Revisiting Time-Space Tradeoffs for Function Inversion

Spencer Peters

Noah S.D.

Siyao Guo

Sasha Golovnev

Function Inversion

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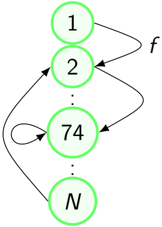

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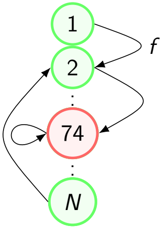

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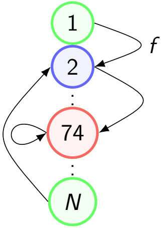

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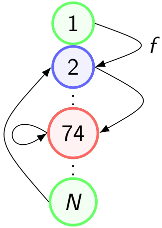

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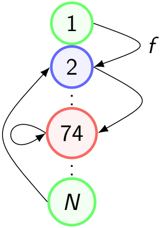

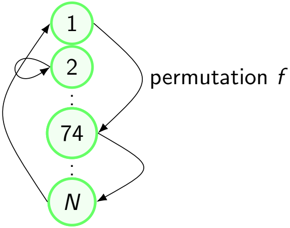

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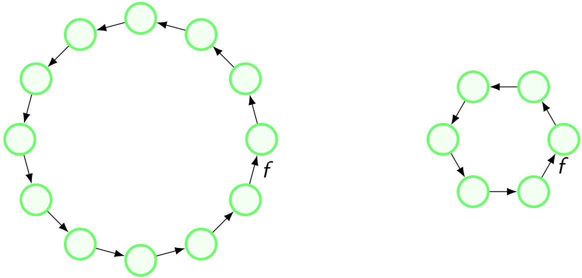

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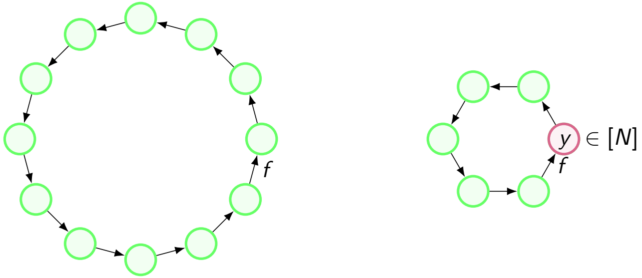

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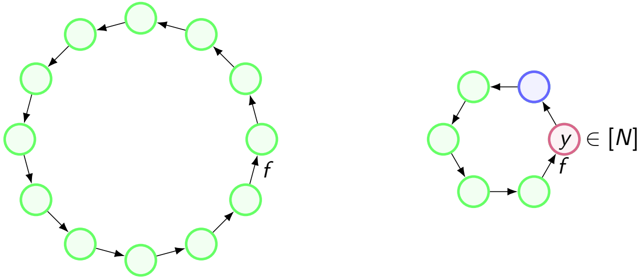

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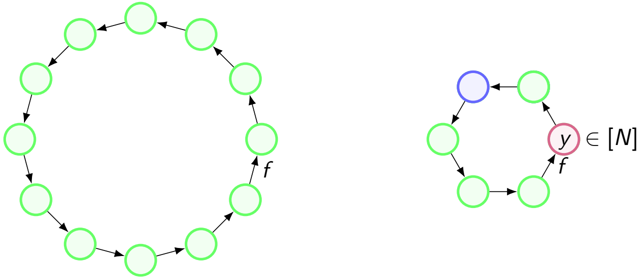

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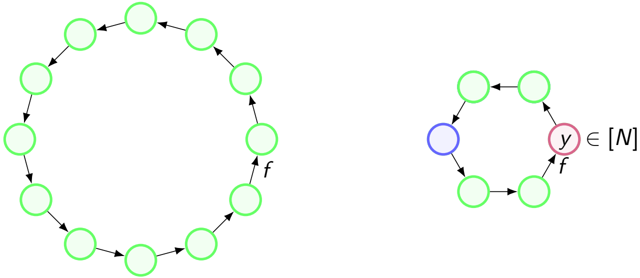

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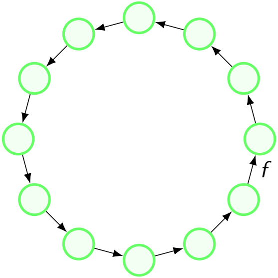

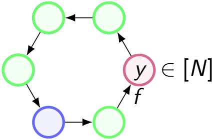

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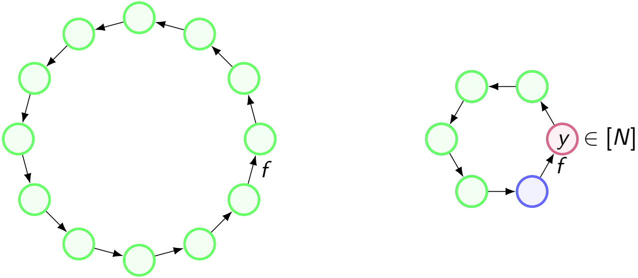

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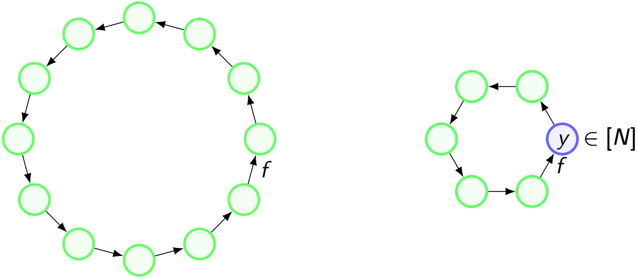

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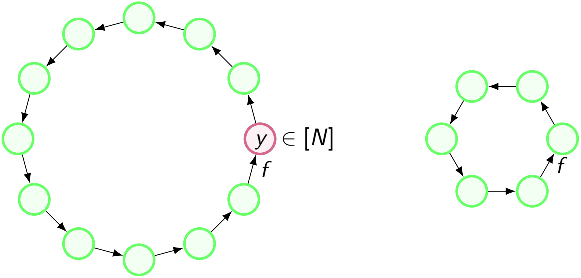

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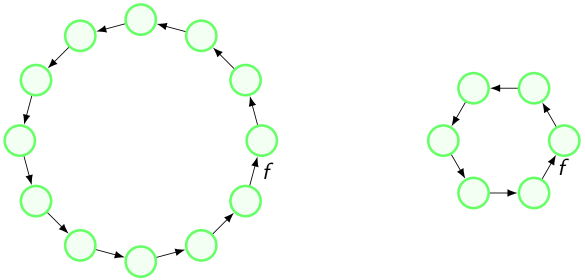

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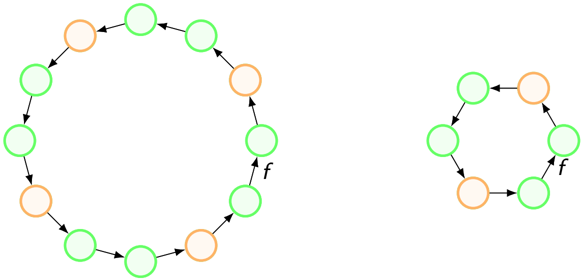

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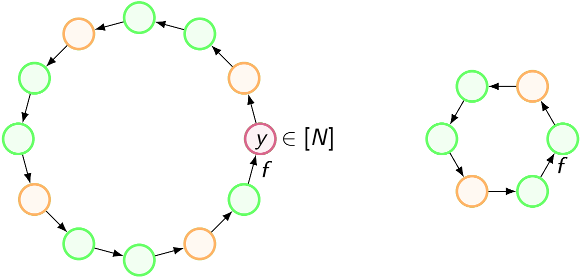

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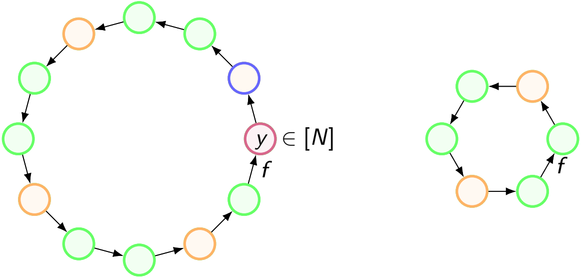

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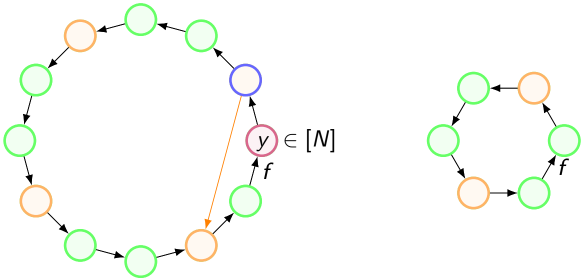

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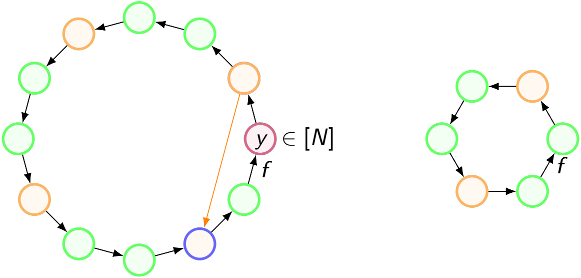

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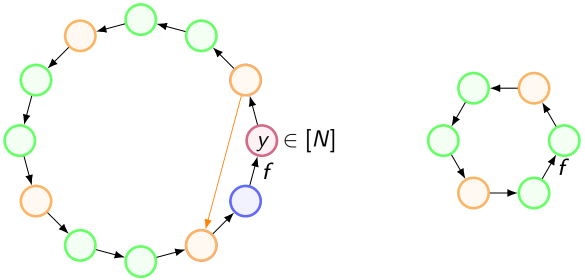

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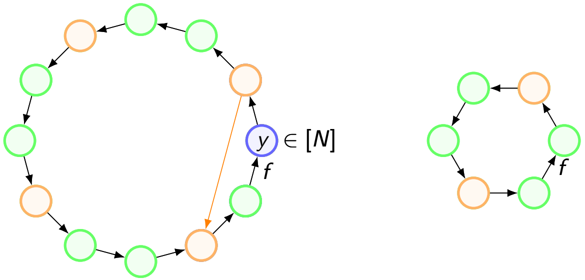

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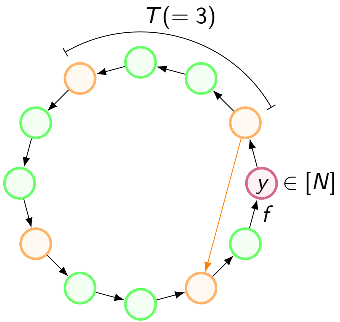

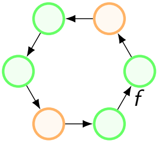

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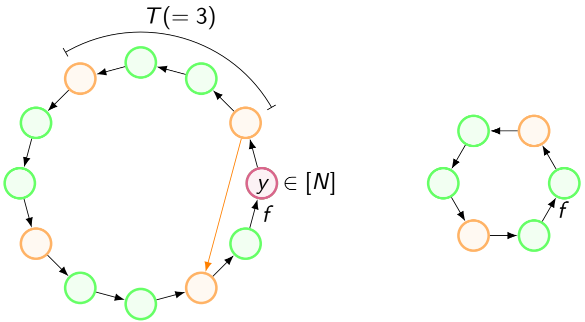

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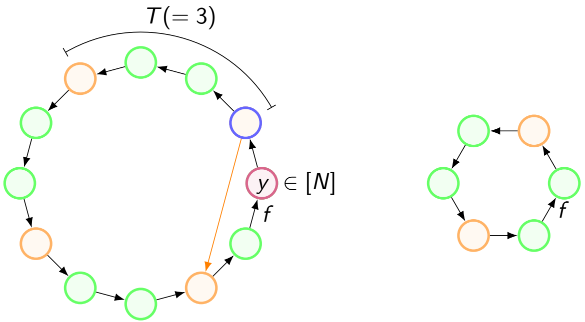

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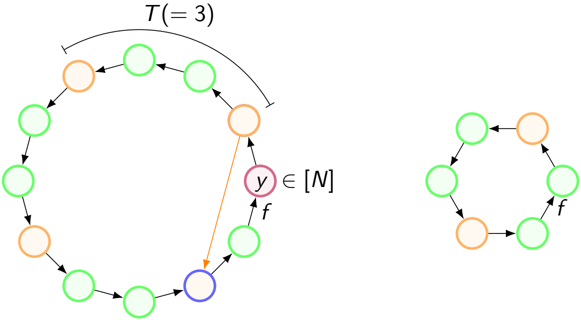

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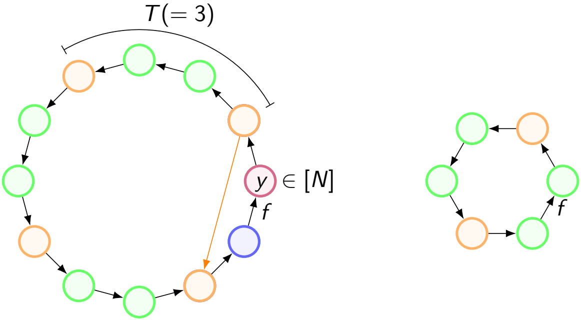

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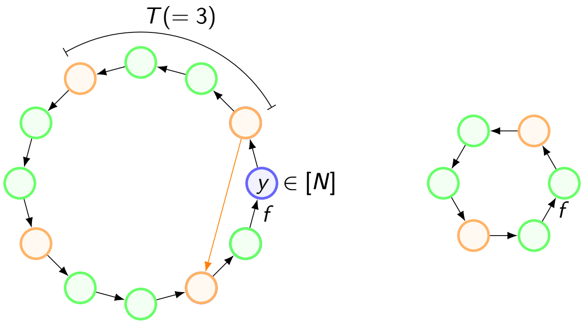

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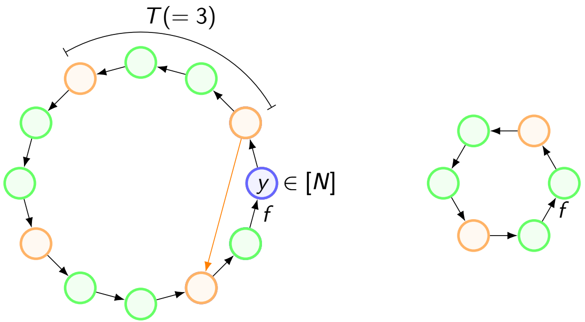

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- We need to store about N/T points total.

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- **Goal:** design a pair of algorithms $(\mathcal{P}, \mathcal{A})$ such that

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- A: Sort of and sort of!

Our Results

- Result 1: A simple improvement
  to Fiat and Naor's algorithm in the regime $T > S$.
- Result 2: A tight lower bound for a natural class of
  non-adaptive function inversion algorithms.
- Not in this talk: equivalences between variants of function
  inversion.

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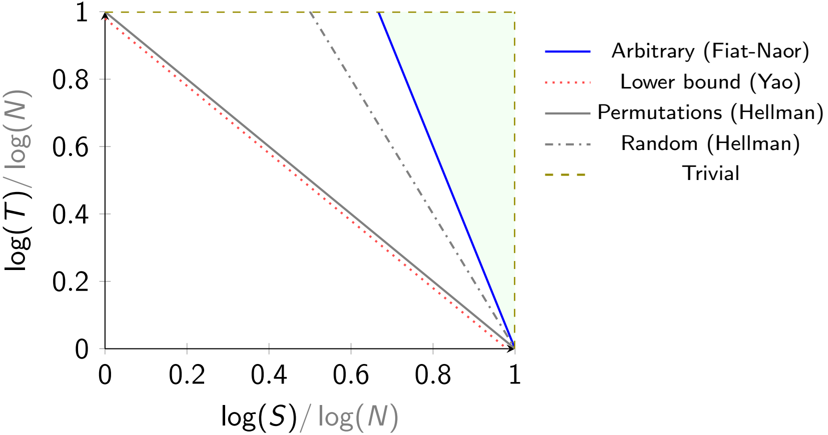

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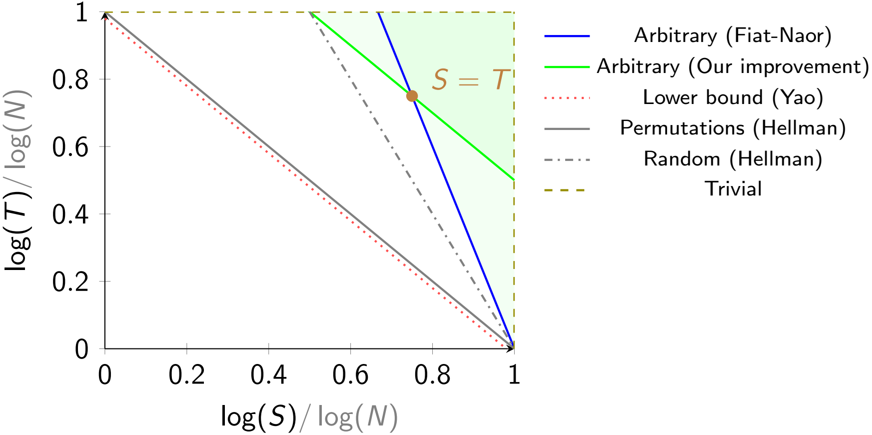

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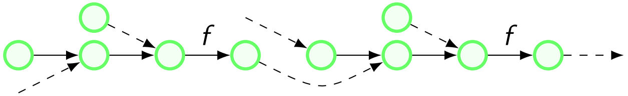

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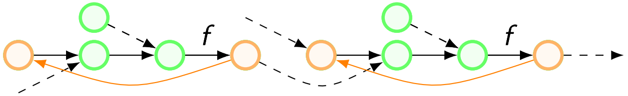

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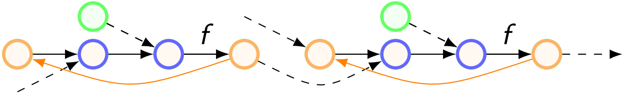

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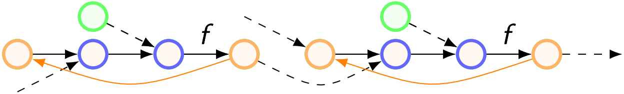

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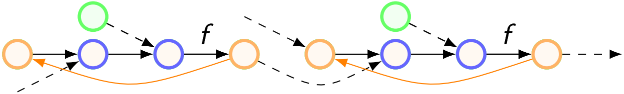

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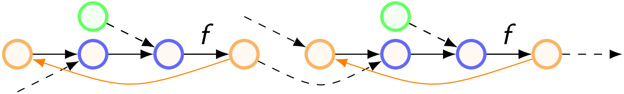

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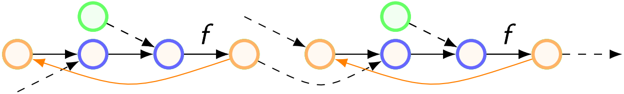

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- So, can repeatedly apply the basic scheme to many compositions $g_i \circ f$, for suitably chosen "rerandomization" functions g_i.

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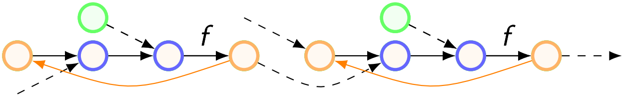

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- To boost the coverage, observe that inverting $g \circ f$ on $g(y)$ is often enough to invert f on y.
- So, can repeatedly apply the basic scheme to many compositions $g_i \circ f$, for suitably chosen "rerandomization" functions g_i.
- For *random* functions, Hellman showed (heuristically) this can be made to work. 

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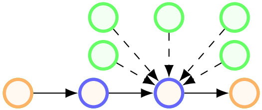
- Hellman's argument fails for arbitrary functions.



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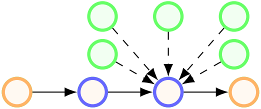

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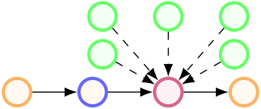

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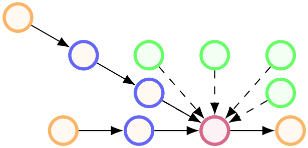

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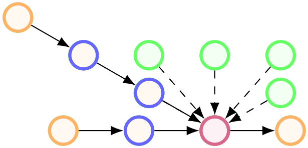

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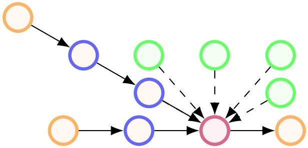

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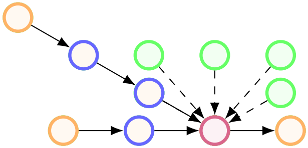

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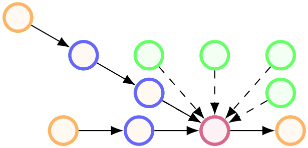

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- More precisely, the "rerandomization" functions are sampled using rejection sampling so that their range is $[N] - L$.

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- How do $\mathcal{A}$ and $\mathcal{P}$ agree on the *same* list of random values x_i?

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- Corrigan-Gibbs and Kogan speculated that there is no non-adaptive algorithm with

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- We show that **the simple algorithm above is asymptotically optimal among guess-and-check algorithms**.

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  - Encoding is $(\alpha, i_1, \dots, i_N)$.
  - For each y, decoder again runs $\mathcal{A}(\alpha, y)$ and receives $x_1, \dots, x_T$. It sets $f^{-1}(y) = x_{i_y}$.

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  - Better algorithms for inverting injective functions?

Thank you!

I'm happy to take additional questions offline.

You can ping me at `speters@cs.cornell.edu`.

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