Finding many collisions via quantum walks

Application to lattice sieving

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Collision-finding

Classical collision algorithms

$$f : \{0,1\}^n \rightarrow \{0,1\}^n$$

A first algorithm

- Create a sorted list $(x_i, f(x_i))$ of size $2^{n/2}$
- Look for collisions
- Also works for collisions between L_1 and L_2

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Improved algorithms

- Memoryless
- Parallelizable
- Same query complexity

Other cases

Finding 2^t collisions instead of 1

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Lower bounds

Matching query lower bound in all cases $\left(\Omega\left(2^{t/2+m/2}\right)\right)$

Quantum collisions

BHT

BHT algorithm

- Take a list $L = (f(y_0), \ldots, f(y_{2^u}))$
- *Search* for an x with $f(x) = f(y_i)$ and $x \neq y_i$
- Cost 2^u memory, $2^u + \sqrt{\frac{2^n}{2^u}}$ time

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Finding 2^t collisions

- Use one larger list of size $2^{n/3+2t/3}$
- Do 2^t quantum searches $\left(\text{cost } 2^t \times \sqrt{\frac{2^n}{2^{n/3-2t/3}}}\right)$

Lower bound [LZ19]

General query lower bound $\Omega\left(2^{m/3+2t/3}\right)$

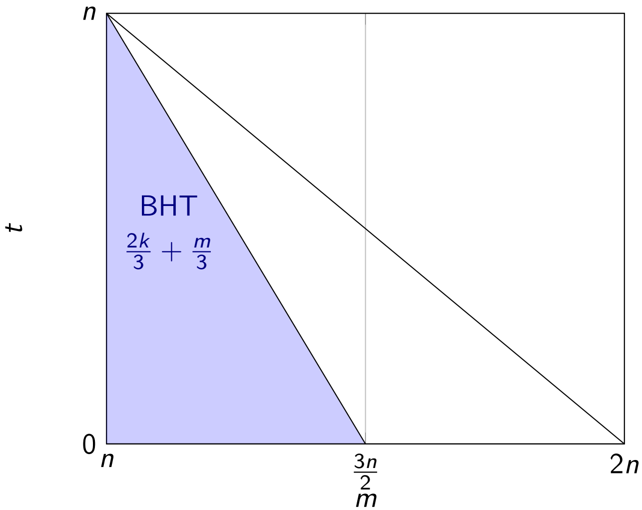
With larger m

BHT algorithm

List of size $2^{m/3}$

- Only 2^{2n-m} inputs are part of a collision
- Need $m/3 \geq m - n$, otherwise the list might contain no relevant input
- $\implies m \leq 3/2n$

Summary



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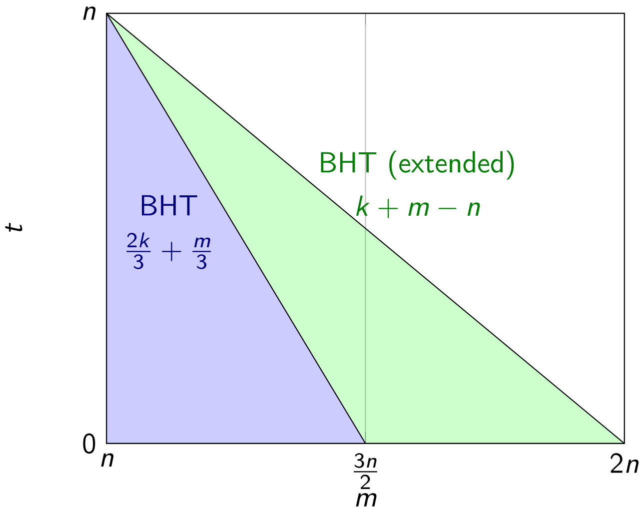

Quantum walks

Random walks

Idea

- Start at a random node
- Walk to adjacent nodes, stop when a good node is found

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Parameters

- Proportion of marked nodes p
- Number of walks steps to sample a random node $1/\varepsilon$
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Total cost

$$S + \frac{1}{p}\left(\frac{1}{\varepsilon}U + T\right)$$

Example: Walk-based collision finding

Definition (Johnson graph)

- Nodes are sets of 2^r elements among 2^n
- N_1 and N_2 are adjacents is $|N_1 \cap N_2| = 2^r - 1$
- $\varepsilon = \frac{2^n}{2^r(2^n - 2^r)} \simeq 2^{-r}$

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Complexity

$$2^r + \frac{1}{2^{2r-n}}(2^r \times 1 + 2^r) \simeq \max(2^r, 2^{n-r})$$

Quantum walks

Principle

Simulate a quantum search on a graph using a walk update operator

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MNRS framework

- Proportion of marked nodes p
- Number of walks steps to sample a uniformly random node $1/\varepsilon$
- Cost to construct the superposition of all nodes S
- Cost to walk to an adjacent node U
- Cost to check is a node is marked T
- Total cost $S + \frac{1}{\sqrt{p}}\left(\frac{1}{\sqrt{\varepsilon}}U + T\right)$

Ambainis algorithm

Problem

$f : \{0,1\}^n \to \{0,1\}^m$, $n < m \leq 2n$, find a collision

MNRS walk in a Johnson graph

- Create a random list of elements of size 2^r
- Apply the walk operator $\sqrt{2^r}$ times
- Test if the node contains a collision

Complexity

- Setup : 2^r
- Fraction of marked nodes : 2^{2r-m}
- Assume Update and Test polynomial
- Cost $2^r + 2^{m/2-r} \times 2^{r/2} \simeq \max(2^r, 2^{m/2-r/2})$

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Aim

Having a procedure that allows us to extract a collision and preserve a useful quantum data structure

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  - With smaller sets (-collisions)
  - In a smaller ambient set (avoid the extracted preimages)

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Efficient history-independent operations

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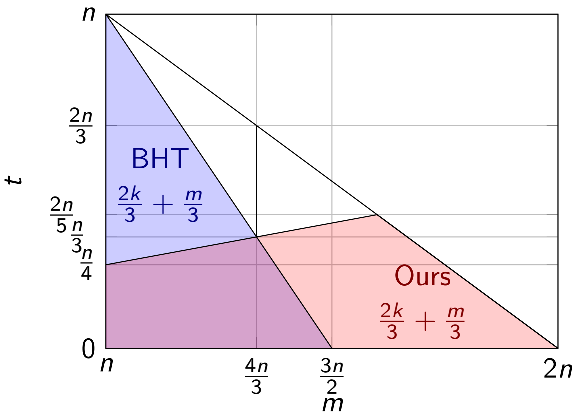
The next quantum walk needs to work

- The quantum states after extraction must be nodes in the graph
- It is fine to start from collision-free nodes
- Nodes with collisions are a small fraction of the nodes

Quantum collisions now



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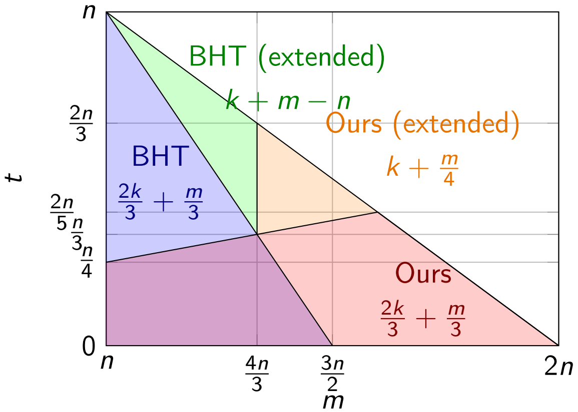

With golden collisions

Golden collisions

Find (x, y) such that $f(x) = f(y)$, plus $P(x, y)$ is true.

Algorithm

The same algorithm works:

- Add the test in the walk
- Count/extract only golden collisions
- Works if a random node contains a *golden* collision with small probability.

Quantum lattice sieving

Lattice sieving

- Start with many vectors v_i
- Find many $v_i \pm v_j$ with smaller norm
- Iterate

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- Find good $v_i \pm v_j$ with a quantum walk
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Improvement

- Original quantum walk $2^{0.2570d+o(d)}$
- Improved quantum walk $2^{0.2563d+o(d)}$

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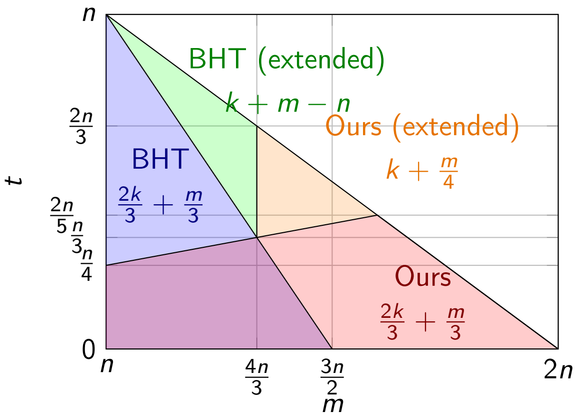

BHT (extended)
$k+m-n$
Ours (extended)
$k+\frac{m}{4}$
BHT
$\frac{2k}{3}+\frac{m}{3}$
Ours
$\frac{2k}{3}+\frac{m}{3}$
n
$\frac{2n}{3}$
$\frac{2n}{5}$
$\frac{n}{3}$
$\frac{n}{4}$
0
t
n
$\frac{4n}{3}$
$\frac{3n}{2}$
$2n$
m