New Bounds on the Local Leakage Resilience of Shamir's Secret Sharing Scheme

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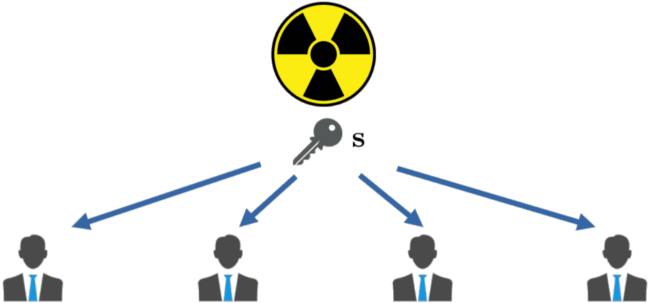
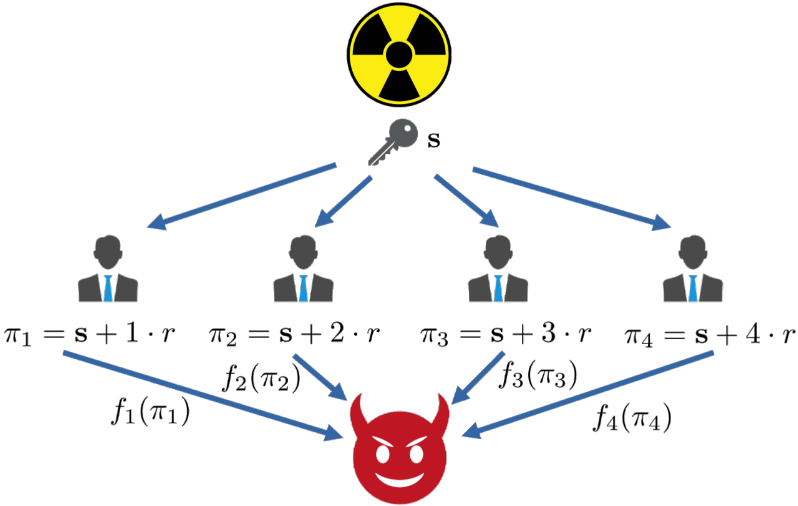
August 21, 2023

Secret Sharing (t out of n)



$\pi_1 = \mathbf{s} + 1 \cdot r$ $\pi_2 = \mathbf{s} + 2 \cdot r$ $\pi_3 = \mathbf{s} + 3 \cdot r$ $\pi_4 = \mathbf{s} + 4 \cdot r$

Local Leakage Resillient Secret Sharing



Applications of Leakage Resilient Secret Sharing

Main Application (Conjectured Benhamouda, Degwekar, Ishai, Rabin '18)

BGW protocol for **MPC** (Ben-Or, Goldwasser, Wigderson '88) is more secure than currently known.

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- Leakage resilient circuit compilers.
- Threshold cryptographic systems.

Definition of Secret Sharing (t out of n)

$\mathbf{s}$ denotes secret, π_i denotes share. Assume $\mathbf{s}, \pi_i \in \mathbb{F}_q$.

Definition (Secret Sharing, t out of n)

A randomized algorithm $\mathbf{s} \mapsto (\pi_1, \ldots, \pi_n)$ s.t.

- **Reconstruction:** Any t shares determine $\mathbf{s}$ uniquely.
- **Indistinguishability:** Knowledge of less than t shares reveals nothing about $\mathbf{s}$.

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Shamir's Secret Sharing Example (3 out of 5)

Let $r_1, r_2 \sim \mathbb{F}_q$ uniformly random.

$$\pi_1 = \mathbf{s} + 1 \cdot r_1 + 1^2 \cdot r_2,$$
$$\pi_2 = \mathbf{s} + 2 \cdot r_1 + 2^2 \cdot r_2,$$
$$\vdots$$
$$\pi_5 = \mathbf{s} + 5 \cdot r_1 + 5^2 \cdot r_2.$$

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Definition (Leakage Resilient SS)

SS is resilient against leakage functions $f_1, \ldots, f_n$ if
knowledge of $\mathbf{Leak} = (f_1(\pi_1), \ldots, f_n(\pi_n))$ reveals almost nothing about $\mathbf{s}$.

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Various Security Models

- f_i outputs a few bits.
- f_i depends on several shares.
- f_i easy to compute.

Applications vs. Previous Results

What is missing for MPC application?

Shamir's secret sharing is resilient for $t = (1-\epsilon)n/2$. (Security against passive adversary.)
Shamir's secret sharing is resilient for $t = (1-\epsilon)n/3$. (Active adversary.)

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Theorems

Shamir's SS is $\exp(-n^c)$ leakage resilient if:

- $f_i : \mathbb{F}_p \to \{0,1\}^{\lg(p)/4}$ and $t \geq n - n^{1/4}$. (Benhamouda, Degwekar, Ishai, Rabin '18)

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- Random linear SS is resilient against $f_i : \mathbb{F}_p \to \{0,1\}$ if $t \geq (0.5+\epsilon)n$. (MPSW '20)
- $\exists$ non-linear SS against $f_i : \mathbb{F}_p \to \{0,1\}^{0.99 \lg(p)}$, $\forall$ access structure. (Srinivasan, Vasudevan '19)

• • •

Limitations of Linear Leakage Resilience

Shamir SS with small t is not leakage resilient

$\forall$ linear SS with threshold t, $\exists$ one-bit leakage functions with $I(\mathbf{s}; \mathbf{Leak}) \geq \exp(-t)$.

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t and n are of same order of magnitude (Nielsen Simkin '19)

$\exists c > 0$ s.t. if $t < cn/\log(n)$, $\exists$ one-bit leakage functions with $I(\mathbf{s}; \mathbf{Leak}) \geq c$.

$\implies$ Leakage resilience essentially requires $t = \Omega(n)$.

Our results

Let $s \mapsto \pi$ be linear SS. **Leak** $= (f_1(\pi_1), \ldots, f_n(\pi_n))$ for $f_1, \ldots, f_n \colon \mathbb{F}_p \to \{0,1\}$. $p = 2^{o(n)}$.

Theorem (Main)

For all secrets $s_1, s_2 \in \mathbb{F}_p$,

$$\mathrm{SD}(\textbf{Leak} \mid \mathbf{s} = s_1 \ , \ \textbf{Leak} \mid \mathbf{s} = s_2) \leq \text{New Proxy}.$$

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Corollary (Going below $t = n/2$)

If $\Pr[f_i = 0] < \epsilon$ then Shamir's secret sharing is Leakage Resilient once $t \geq 0.01n$.

Main Result

Let $f_1, \ldots, f_n \colon \mathbb{F}_p \to \{0,1\}$. Let $I \subseteq [n]$. Define

$$f(I) := \max_{s \in \mathbb{F}_p} \left| \Pr\left[\bigoplus_{i \in I} f_i(\pi_i) = 0 \,\middle|\, \mathbf{s} = s \right] - \Pr\left[\bigoplus_{i \in I} f_i(\pi_i) = 0 \right] \right| .$$

Theorem (New Proxy)

$$\mathrm{SD}(\mathbf{Leak} \mid \mathbf{s} = s_1 \ , \ \mathbf{Leak} \mid \mathbf{s} = s_2)^4 \le p^{O(1)} \sum_{I \subseteq [n]} f(I)^2.$$

Previous Proxy (BDIR '19)

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Previous Barrier (MPSW '19)

$\exists$ functions $f_1, \dots, f_n$ with $\mathit{Proxy} \ge 1$ whenever $t \le n/2$. Even if $\Pr[f_i = 0] \approx 0$.

(Heuristic) Interpretation of Main Result

Suppose attackers wish to distinguish

$$\mathbf{s} = 0 \qquad \text{and} \qquad \mathbf{s} = 1.$$

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$$\mathrm{Cov}(\mathbf{s}, s_j) = \epsilon_j, \qquad j = 1 \ldots \ell.$$

Aggregating all information, maximum likelihood of $\mathbf{s}$ gives (optimal) advantage

$$\mathop{\mathbb{E}}_{s_1,\ldots,s_\ell} \left| \Pr\left[\mathbf{s} = 0 \mid s_1, \ldots, s_\ell\right] - \Pr\left[\mathbf{s} = 1 \mid s_1, \ldots, s_\ell\right] \right| = \Theta\left(\sum_j \epsilon_j^2\right).$$

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Suggested attack

Given the leakage, compute $s_I = \bigoplus_{i \in I} f_i(\pi_i)$. Covariance of s_I with $\mathbf{s}$ is $\epsilon_I = p^{O(1)} f(I)$.
Using MLE, advantage is

$$\sum_I \epsilon_I^2 = p^{O(1)} \sum_I f(I)^2.$$

Open problems

Conjecture (Benhamouda, Degwekar, Ishai, Rabin '18)

Shamir SS over $\mathbb{F}_p$ with $t/n = \alpha > 0$ and **arbitrary** 1-bit leakage from each share satisfies

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Under same conditions, prove

$$f([n]) = \max_{s \in \mathbb{F}_p} \left| \Pr\left[\bigoplus_{i=1}^{n} f_i(\pi_i) = 0 \,\middle|\, \mathbf{s} = s \right] - \Pr\left[\bigoplus_{i=1}^{n} f_i(\pi_i) = 0 \right] \right| = \exp(-O_\alpha(n)).$$

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Problem (Generalize to multi-bit leakage)

Find useful bound for (1) when $f_1, \ldots, f_n \colon \mathbb{F}_p \to \{0,1\}^m$ for $m > 1$.

Thank You!