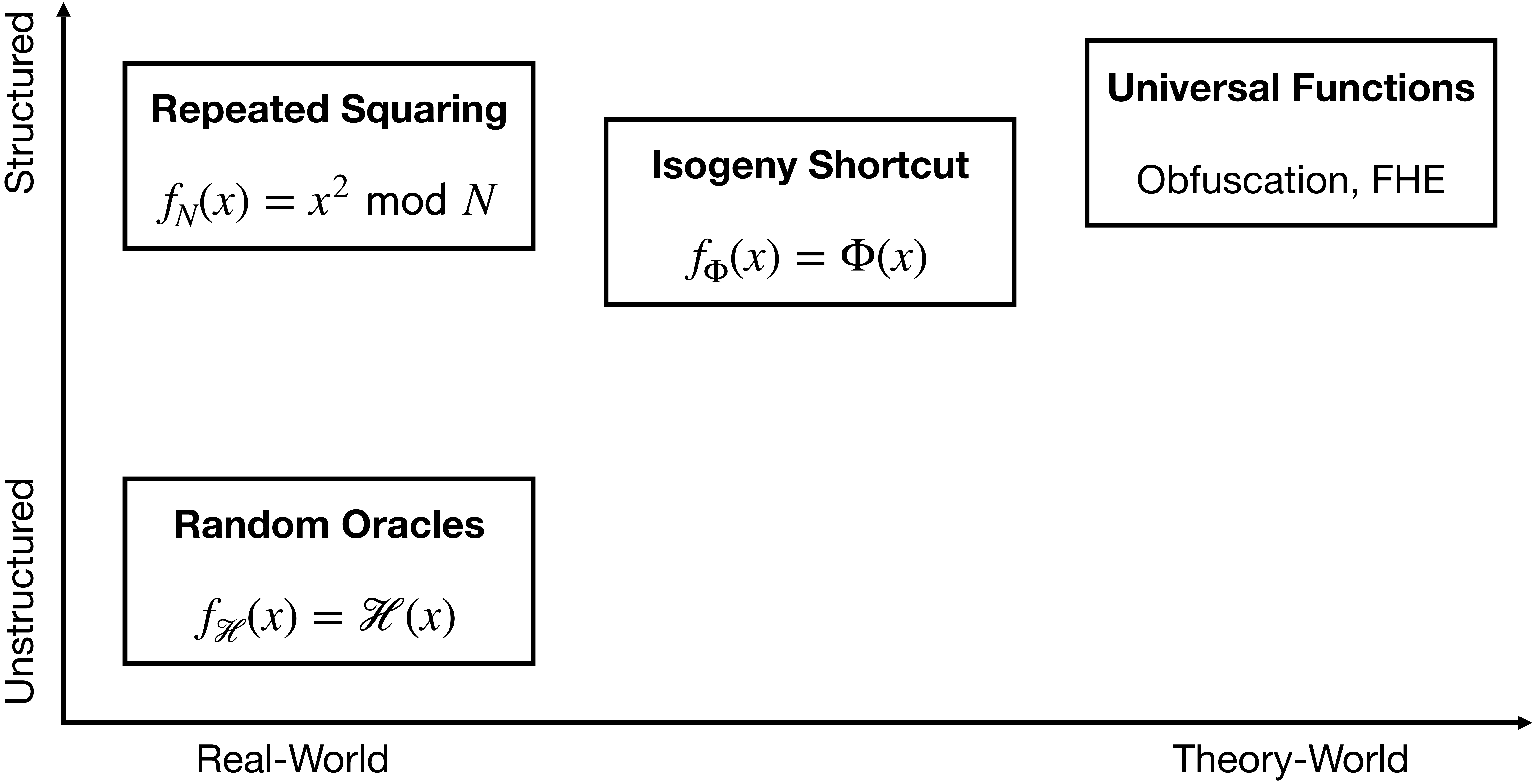
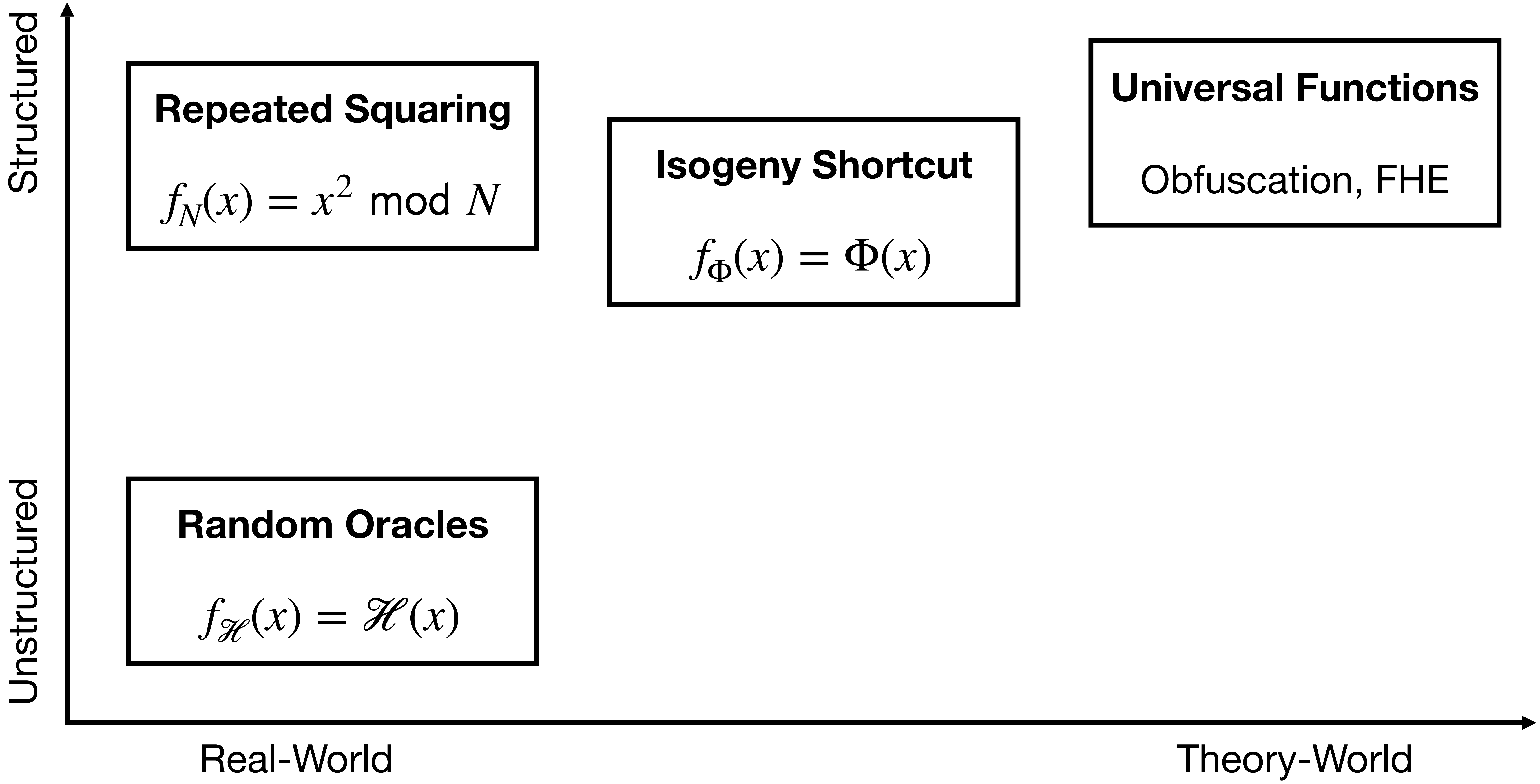


Enter Quantum Computing



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Isogeny Shortcut

$f_\Phi(x) = \Phi(x)$

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Can we construct efficient *post-quantum* timed cryptography?

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- Open problems

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- Input: a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ and a vector $\mathbf{x} \in \mathbb{Z}_q^n$

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$$f_{\mathbf{A}}(\mathbf{x}) = \mathbf{y} = \mathbf{A} \cdot \mathbf{G}^{-1}(\mathbf{x}) \iff \begin{bmatrix} \mathbf{G} \\ \mathbf{A} \end{bmatrix} \cdot \mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

$$\mathbf{u} \in \{0,1\}^m$$

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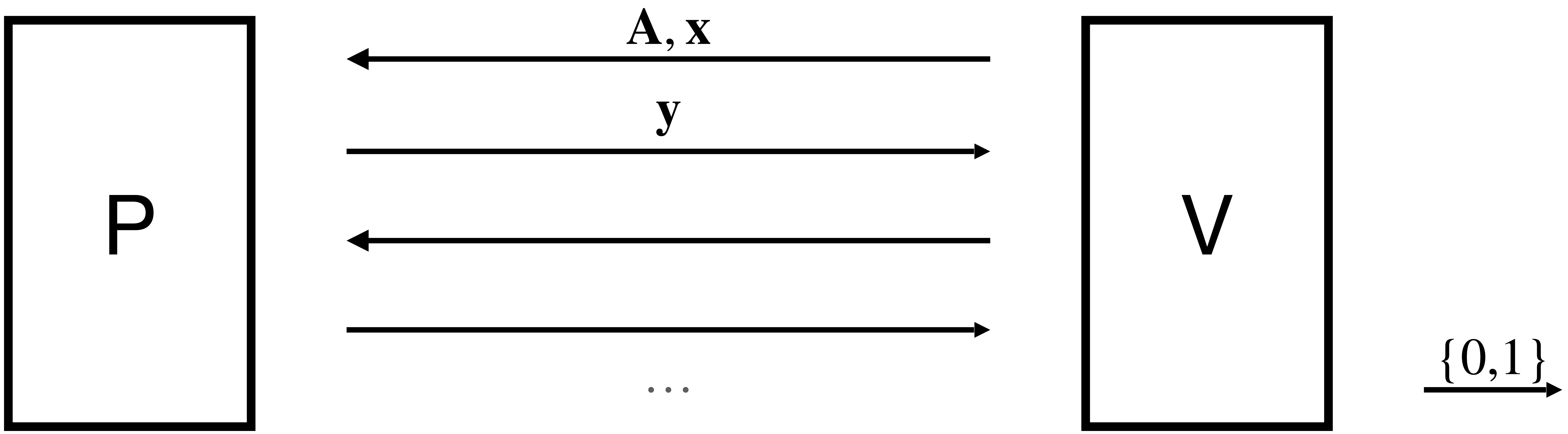
- More heuristic evidence & cryptanalysis (see paper)

Proofs of Sequential Work



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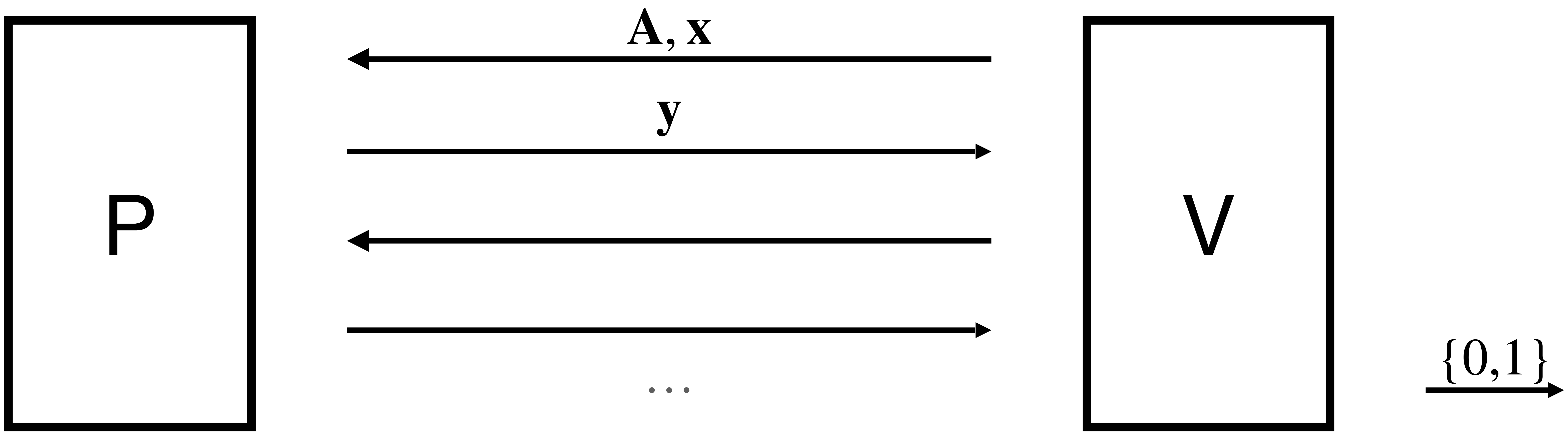

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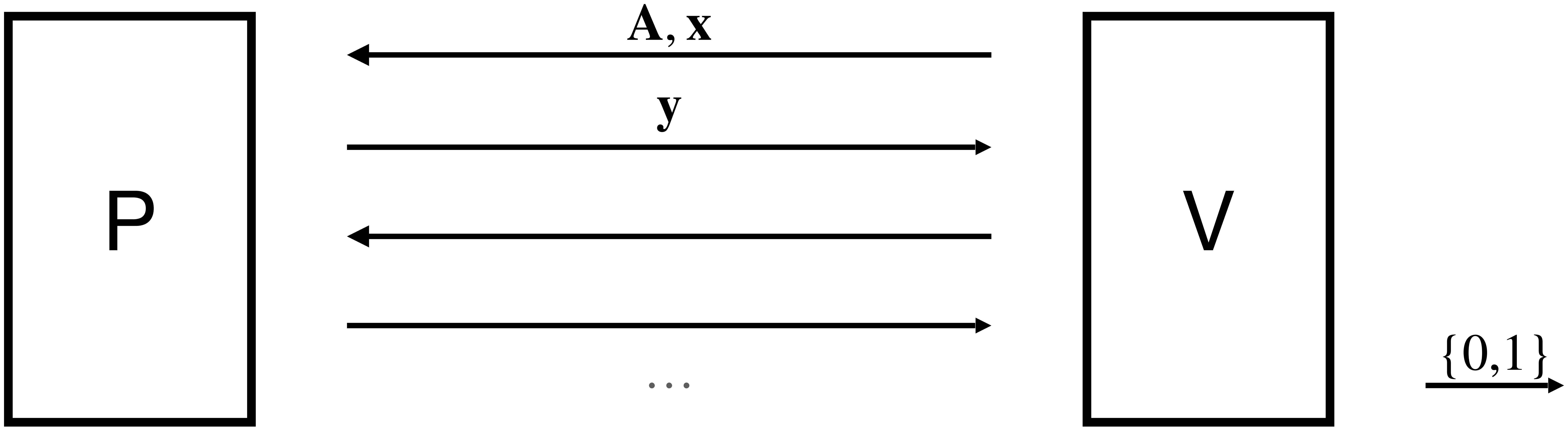

- Succinctness: V's work is ~logT
- Soundness: No prover of depth << T can pass the verification

Self-Symmetry

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$$\mathbf{A}_T = \begin{bmatrix} -\mathbf{G} & & & & \\ \mathbf{A} & -\mathbf{G} & & & \\ & \mathbf{A} & & & \\ & & \ddots & & \\ & & & & -\mathbf{G} \\ & & & & \mathbf{A} \end{bmatrix}$$

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$$T = 2t + 1$$

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- Verifier's runtime ~log(T)

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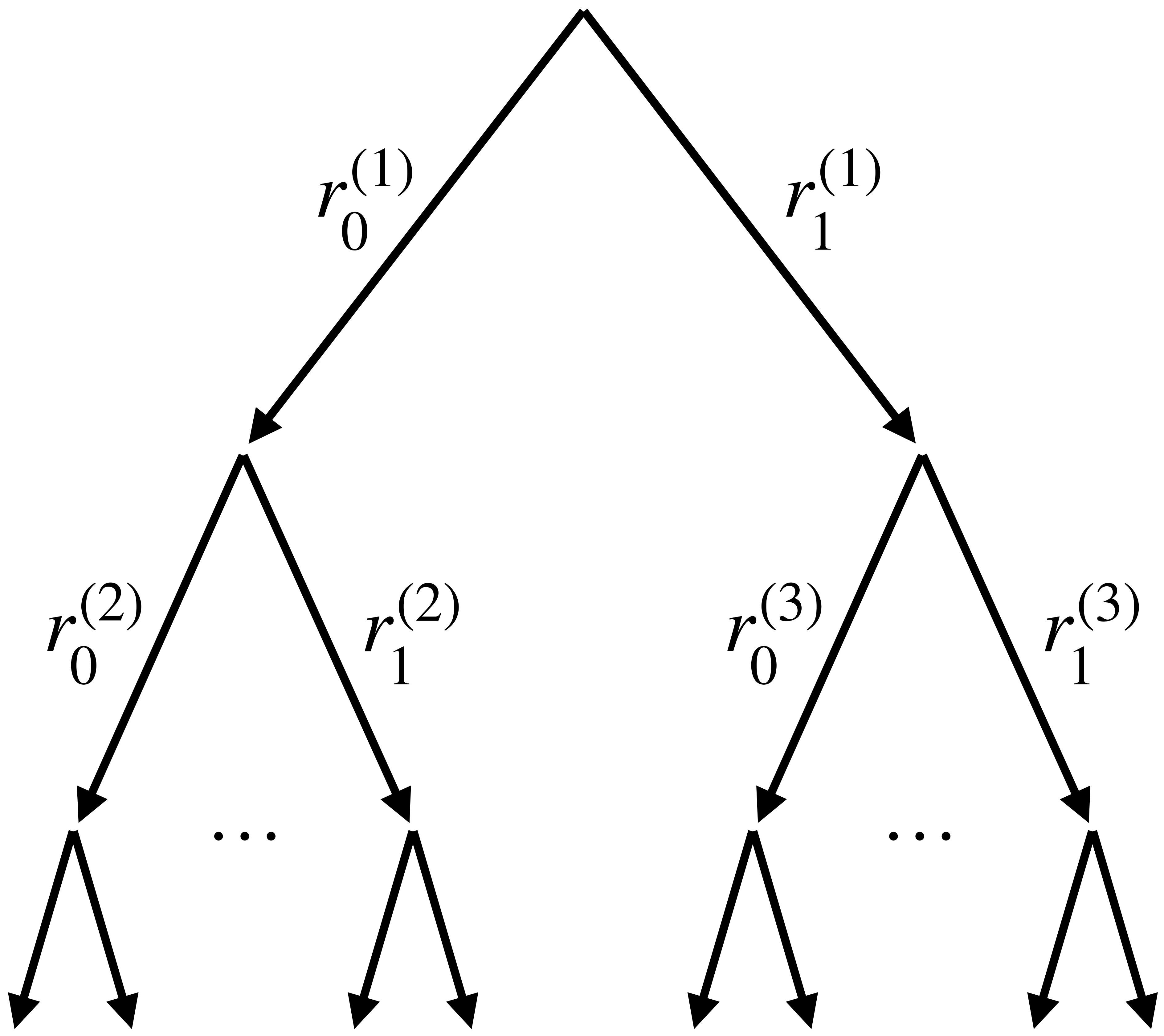

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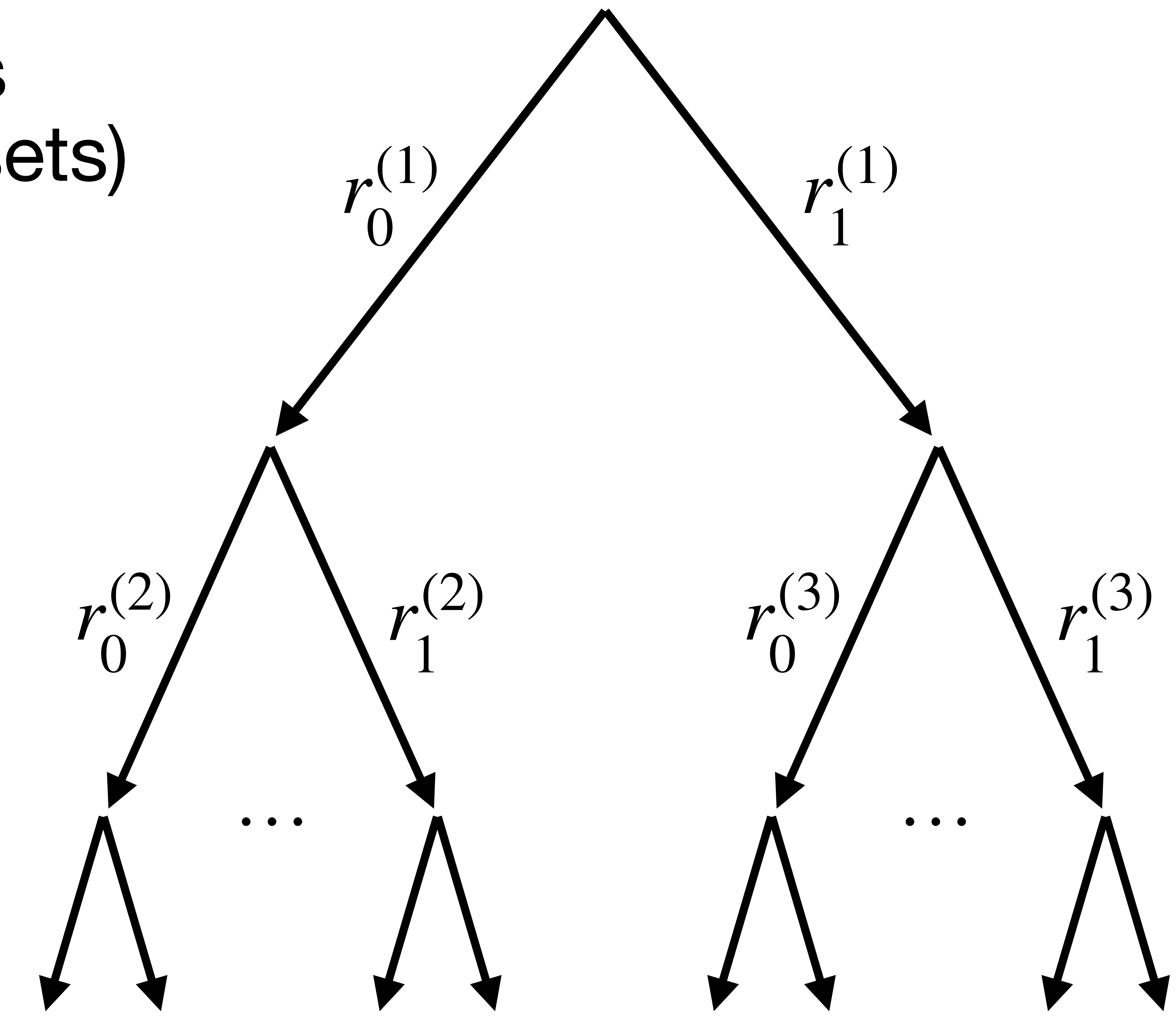

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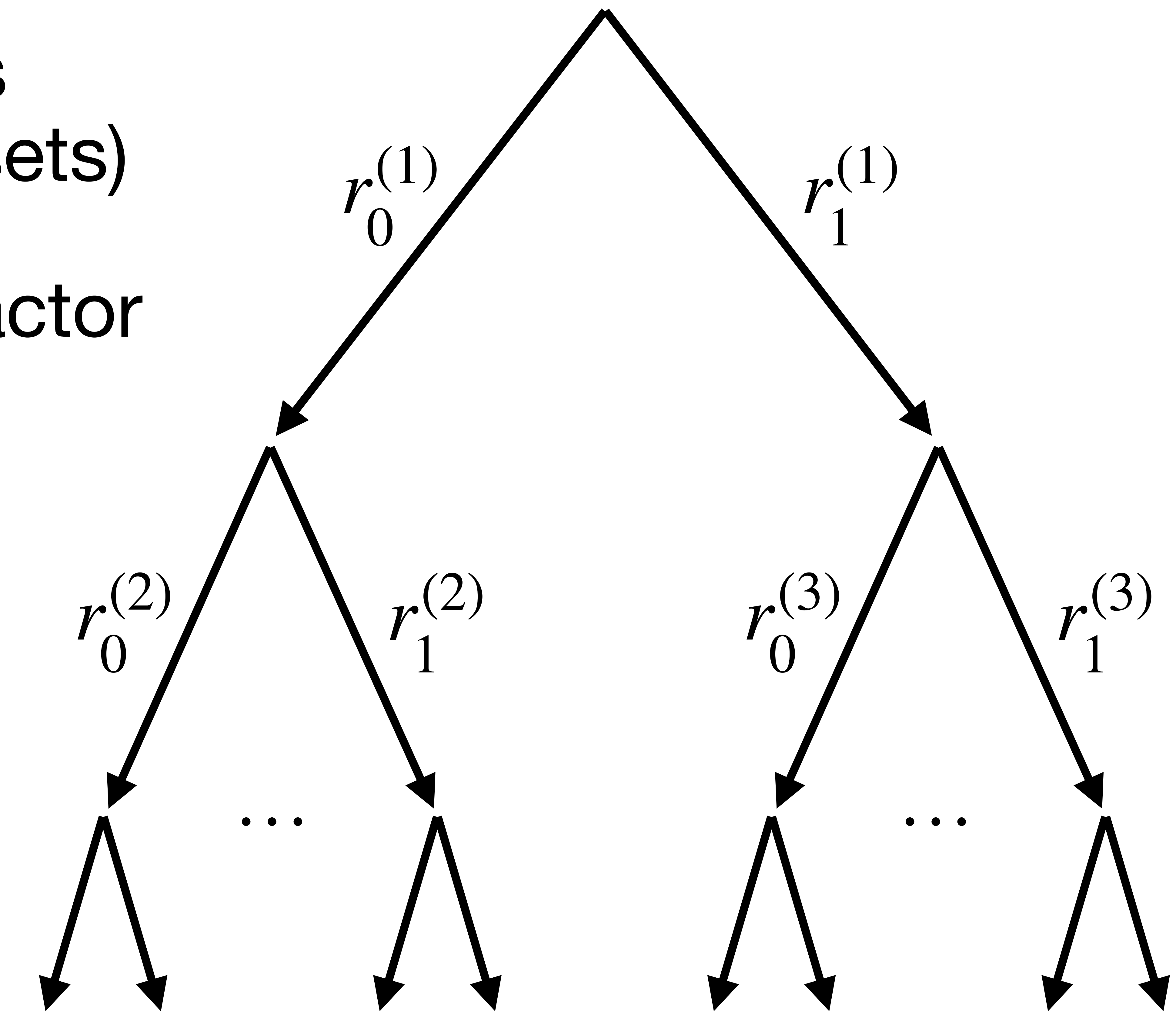

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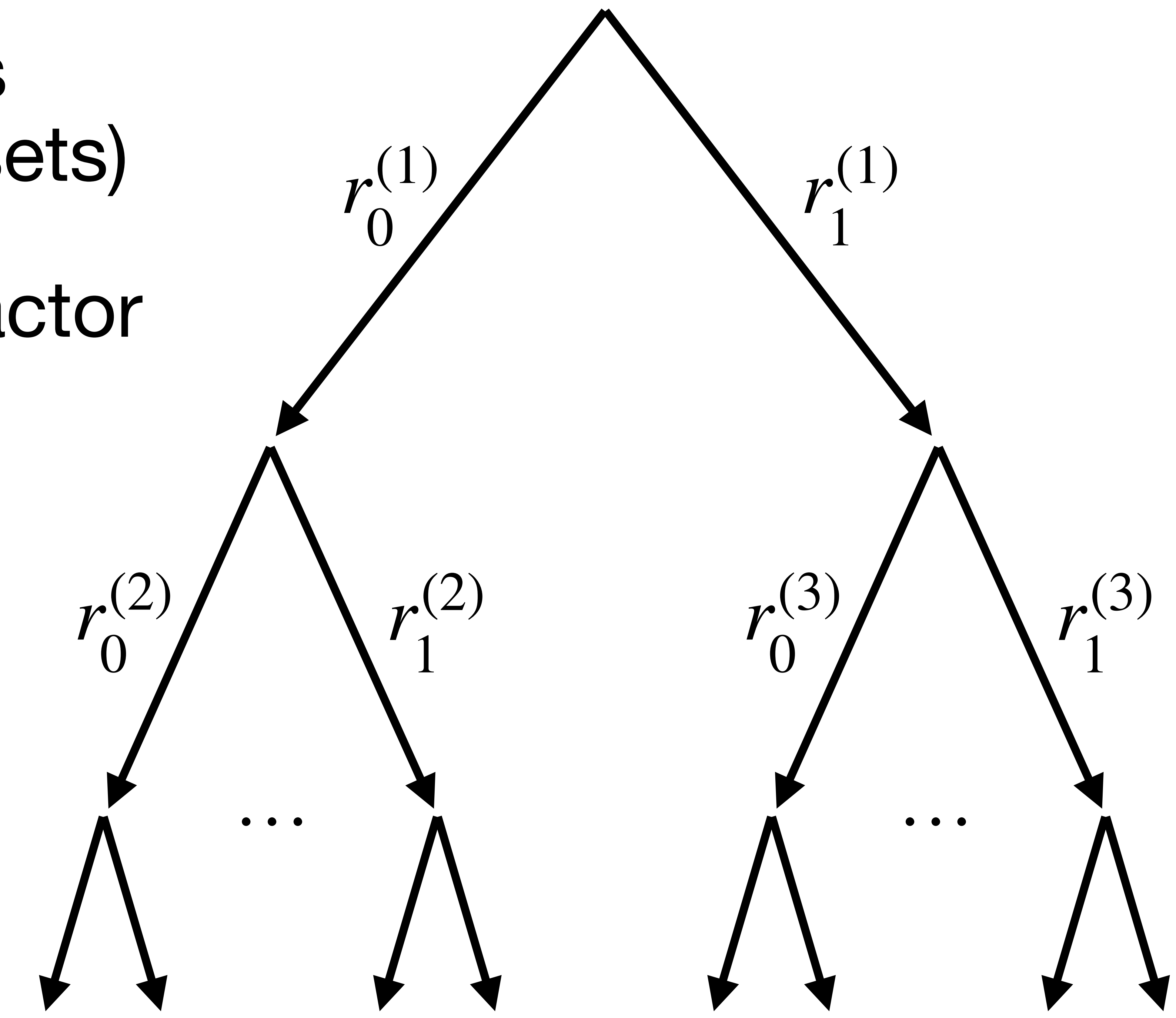

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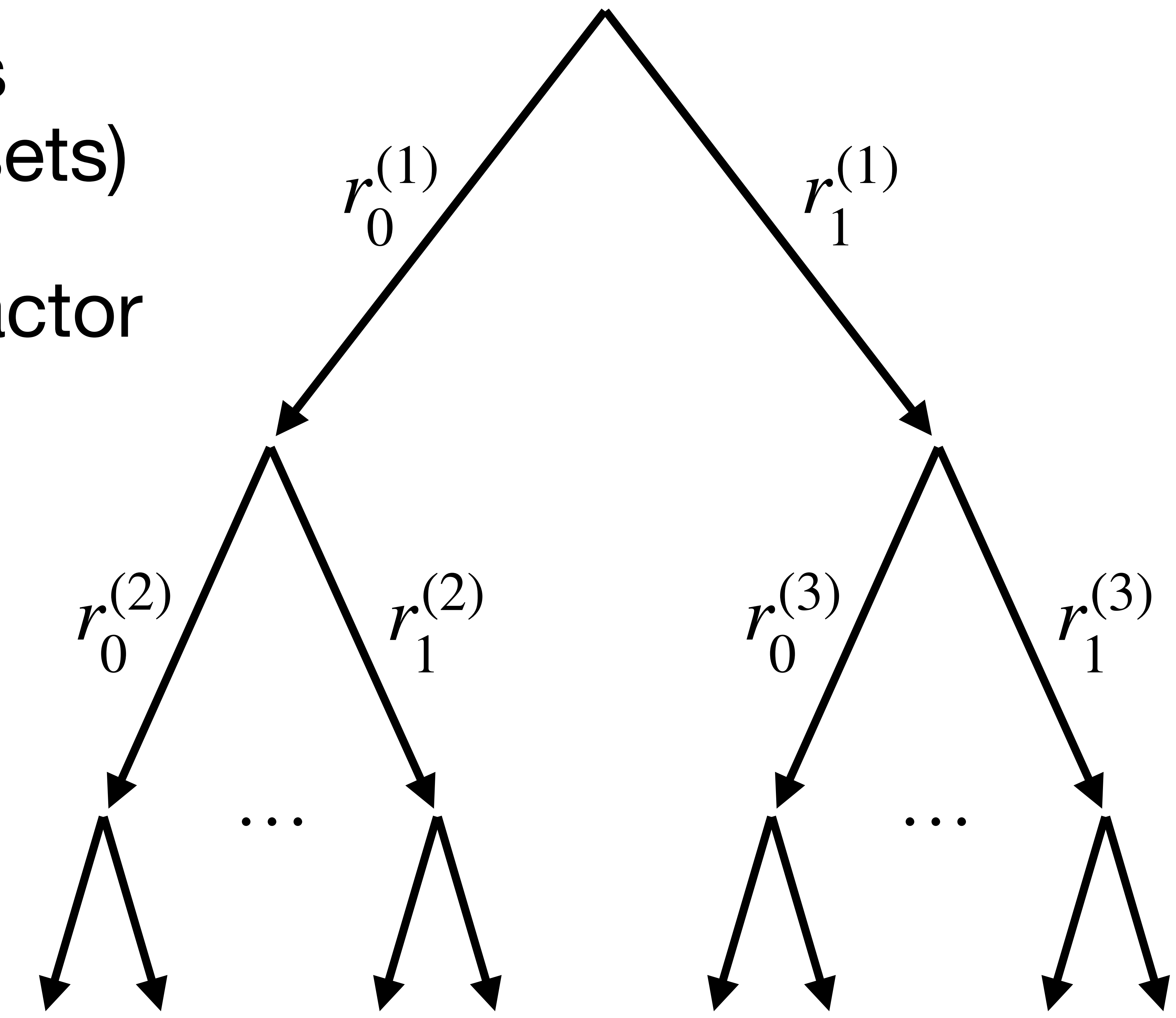
    - Contradiction!



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