Multi-party Homomorphic Secret Sharing & Sub-linear MPC from Sparse LPN

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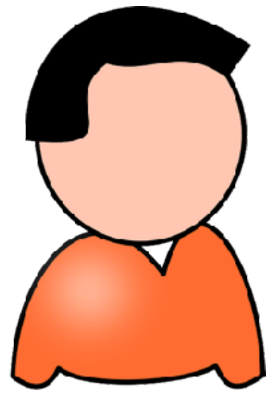

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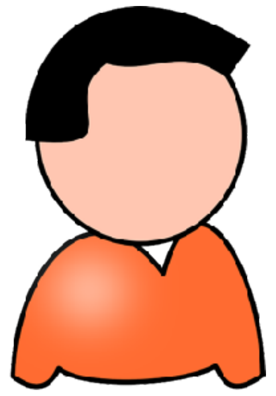

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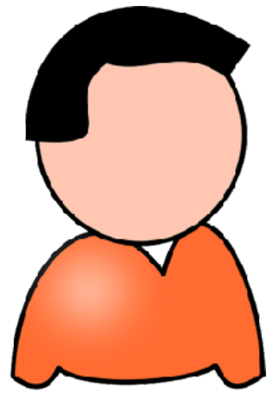

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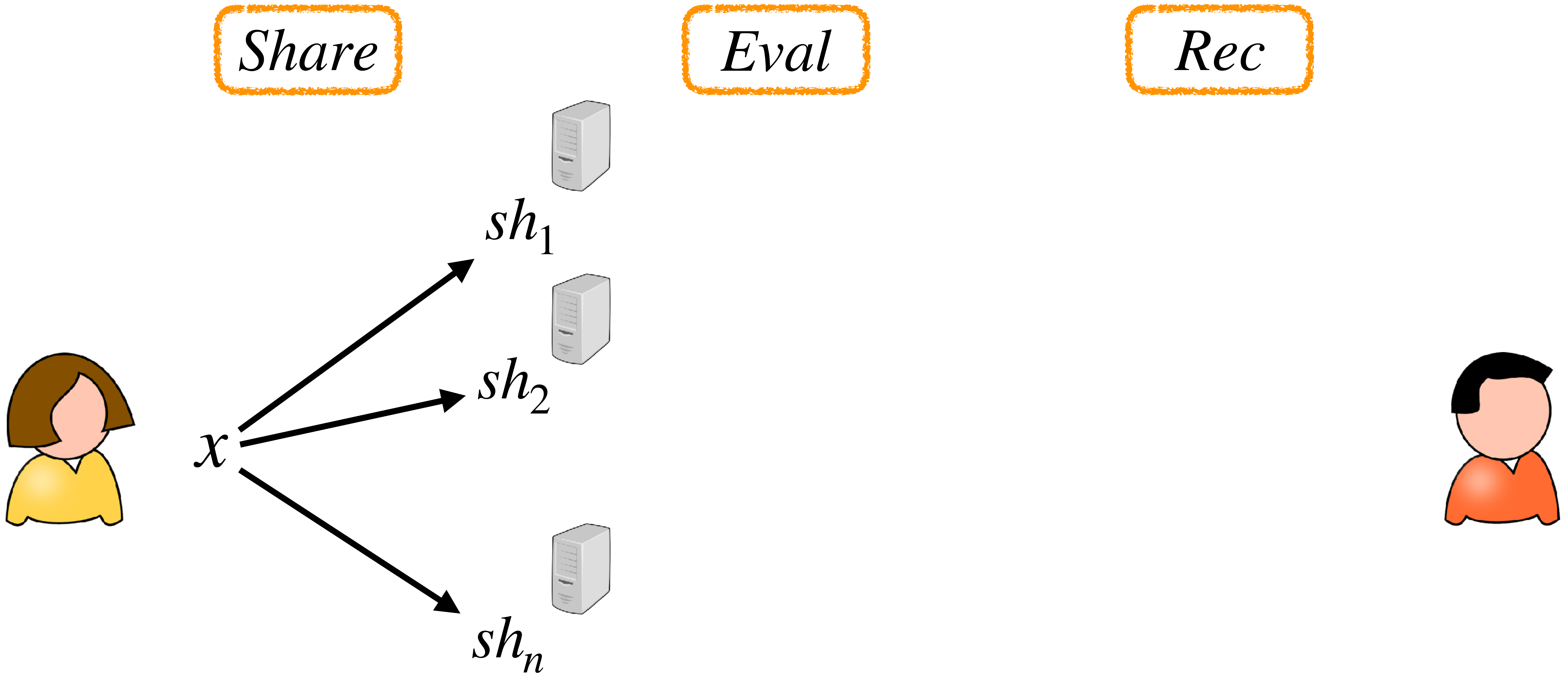

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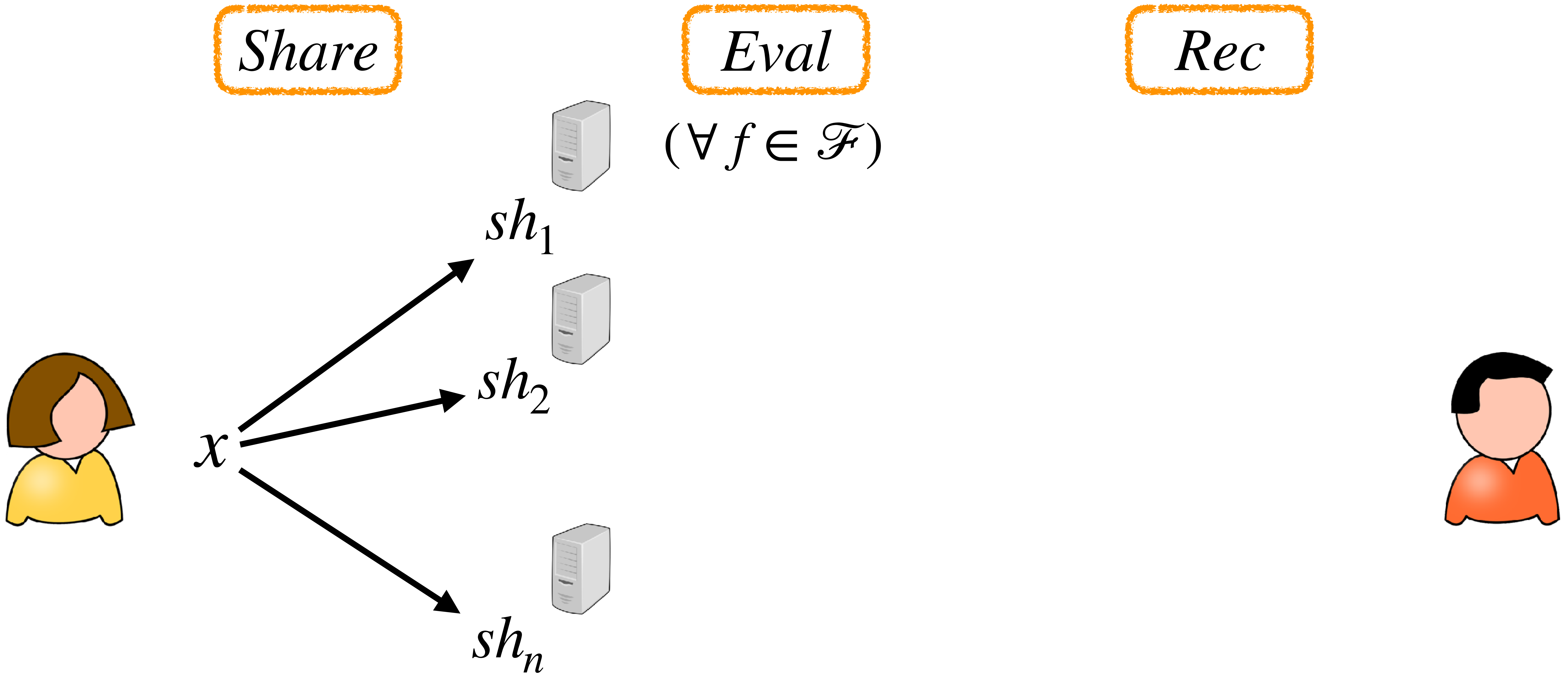

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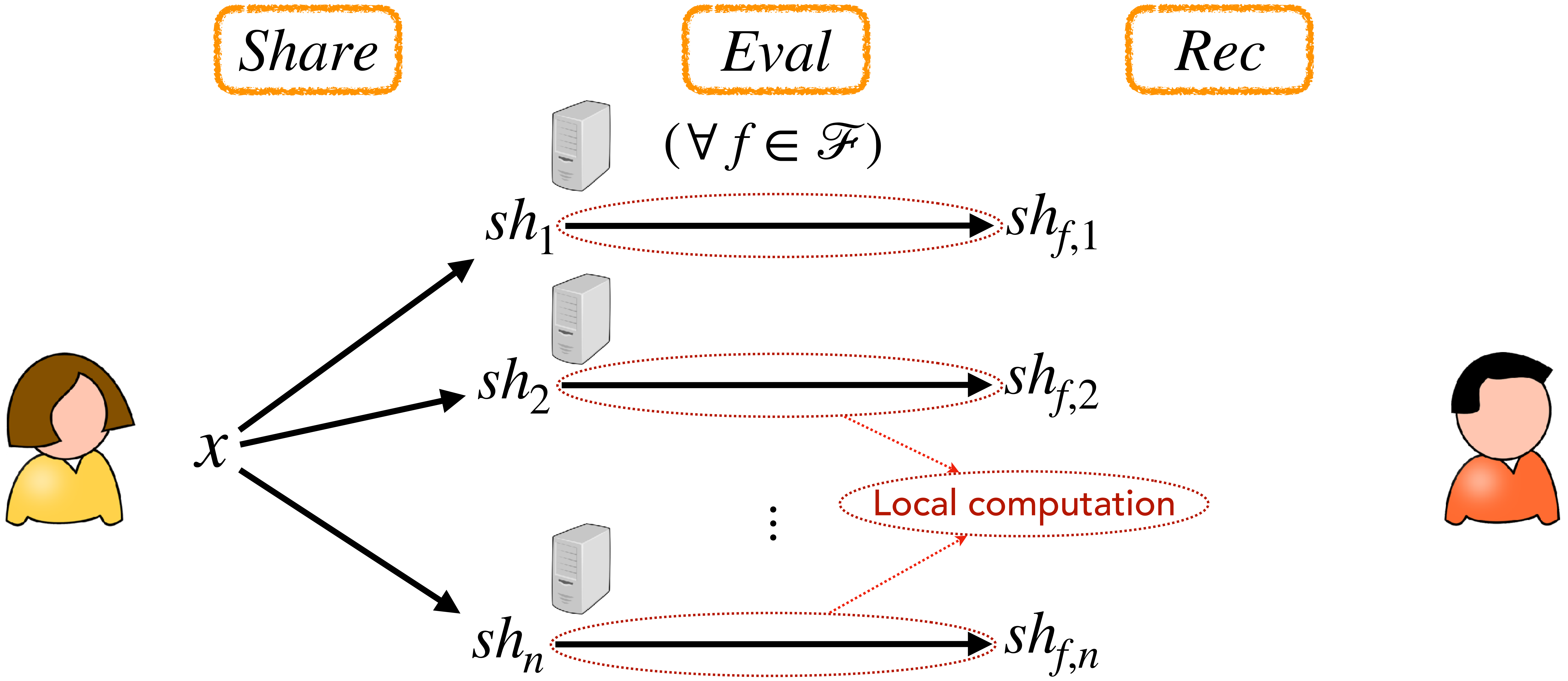

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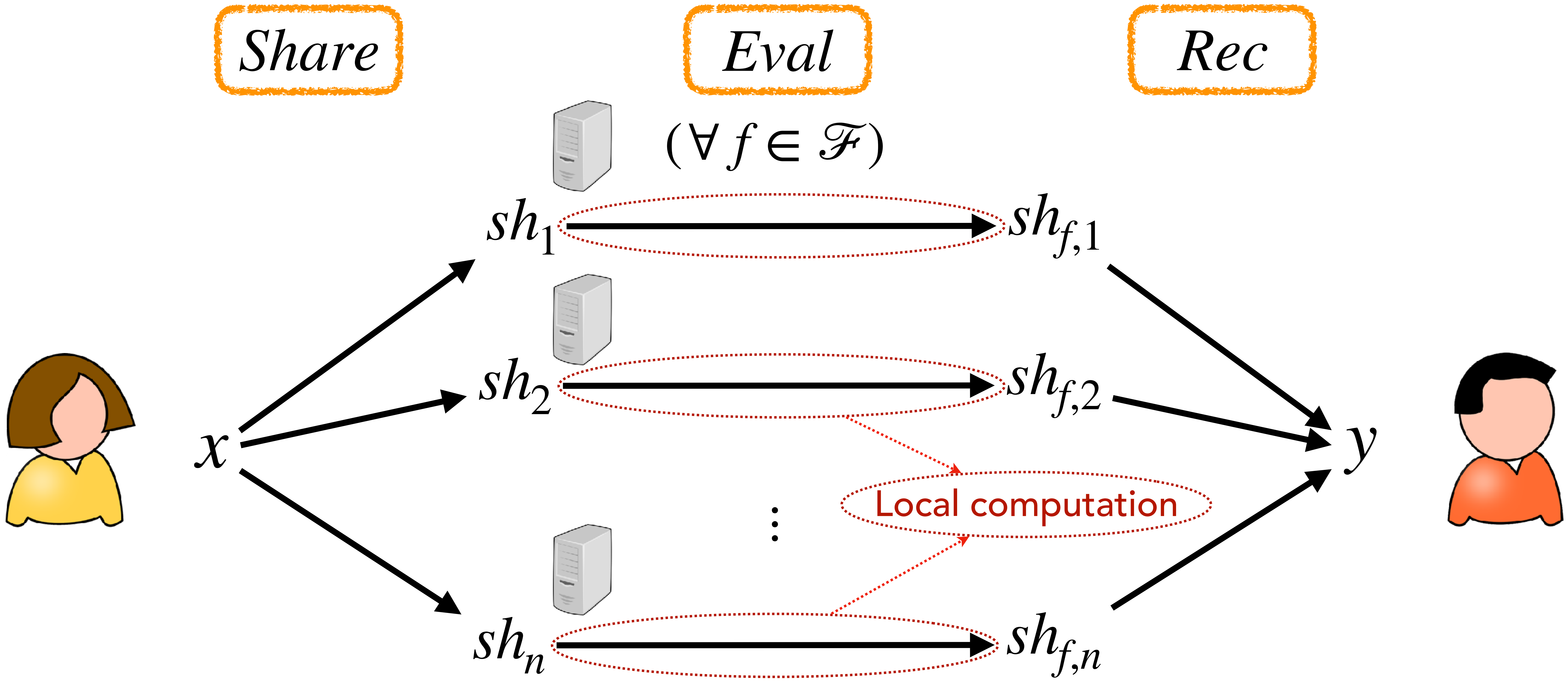

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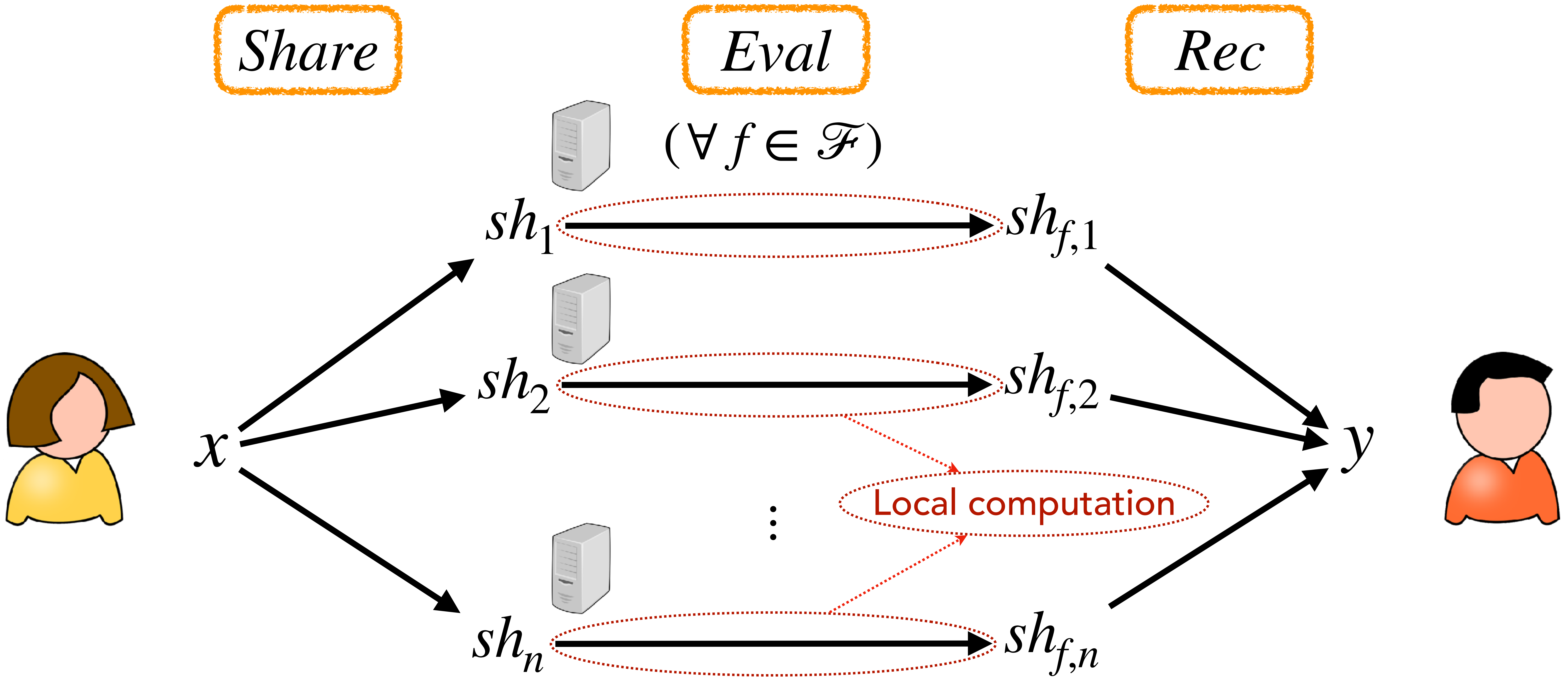

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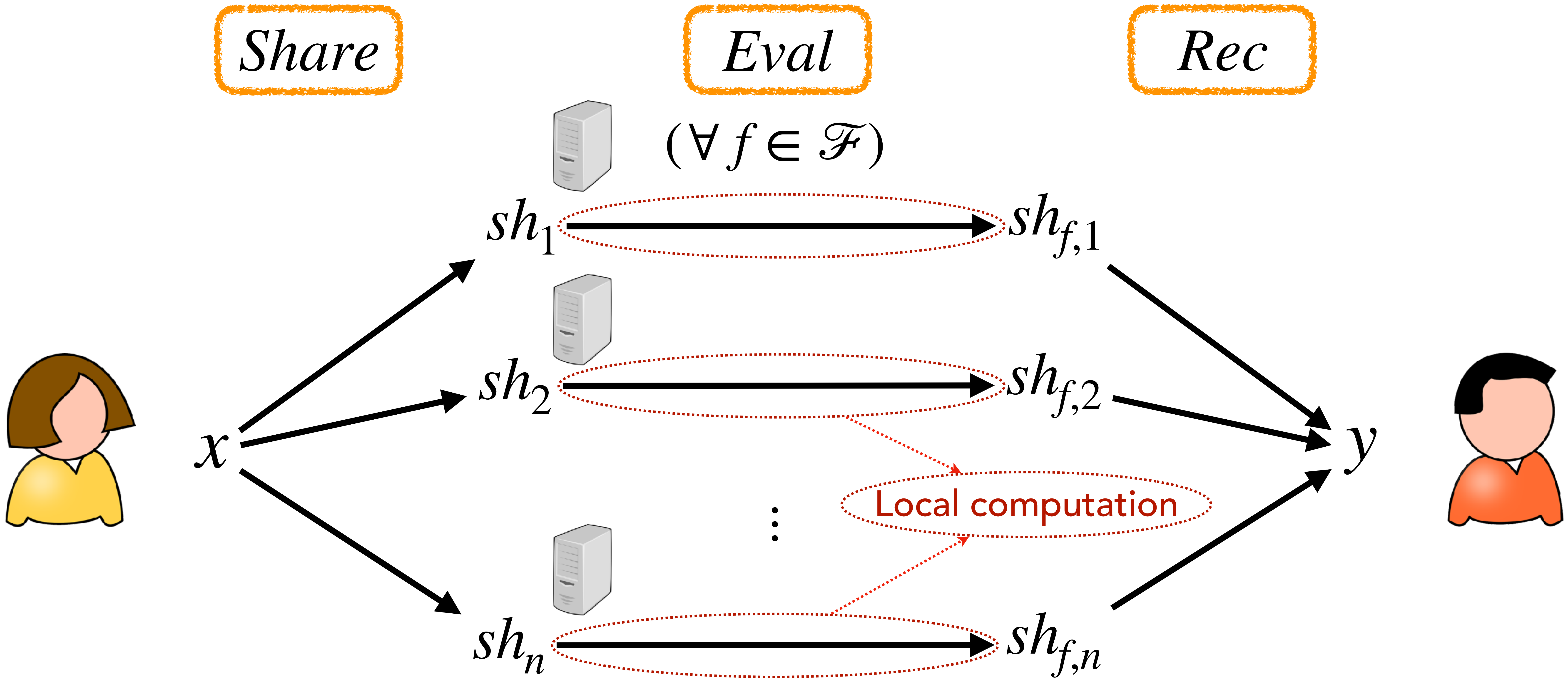

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- **Compactness:** $|\mathsf{sh}_{f,i}| \ll |f|$
- **Linear reconstruction:** *(Default)*
  Rec is a linear function

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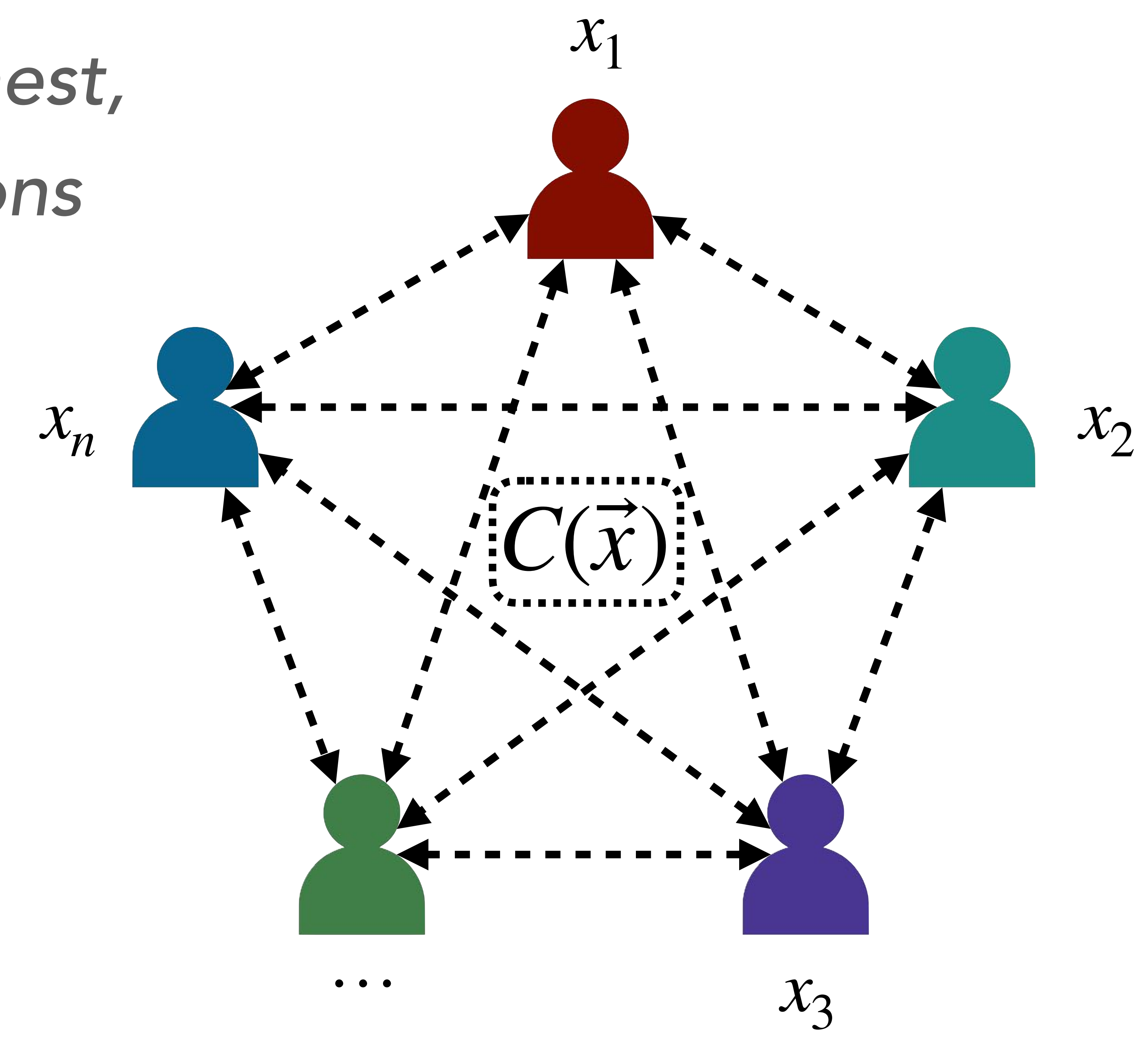

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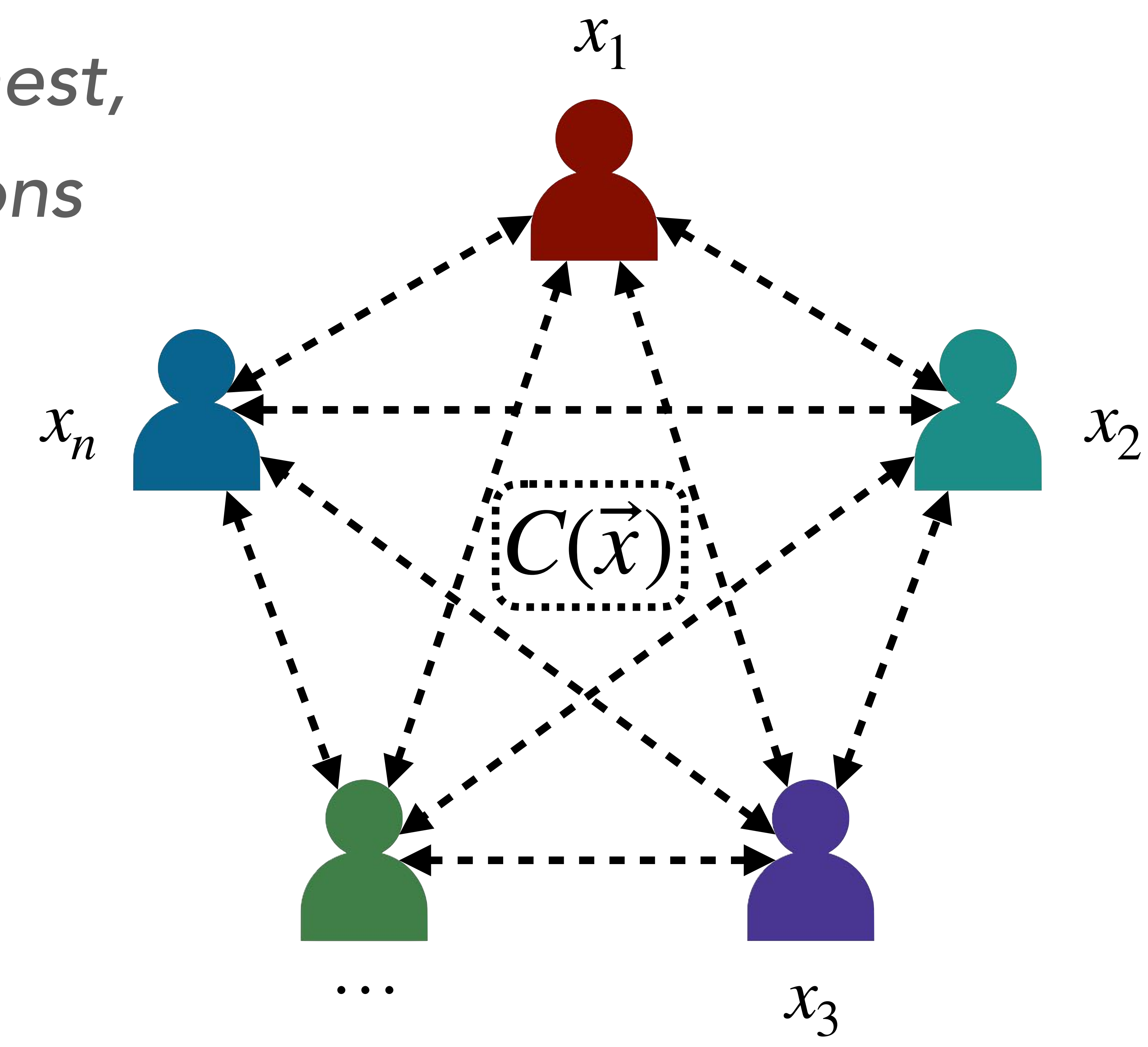

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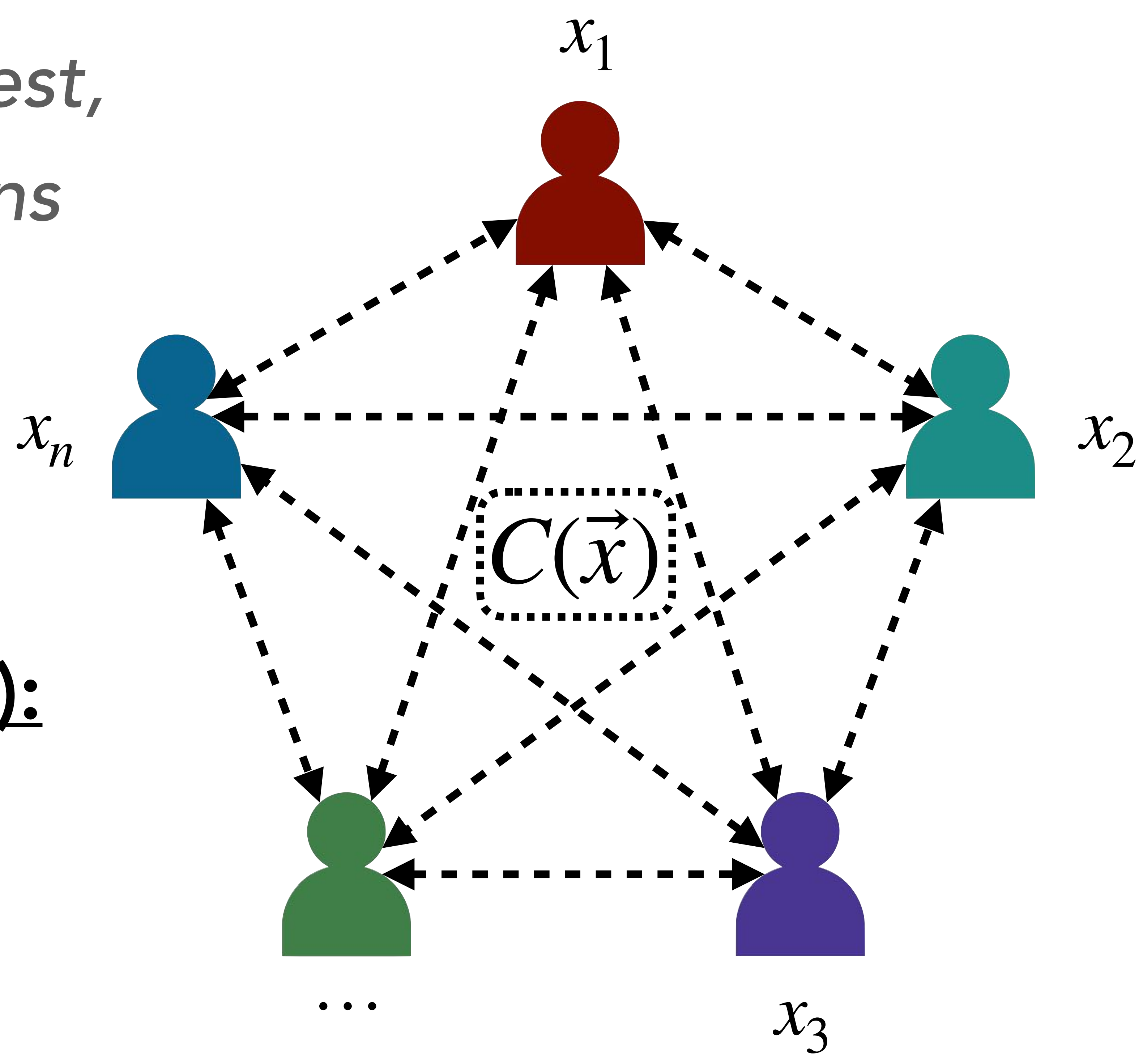

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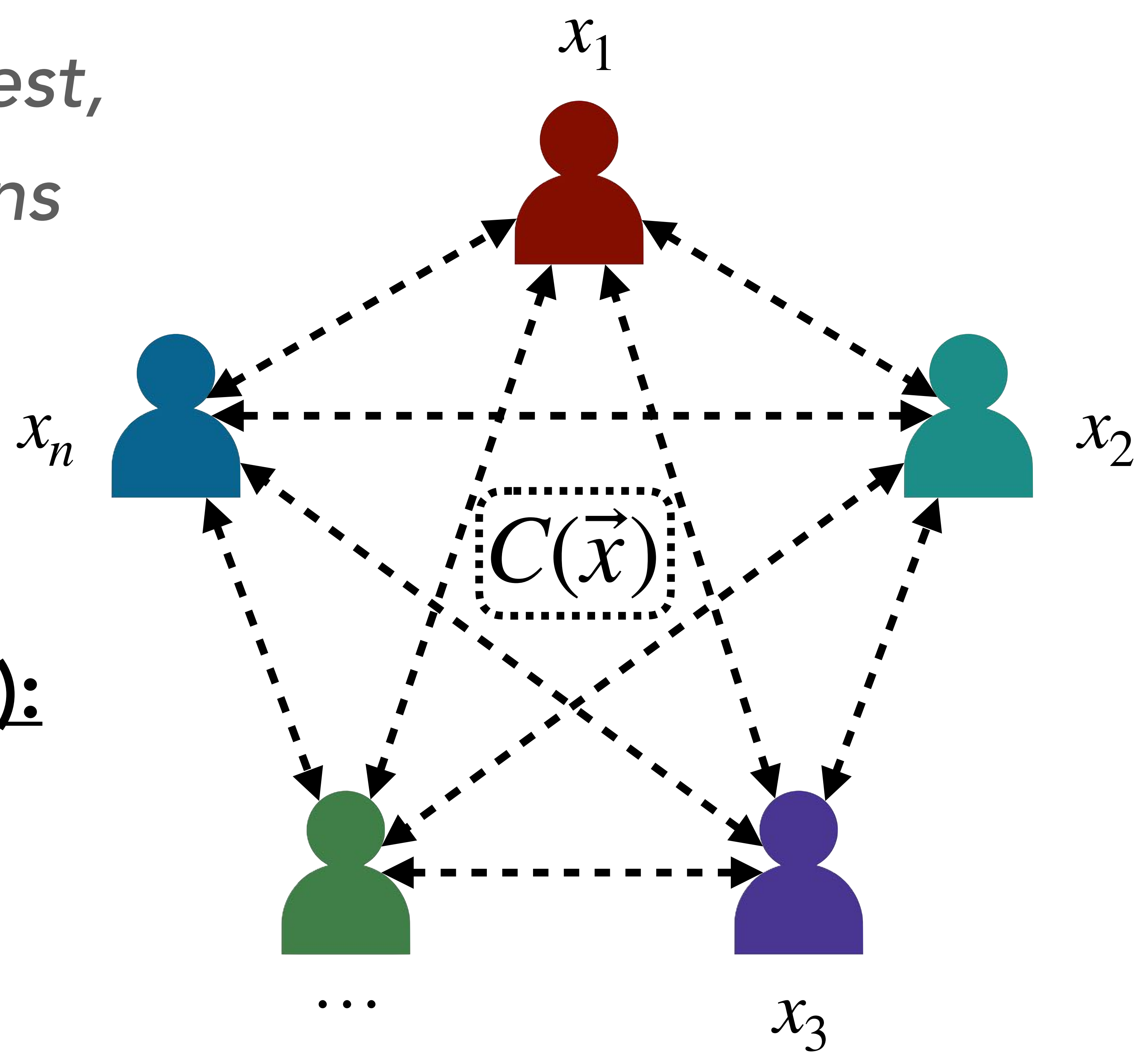

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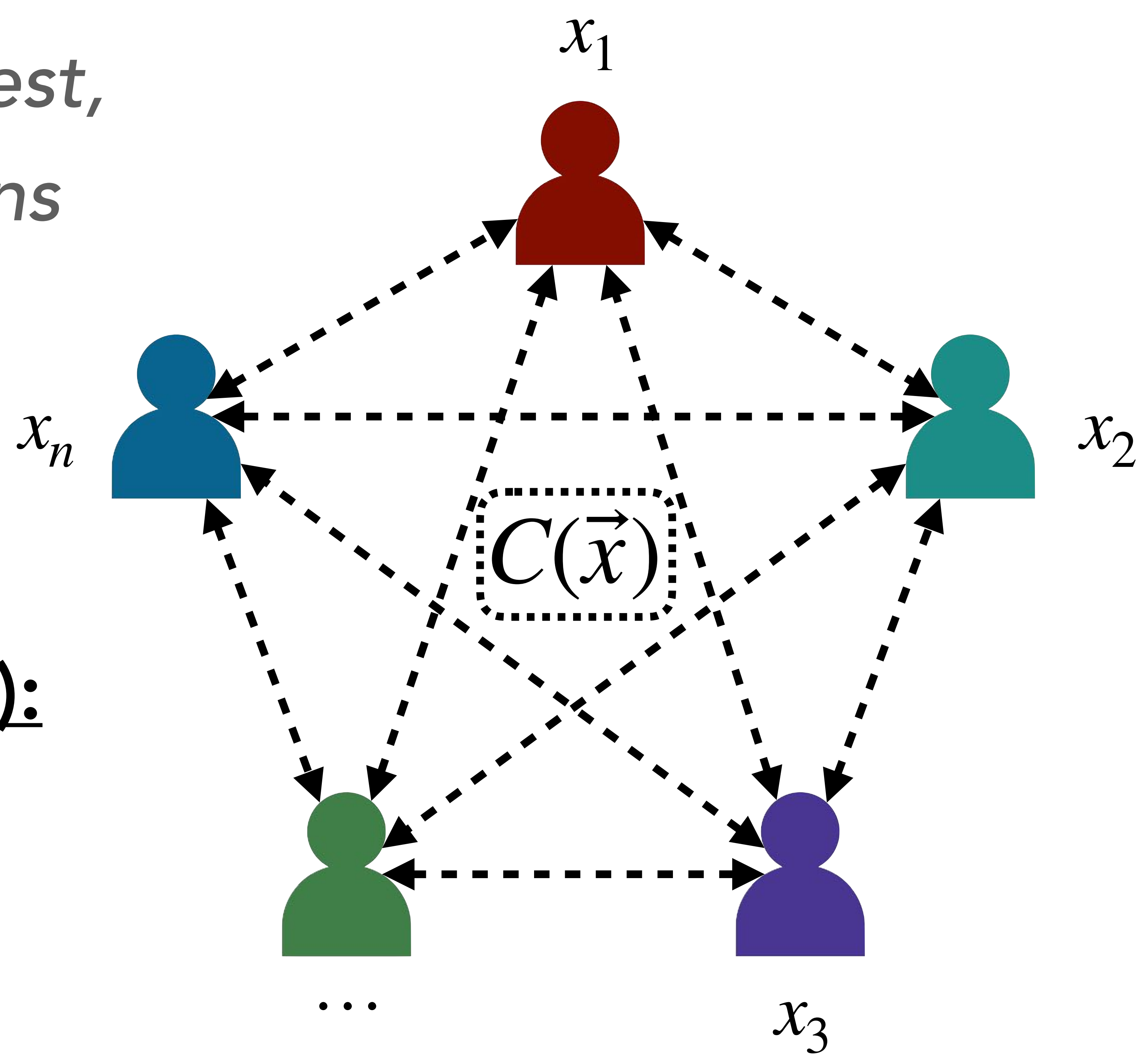

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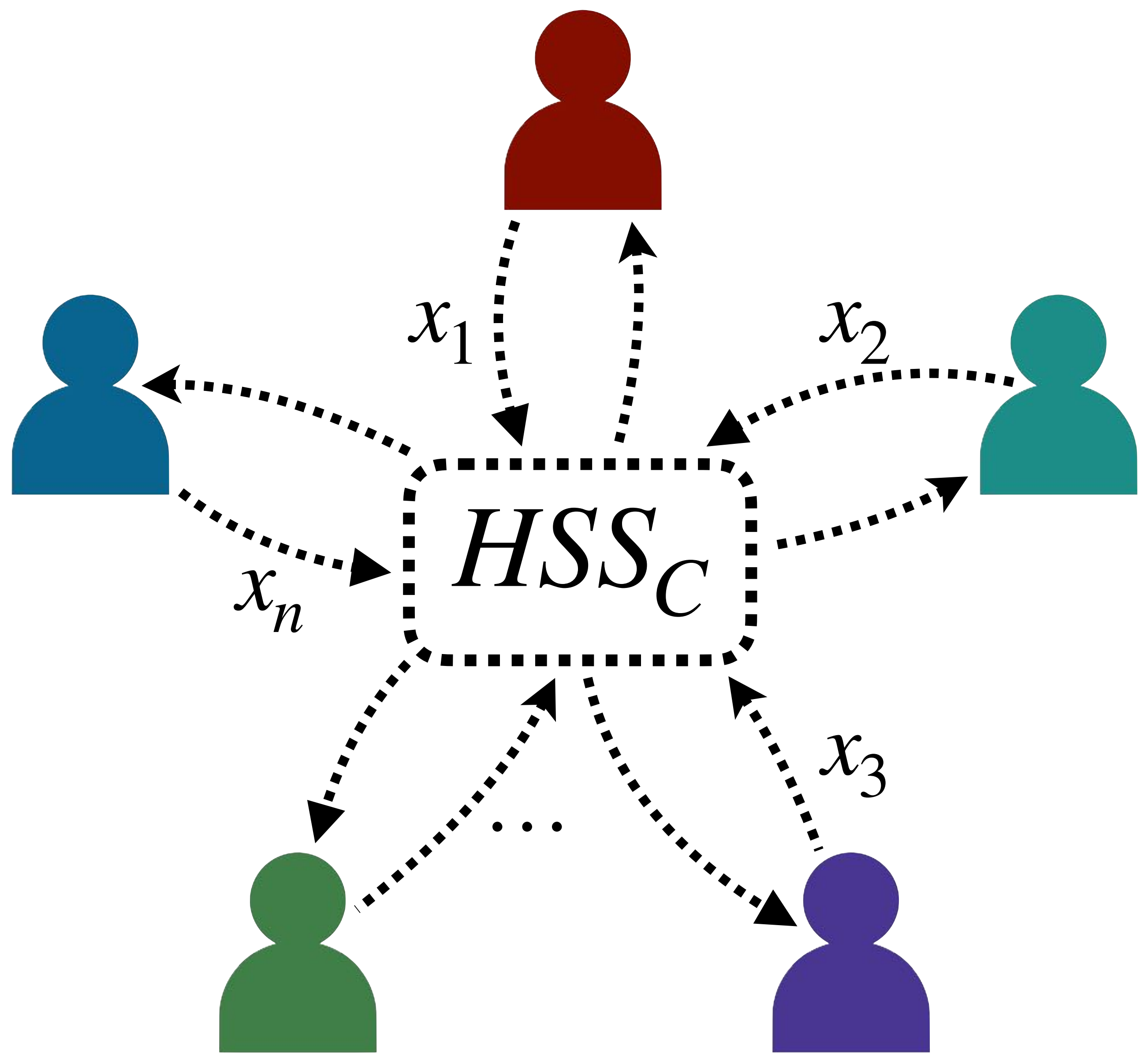

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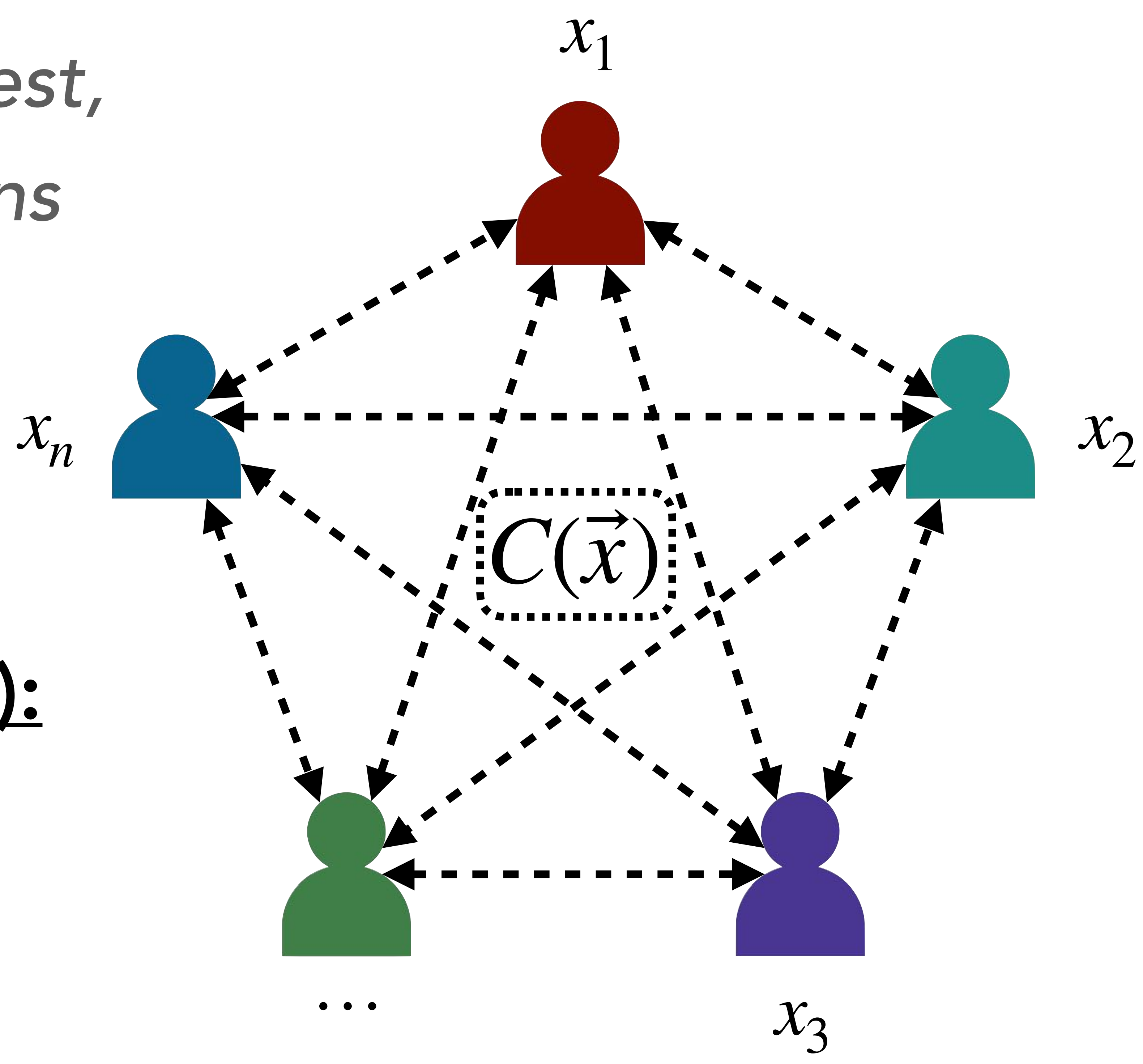

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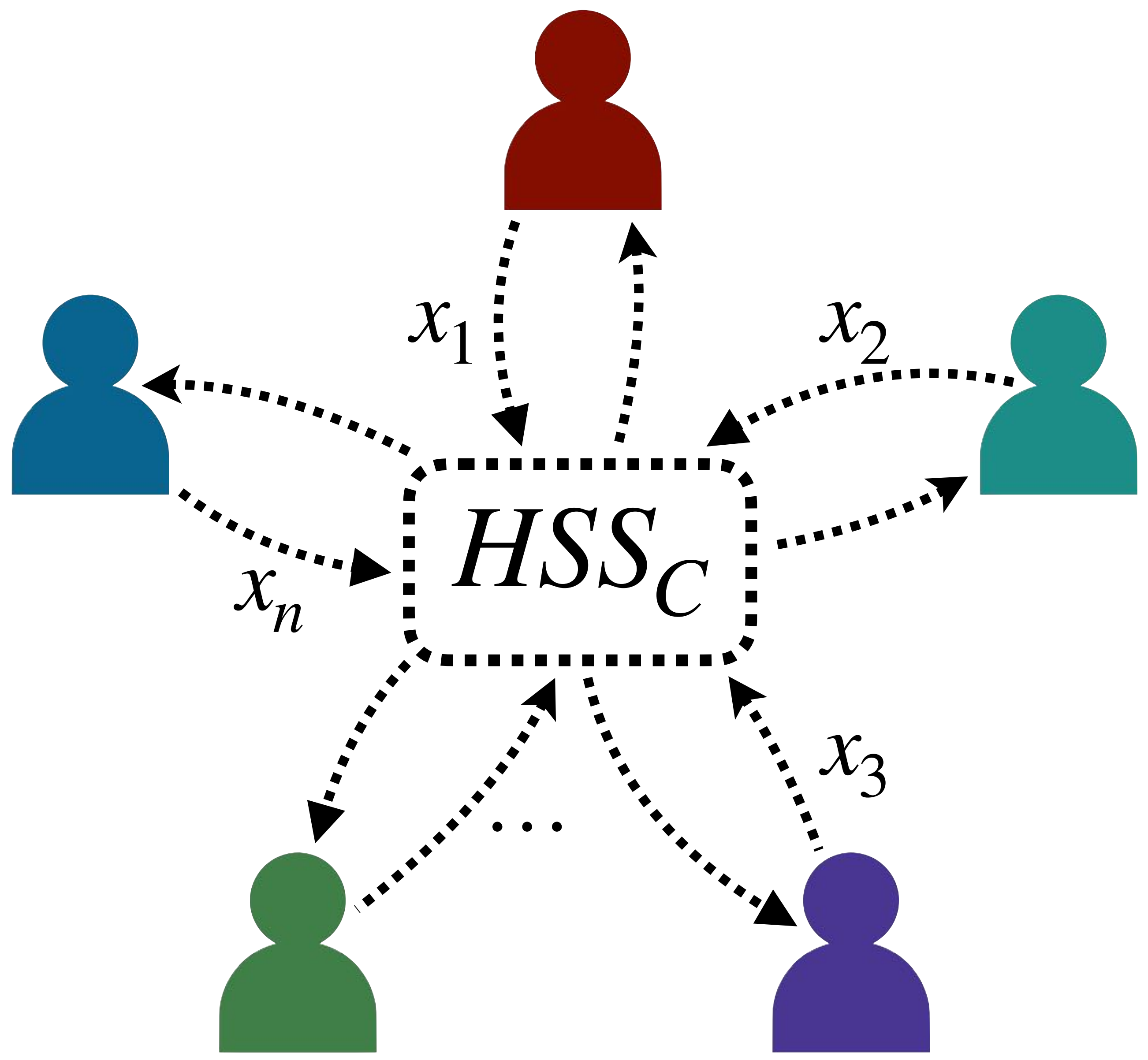

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Can we achieve HSS and sublinear MPC for arbitrary number of parties, without using iO or FHE?

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<u>Theorem 1:</u> Assuming ***<u>Sparse LPN</u>*** (over $\mathbb{F}$), there exists HSS for ***<u>arbitrary</u>*** number of parties, with $1/\text{poly}(\lambda)$ error and linear reconstruction*, for the following function classes:

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<u>Theorem 2:</u> Assuming ***<u>Sparse LPN</u>*** and OTs*, there exists ***<u>sublinear</u>*** MPC for *<u>layered</u>* Boolean circuits, with per-party communication $\approx \omega(1)\cdot S/\log\log S$ for a layered circuit of size S.

* known from LPN with noise $1/\sqrt{n}$ [Ale03], or a *<u>specific</u>* parameter setting for sparse LPN [ABW10]

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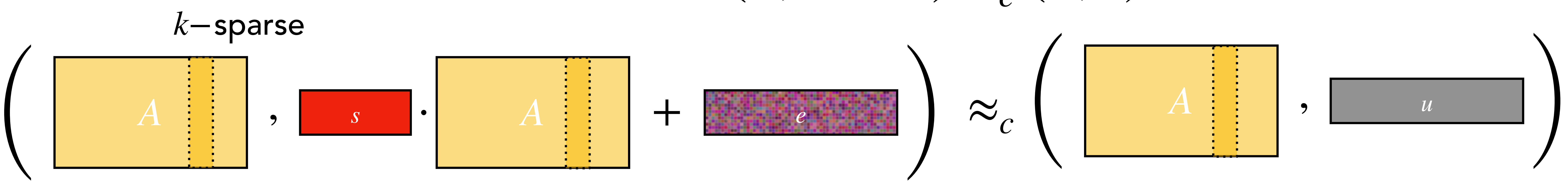

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- When $|\mathbb{F}| = 2$, this problem (and close variants) have been studied extensively in works on average-case complexity [Gol00, CM01, Fei02, MST03, FKO06, AOW15, AL16, KMOW17].

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- **Prior Applications:** hardness of approximation [Ale03], linear-stretch PRGs with constant locality [AIK06], constant-overhead commitments [IKOS08], PKE and semi-honest OT [ABW10], pseudorandom correlation generators (PCGs) [BCG+18, BCG+19], and constant-rate VOLEs [ADI+17, AK23]

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- This parameter regime is not known to imply PKE [ABW10] $\Longrightarrow$ multi-party HSS* potentially ***weaker*** than PKE.

* we consider secret-key HSS in this work, public-key HSS necessarily implies PKE

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- **Invariant:** any intermediate value y is stored as ***noisy*** shares $\left[y + e_y\right], \left[y \cdot s + e_{y \cdot s}\right]$.

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Limitation: ***Distributed rounding*** procedure only works for 2 parties.

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Problem: Noise grows by factor of $O(n) \Longrightarrow$ too fast!

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Noise growth: only $O(k)$ each time $\Longrightarrow$ for degree-d monomials, noise grows by $k^{O(d)}$.

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- Small computation overhead $O(k) = \text{poly} \log n$ for each multiplication.

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- Secret shares are indistinguishable due to t-privacy of $[\,\cdot\,]$.

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- **Technical Issue:** our proof only works for $|\mathbb{F}| > 2$! $\Longrightarrow$ HSS for $\mathbb{F}_2$ can be done in $\mathbb{F}_4$

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Thank you! Questions?