A Direct Key Recovery Attack on SIDH

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Contribution

- Merge of two papers:
  - M.–Martindale (2022), independent from Castryck–Decru (2022)
  - Wesolowski (2022)
- Proof-of-Concept Implementation — mainly by Panny and Pope — available at

  https://github.com/Breaking-SIDH/direct-attack

Supersingular Isogeny with Torsion (SSI-T)

SSI-T

Let p be a large prime. There are public elliptic curves E_{start} and E_{Bob}, and a secret isogeny $\varphi_{\text{Bob}} : E_{\text{start}} \to E_{\text{Bob}}$ of degree 3^b.

$$\text{Given } \varphi_{\text{Bob}}(E_{\text{start}}[2^a]),$$

compute φ_{Bob}.

History of the SIDH problem

2011 De Feo, Jao, and Plût introduce SIDH

2016 Galbraith, Petit, Shani, Ti give an active attack

2017 Petit describes passive attacks on some parameter sets - SIDH not affected

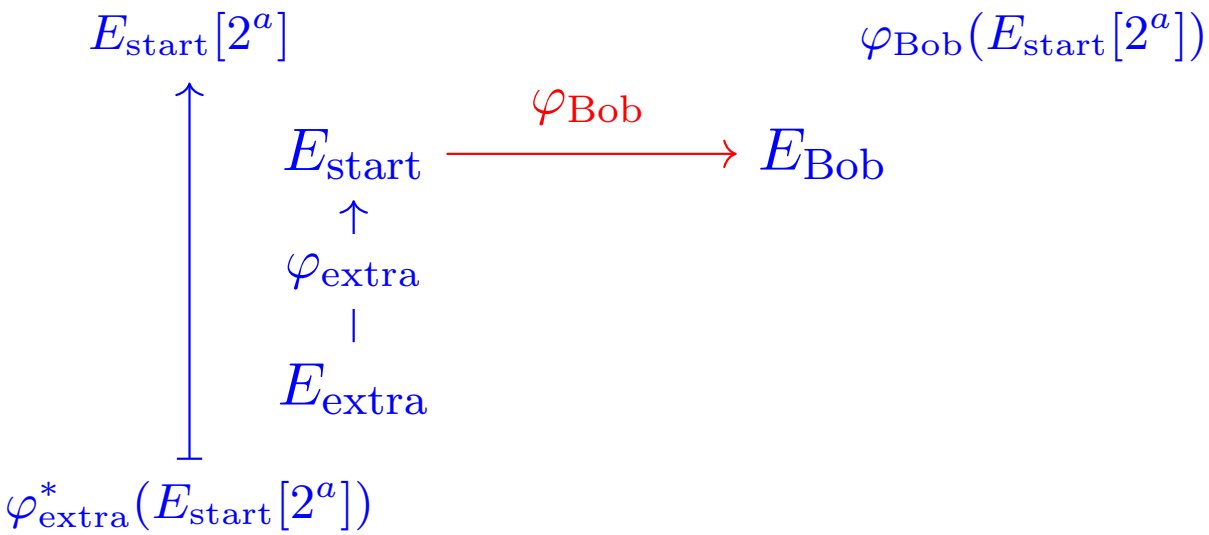
2020 de Quehen, Kutas, Leonardi, Martindale, Panny, Petit, Stange give passive attacks on more parameter sets

The attack

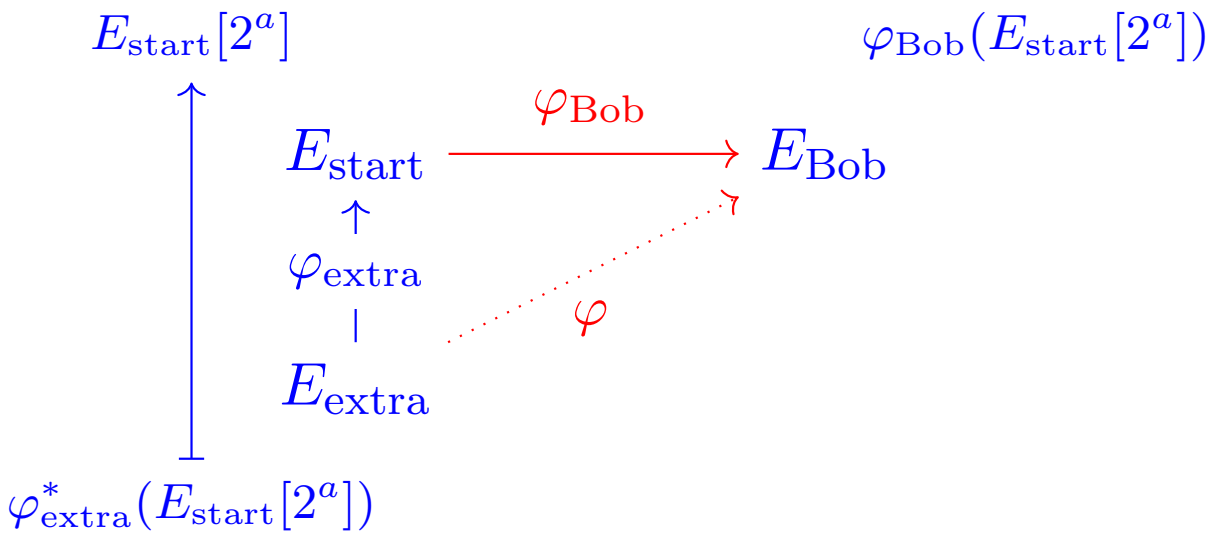
$E_{\text{start}}[2^a]$ $\qquad\qquad$ $\varphi_{\text{Bob}}(E_{\text{start}}[2^a])$

$$E_{\text{start}} \xrightarrow{\varphi_{\text{Bob}}} E_{\text{Bob}}$$

The attack

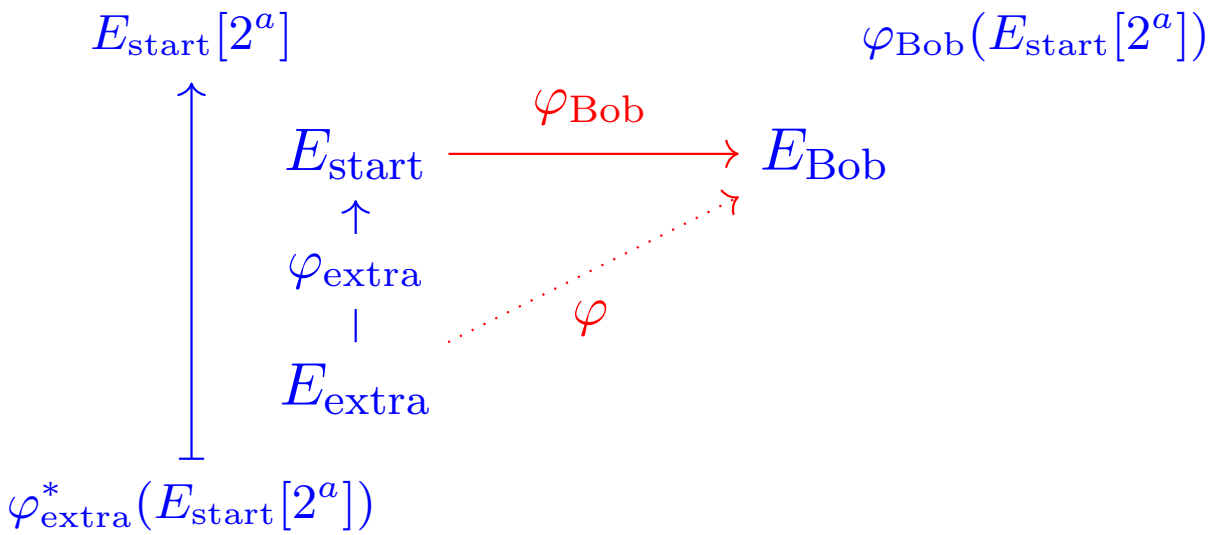

$E_{\text{start}}[2^a]$
$\varphi_{\text{Bob}}(E_{\text{start}}[2^a])$
φ_{Bob}
E_{start}
E_{Bob}
φ_{extra}
E_{extra}
$\varphi^*_{\text{extra}}(E_{\text{start}}[2^a])$

The attack



$$\Phi = \begin{pmatrix} \varphi_{\text{extra}} & -\widehat{\varphi}_{\text{Bob}} \\ * & * \end{pmatrix} : E_{\text{extra}} \times E_{\text{Bob}} \to E_{\text{start}} \times E_{\text{known}}$$

The attack



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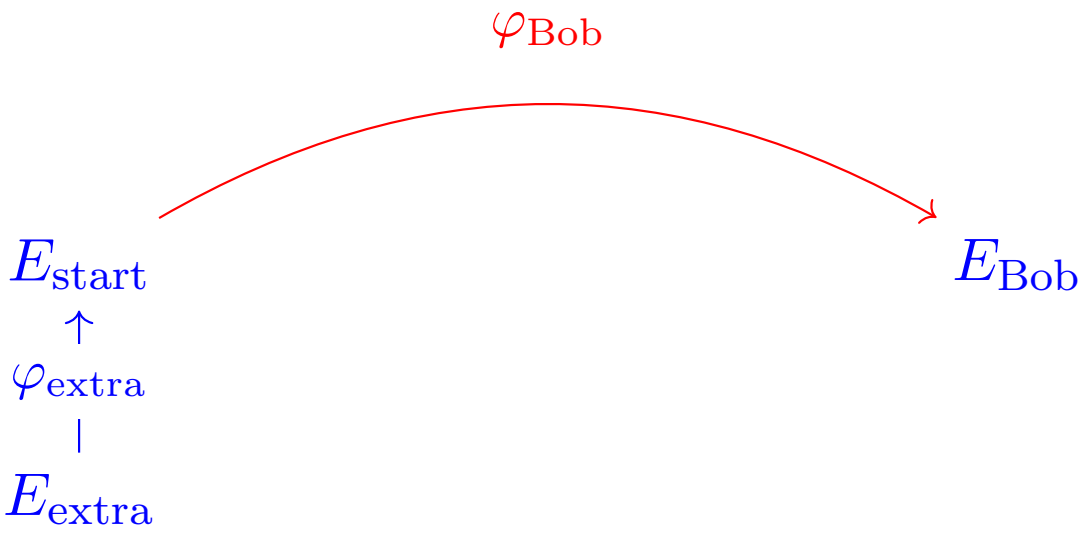
$$\text{Ker}(\Phi) = \left(3^b E_{\text{extra}}[2^a], \varphi(E_{\text{extra}}[2^a])\right)$$

Unknown Endomorphism Ring

- $\deg(\varphi_{\text{extra}}) = 2^a - 3^b$ may not be smooth
- $\deg(\varphi_{\text{extra}}) = e2^{a-j} - 3^{b-i}$ for some small e, i, and j
- This determines the complexity of the attack ($L_{2^a}(1/2)$).



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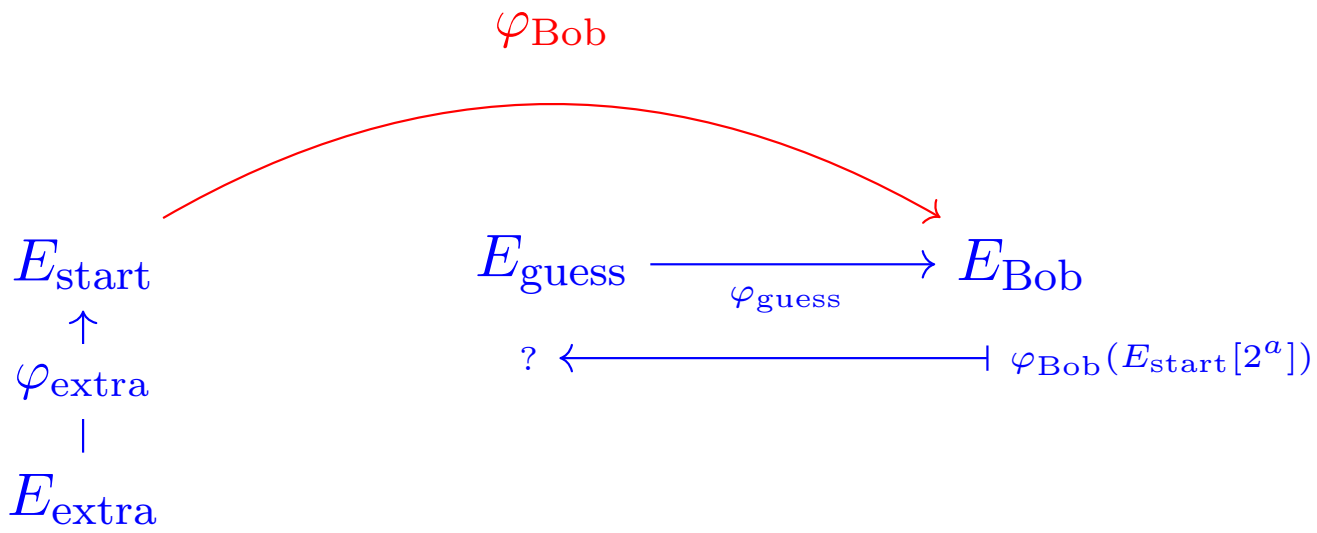

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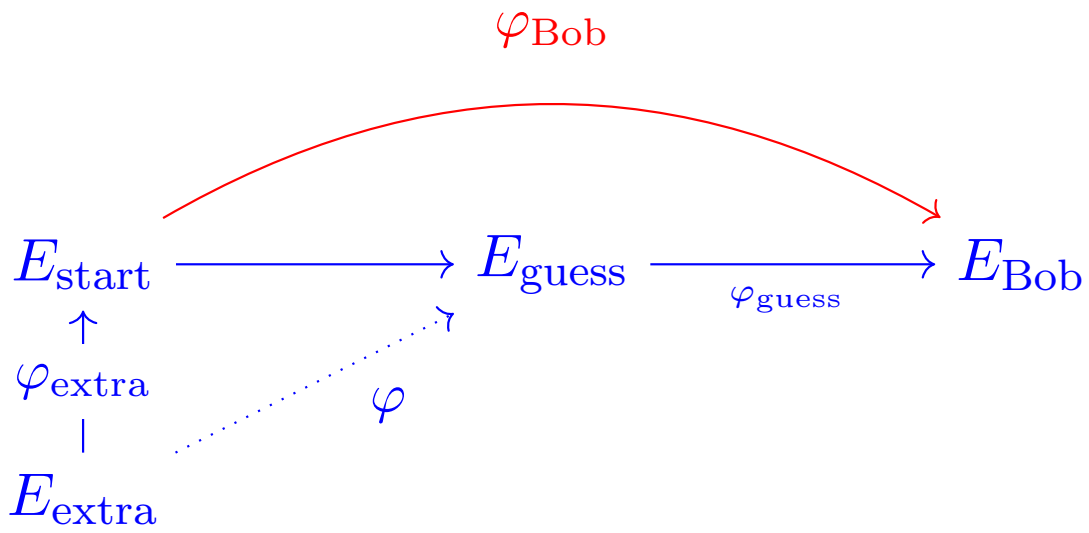

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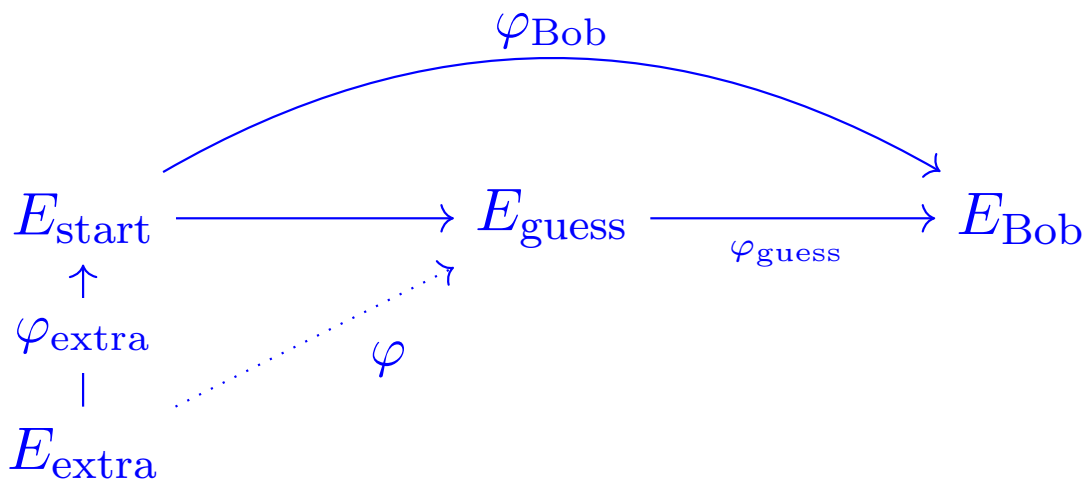

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Theorem

Assume the generalised Riemann hypothesis. Let E_{start} be a supersingular curve, together with a basis $(\alpha_i)_{i=1}^4$ of $\text{End}(E_{\text{start}})$. There exists a polynomial-time algorithm that finds an isogeny $\varphi_{\text{extra}} : E_{\text{extra}} \to E_{\text{start}}$ of degree $2^a - 3^b$.

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The End

Thanks for your attention!

Questions?