Cryptography with Weights:
MPC, Encryption and Signatures

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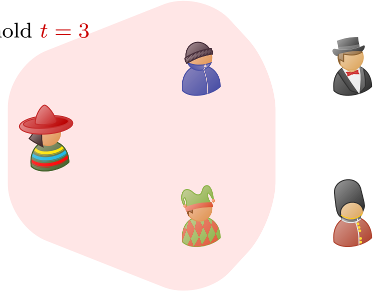
Privacy threshold $t = 3$



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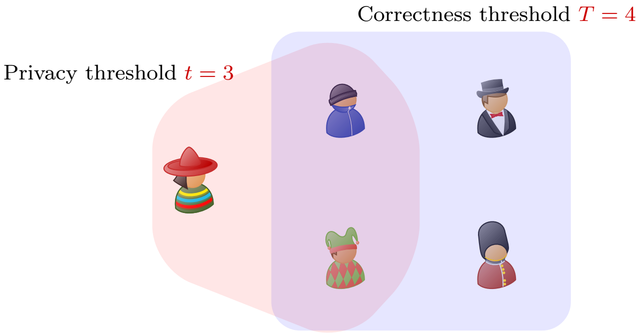

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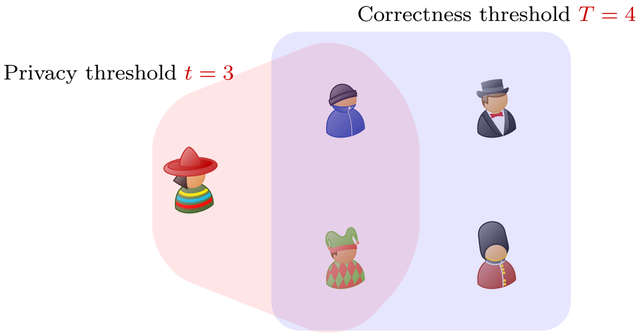
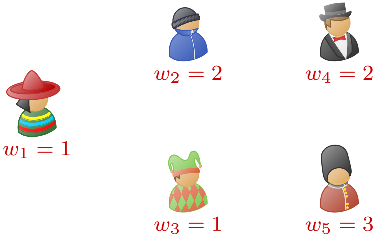
  - Sharp threshold: $T = t + 1$; Ramp setting: $T > t + 1$



One-party-one-vote may not suffice

- Parties are naturally asymmetric in some emerging applications
  - threshold signature in a stake-based blockchain setting



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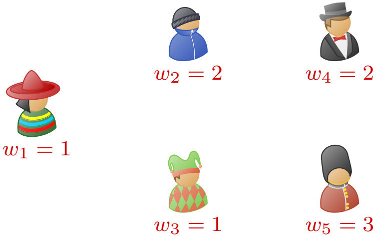

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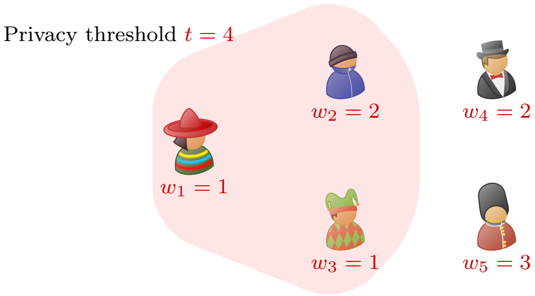

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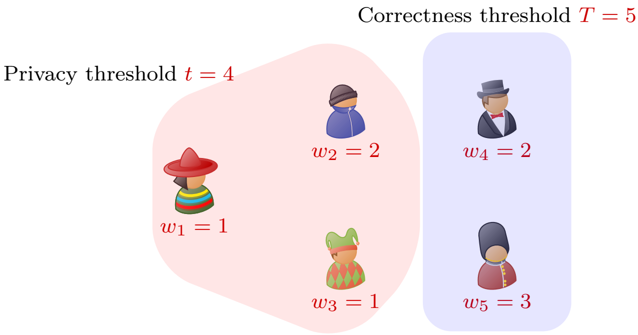

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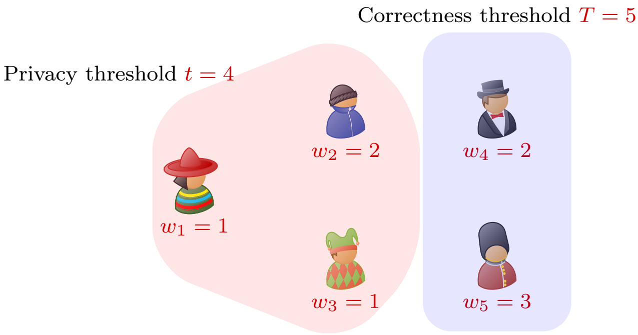
  - Motivated by real-world scenarios, small weight regime $w_i = \mathsf{poly}(\lambda)$.



Existing Solutions: Naïve Virtualization

- Party with weight w_i is treated as w_i virtual parties.
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This work: take-home message

The answer is yes if there is a sufficient gap between reconstruction threshold T and privacy threshold t,

$$T - t = \Omega(\lambda).$$

Technical Core

Efficient Weighted Ramp Secret Sharing (WRSS)

Let $(w_1, \ldots, w_n, T, t)$ define a weighted access structure.

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There exists a weighted ramp secret sharing scheme for λ-bit secret such that

- The share size of a party with weight w_i is $O(w_i)$.
  - Comparison to Shamir $w_i \cdot \lambda$ for a λ-bit secret
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Applications

- Applicable to MPC, threshold encryption, and threshold signature.
- The application inherits the efficiency gain of the secret-sharing schemes.
- WRSS is *non-linear*, which presents some technical challenges

Prior Works

[Beimel-Weinreb'05, Beimel-Tassa-Weinreb'05]

- Computational setting (OWF), Sharp threshold
- poly(n) share size, independent of the weights w_i
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Concurrent Work

[Benhamouda-Halevi-Stambler ITC'22]

- Information-theoretic and ramp setting, where $T = \beta \cdot W,\ t = \alpha \cdot W$ with constants $\beta > \alpha$.
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Compare to Our Work

- Our scheme still depends on the weights w_i, trade-off depends on the weights
- Our scheme preserves the *algebraic structure* of the secrets, render it applicable to threshold crypto and MPC

CRT-based Secret Sharing [Mignotte'83, Asmuth-Bloom'83]

- Suppose secret $s \in \mathbb{F}$, where $|\mathbb{F}| = p_0 \approx 2^\lambda$.
- Parties are associated with integers $p_1, \ldots, p_n$.
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To reconstruct a secret

- Given the secret shares $\{s_i\}_{i \in A}$ from an authorized set A
- Invoke Chinese remaindering theorem to find the integer S such that

$$\forall i \in A, \qquad S \mod p_i = s_i.$$

- Reconstruct the secret s as $s = S \mod p_0$.

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- Authorized set A satisfies $\sum_i w_i > T$. Enough information to construct!
- Unauthorized set B satisfies $\sum_i w_i < t$. Small enough such that no information of the secret is leaked.

Weighted MPC

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- Integer grows *quickly* for $\times$. Every multiplication doubles the length of the integer.
- "degree-reduction" protocol after each multiplication!

Applications to Threshold Crypto

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Challenges with non-linear secret sharing

To reconstruct a secret s,

$$s = \Bigg(\underbrace{(s_1 \cdot \lambda_1 + s_2 \cdot \lambda_2 + \cdots + s_n \cdot \lambda_n) \overbrace{\bmod P}^{\text{non-linear}}}_{S} \Bigg) \bmod p_0.$$

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Suppose parties want to reconstruct g^s by broadcasting g^{s_i}. Note that

$$(g^{s_1})^{\lambda_1} \cdots (g^{s_n})^{\lambda_n} \neq g^s$$

as

$$\left(s_1 \cdot \lambda_1 + s_2 \cdot \lambda_2 + \cdots + s_n \cdot \lambda_n\right) \mod p_0 \neq \Big(\left(s_1 \cdot \lambda_1 + s_2 \cdot \lambda_2 + \cdots + s_n \cdot \lambda_n\right) \mod P\Big) \mod p_0$$

Our Solution

We change the reconstruction to be

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Let r_i be $(s_i \cdot \lambda_i) \mod P$. Note that

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for some $\alpha \in \{0, 1, \ldots, n-1\}$.

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- Suppose parties broadcast g^{r_i}.
- Now, parties know

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Weighted Threshold Encryption/Signature

- Threshold ElGamal: The encryptor will send additional information to help parties recover α.
- We also constructed weighted threshold ECDSA. Refer to the paper for details.

Follow-up Works

- Weighted (sharp-)Threshold Signature
  - [Garg-Jain-Mukherjee-Sinha-Wang-Zhang S&P'24] **ia.cr/2023/567**
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Thanks! Questions?